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On the Relativistic Tunneling and Above Barrier Transmission in Some One-dimensional Structured Barrier Systems

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Abstract. In this paper we present the transmission and tunneling coefficients computed on the basis of a relativistic transfer matrix approach (1-dimensional Dirac equation) for a structured potential barrier, built of $(N_1 + N_2 + 2)$ potential scatterers: two rectangular potential steps and two kinds of δ-like wells. In the particular case of a rectangular barrier the reflectionless condition is solved and the expressions of admissible energies are given explicitly.

1 Introduction

In 1956 Callaway and in 1957 Callaway and Woods [1, 2] introduced relativistic considerations into condensed matter physics. After the works of Glasser and Davison [3, 4] the interest towards relativistic studies in solvable one-dimensional models based on the Dirac equation increased strongly [5–10]. In result the bases of a new field of research in the solid state physics called "relativistic condensed matter physics" were laid down [9]. We note that this new field naturally comprises the processes of relativistic tunneling [11] and above barrier transmission [12]. The non-relativistic tunneling through barriers in which the structure of the tunneling region is taken into account [13, 14] is of definite interest for the explanation of the electron tunneling paths in certain biological molecules [15, 16]. These considerations naturally bring us to the idea to carry out the corresponding relativistic study for the same kind of barriers. So this paper presents some results of the relativistic above-barrier transmission and tunneling through one-dimensional time-independent structured barriers, being combinations of rectangular and δ-like potentials. In order to compute the corresponding relativistic coefficients of transmission and tunneling we are going to use the transfer-matrix approach adapted to the relativistic considerations based on the Dirac equation by Glasser and Davison [3, 4] and Subramanian and Bhagwat [5].

2 Transfer matrix approach to relativistic coefficients of transmission and reflection

First we shall define the relatistic coefficients of transmission T and reflection R in terms of the relativistic transfer-matrix M.

The time-independent one-dimensional Dirac equation for an electron with proper mass m_0 and energy E in the potential field $V(x)$ reads [3, 4, 17]:

$$\left[i\hbar c\sigma_x \frac{d}{dx} - m_0 c^2 \sigma_z + V(x)\right]\Psi(x) = E\Psi(x), \tag{1}$$

where $\Psi(x)$ is a 2-component spinor wave function

$$\Psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix} \quad \text{and} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For a localized potential having constant values V_1 in the region $(-\infty, x_1)$ and V_3 in the region (x_2, ∞), where $x_1 < x_2$, the solutions

$$\Psi^j = \begin{pmatrix} \psi_j^{(1)} \\ \psi_j^{(2)} \end{pmatrix}, \quad j = 1, 3$$

of equation (1) in these regions are [3, 4]

$$\Psi^j(x) = A^j \exp(i\kappa_j x) + B^j \exp(-i\kappa_j x), \quad j = 1, 3.$$

For the amplitudes we have

$$A^j = \begin{pmatrix} \alpha_j^{(1)} \\ \alpha_j^{(2)} \end{pmatrix} = \begin{pmatrix} -\gamma_j \\ 1 \end{pmatrix} \alpha_j^{(2)}, \quad B^j = \begin{pmatrix} \beta_j^{(1)} \\ \beta_j^{(2)} \end{pmatrix} = \begin{pmatrix} \gamma_j \\ 1 \end{pmatrix} \beta_j^{(2)}, \tag{2}$$

where $\kappa_j^2 = (\hbar c)^{-2}(\varepsilon - V_j)(\varepsilon - V_j + 2m_0c^2)$, $\varepsilon = E - 2m_0c^2$,

$$\gamma_j = (\varepsilon - V_j)(\hbar c\kappa_j)^{-1}$$

and E is the relativistic energy of the electron.

In the same way as in the non-relativistic considerations [18, 19] the transfer-matrix $M(E)$, depending on the localized potential $V(x)$, and connecting the spinor amplitudes A^3 and B^3 with A^1 and B^1 on the two sides of the potential is defined by the matrix equation

$$\begin{pmatrix} \alpha_3^{(2)} \\ \beta_3^{(2)} \end{pmatrix} = M \begin{pmatrix} \alpha_1^{(2)} \\ \beta_1^{(2)} \end{pmatrix}. \tag{3}$$

In analogy with the nonrelativistic considerations in [20, 21] we introduce the relativistic reflection coefficient R and the relativistic transmission coefficient T by the following expressions

$$R = |j_r||j_i|^{-1}, \quad T = |j_t||j_i|^{-1},$$

where j_r, j_i and j_t are the probability current densities of the incident, reflected and transmitted waves denoted by $A^1 \exp(i\kappa_1 x)$, $B^1 \exp(-i\kappa_1 x)$ and $A^3 \exp(i\kappa_3 x)$ respectively. For the Dirac equation (1) $j = -c\Psi^+\sigma_x\Psi$ [17], and we have

$$j_i = 2c\gamma_1|\alpha_1^{(2)}|^2, \quad j_r = -2c\gamma_1|\beta_1^{(2)}|^2 \quad \text{and} \quad j_t = 2c\gamma_3|\alpha_3^{(2)}|^2,$$

which implies

$$R = |\beta_1^{(2)}|^2|\alpha_1^{(2)}|^{-2}, \tag{4}$$

$$T = \gamma_3\gamma_1^{-1}|\alpha_3^{(2)}|^2|\alpha_1^{(2)}|^{-2}. \tag{5}$$

Now from (3), (4) and (5) if $\beta_3^{(2)} = 0$ (there are only transmitted waves in the region (x_2, ∞)) it follows that

$$T = \begin{cases} |M_{22}|^{-2} \det M & \text{at } V_1 \neq V_3 \\ |M_{22}|^{-2} & \text{at } V_1 = V_3 \end{cases} \tag{6}$$

$$R = \begin{cases} T|M_{21}|^2 (\det M)^{-1} & \text{at } V_1 \neq V_3 \\ T|M_{21}|^2 & \text{at } V_1 = V_3 \end{cases} \tag{7}$$

and the condition for transmission without reflection ($R = 0$) is

$$|M_{21}|^2 = 0 \quad \text{or} \quad |M_{22}|^2 = \det M.$$

Clearly in order to compute the relativistic coefficients R and T it is necessary to know the components of M as functions of the system's parameters. Further we shall deal only with the coefficient T (6), which in case of above barrier transmission will be denoted by D and in case of tunneling it will be denoted by J. In the next section D and J will be worked out as functions of the system's parameters.

3 The model and its total transfer matrix

We consider 1-dimensional structured barrier $V(x)$ with length L built of $(\mathrm{N}_1 + \mathrm{N}_2 + 2)$ individual potential scatterers (two rectangular potential steps and two kinds $(\mathrm{N}_1 + \mathrm{N}_2)$ of δ-like potential wells) located at the points Z

$$Z = \begin{cases} 0 & v = 0 \\ Z_v^{(1)} = 2b_1 + 2a_1(v-1), & v = 1, \ldots, \mathrm{N}_1 \\ Z_v^{(2)} = 2b_1 + 2(\mathrm{N}_1 - 1)a_1 + 2a + 2(v-1)a_2, & v = 1, \ldots, \mathrm{N}_2 \\ L = 2\sum_{k=1}^{2} [b_k + (\mathrm{N}_k - 1)a_k] + 2a, & v = \mathrm{N}_1 + \mathrm{N}_2 + 2, \end{cases}$$

where $2b_1$ is the distance between the potential step at $Z = 0$ and the first scatterer of the 1-dimensional chain of N_1 scatterers with lattice constant $2a_1$; $2a$ is the distance between the last scatterer of the first chain and the first one of the second 1-dimensional chain of N_2 scatterers with lattice constant $2a_2$; $2b_2$ is the distance between the last scatterer of the second chain and the potential step at $Z = L$. For the potential energy $V(x)$ in the 1-dimensional Dirac equation (1) we can write

$$V(x) = \begin{cases} V_1, & x \in \Omega^1 = (-\infty, 0) \\ V_2 - 2a_1\xi_1 \sum_1^{\mathrm{N}_1} \delta[x - 2b_1 - 2(v-1)a_1] - 2a_2\xi_2 \sum_1^{\mathrm{N}_2} \delta[x - 2b_1 - (\mathrm{N}_1 - 1)2a_1 \\ \quad - 2a - (v-1)2a_2], & x \in \Omega^2 = (0, L) \\ V_3, & x \in \Omega^3 = (L, \infty), \end{cases} \tag{8}$$

where $2a_k\xi_k = (2m_0)^{-1}\hbar^2 2a_k\eta_k$, $\eta_k > 0$ are the strengths of the δ-potential wells $(k = 1, 2)$ and having dimension $(\text{length})^{-2}$, the constants of normalization $2a_k$ being introduced for reasons of convenience [22], and $V_2 > V_1$, $V_2 > V_3$. The δ-like scatterers at the points $Z_v^{(k)}$ divide $\Omega^2 = (0, L)$ additionally into $(\mathrm{N}_1 + \mathrm{N}_2 + 1)$ subregions with constant potential V_2. Then v takes the following values:

$$v = \begin{cases} 0 & \text{for } x \in \Omega^1 \\ 1, \ldots, \mathrm{N}_1 + \mathrm{N}_2 + 1 & \text{for } x \in \Omega^2 \\ \mathrm{N}_1 + \mathrm{N}_2 + 2 & \text{for } x \in \Omega^3. \end{cases}$$

So, for every v the constant potential solutions

$$\Psi_v^j = A_v^j \exp(i\kappa_j x) + B_v^j \exp(-i\kappa_j x), \quad j = 1, 2, 3$$

of the Dirac equation (1) have amplitudes

$$A_\nu^j = \begin{pmatrix} -\gamma_j \\ 1 \end{pmatrix} \alpha_\nu^{(2)}, \quad B_\nu^j = \begin{pmatrix} \gamma_j \\ 1 \end{pmatrix} \beta_\nu^{(2)},$$

where κ_j and γ_j are given in (2).

In order to compute the coefficients D and J according to (6) for the above introduced barrier $V(x)$ we need the corresponding transfer matrix M. From (3) and from the known group properties of the relativistic transfer matrices [12, 23, 24] it immediately follows that the total transfer matrix M

$$\begin{pmatrix} \alpha^{(2)}_{N_1+N_2+2} \\ \beta^{(2)}_{N_1+N_2+2} \end{pmatrix} = M \begin{pmatrix} \alpha_0^{(2)} \\ \beta_0^{(2)} \end{pmatrix}$$

is given as a matrix-product of the transfer matrices $M^{(Z)}$ for the two potential steps at $Z = 0, L$ and the transfer matrices $M_\nu^{(k)}$, $k = 1, 2$, for the δ-like wells at the points $Z_\nu^{(k)}$, i.e.

$$M = M^{(L)} \prod_{\nu = N_1+N_2}^{N_1+1} M_\nu^{(2)} \prod_{\nu = N_1}^{1} M_\nu^{(1)} M^{(0)}. \tag{9}$$

Now from the well known continuity conditions [3, 4, 5] on the spinor wave solutions $\Psi_\nu^j(x)$ at every point Z of finite or infinite discontinuity of the potential (8) we compute the matrix elements of the corresponding relativistic matrices in the above matrix product.

For the above-barrier relativistic transmission coefficient D, which means that κ_2 in (6) is real, we are going to use the general expression obtained in [12].

Passing to the tunneling coefficient J we note first that this consideration is a generalization of the non-relativistic consideration for the same potential given in [14]. As in the non-relativistic case a tunneling process takes place if $i\kappa_2 = \kappa$ in (6) is real (in accordance with [11]), which leads to $\gamma_2 = i\gamma$ and γ is real. From the continuity condition of Glasser and Davison [3, 4], imposed at the points $Z = 0, L$, we obtain for $M^{(L)}$ and $M^{(0)}$ the following expression:

$$M^{(L)} = \mathscr{T}^{-1}(\kappa_3, L) R^{(L)} \mathscr{T}(-i\kappa, L), \quad M^{(0)} = \mathscr{T}^{-1}(-i\kappa, 0) R^{(0)} \mathscr{T}(\kappa, 0),$$

where the translational matrices $\mathscr{T}(k, z)$ are of the form

$$\mathscr{T}(k, z) = \begin{pmatrix} \exp(ikz) & 0 \\ 0 & \exp(-ikz) \end{pmatrix},$$

the relativistic R-matrices $R^{(Z)}$ are

$$R^{(0)} = (2\gamma)^{-1} \begin{pmatrix} \gamma - i\gamma_1 & \gamma + i\gamma_1 \\ \gamma + i\gamma_1 & \gamma - i\gamma_1 \end{pmatrix}, \quad R^{(L)} = (2i\gamma_3)^{-1} \begin{pmatrix} i\gamma_3 - \gamma & i\gamma_3 + \gamma \\ i\gamma_3 + \gamma & i\gamma_3 - \gamma \end{pmatrix},$$

and $\kappa^2 = (\hbar c)^{-2}(V_2 - \varepsilon)(\varepsilon - V_2 + 2m_0c^2)$, $\gamma = (\varepsilon - V_2)(\hbar c\kappa)^{-1}$.

The infinite discontinuities, represented by the two kinds of δ-like wells, we take into account through an equivalent boundary condition [5], imposed at $Z_\nu^{(k)}$ and for $M_\nu^{(k)}$

we obtain

$$M_\nu^{(k)} = \mathscr{T}^{-1}(-i\kappa, Z_\nu^{(k)})R^{(k)}\mathscr{T}(-i\kappa, Z_\nu^{(k)}), \quad k=1,2; \quad \nu = 1,\ldots, \mathrm{N}_1+\mathrm{N}_2.$$

The relativistic R-matrices $R^{(k)}$ for the two kinds of δ-like wells are

$$R^{(k)} = \begin{pmatrix} \cos\phi_k + \frac{1}{2}\omega_+ \sin\phi_k & \frac{1}{2}\omega_- \sin\phi_k \\ -\frac{1}{2}\omega_- \sin\phi_k & \cos\phi_k - \frac{1}{2}\omega_+ \sin\phi_k \end{pmatrix},$$

where $\phi_k = 2\,\mathrm{arctg}(a_k\xi_k/\hbar c)$, $\omega_\pm = (\gamma^{-1} \mp \gamma)$.

We substitute the expressions obtained for $M^{(Z)}$ and $M_\nu^{(k)}$ into (9). After some appropriate transformations and making use of the Hamilton-Cayley theorem [12] for the transfer-matrix M we obtain

$$\begin{aligned} M = {} & \mathscr{T}^{-1}(\kappa_3, L)R^{(L)}\mathscr{T}(-i\kappa, 2(b_2-a_2))[C_{\mathrm{N}_2}^{(2)}\mathscr{T}(-i\kappa, 2a_2)R^{(2)} - C_{\mathrm{N}_2-1}^{(2)}I] \\ & \times \mathscr{T}(-i\kappa, 2(a-a_1))[C_{\mathrm{N}_1}^{(1)}\mathscr{T}(-i\kappa, 2a_1)R^{(1)} - C_{\mathrm{N}_1-1}^{(1)}I]\mathscr{T}(-i\kappa, 2b_1)R^{(0)}, \end{aligned} \tag{9a}$$

where I is the unit 2×2 matrix and the real quantities $C_{\mathrm{N}_k}^{(k)}$ look as follows

$$C_{\mathrm{N}_k}^{(k)} = \begin{cases} \sin(N_k\Phi_k)/\sin\Phi_k, & \Phi_k = \arccos(\chi_k/2) \quad \chi_k < 2 \\ \mathrm{N}_k & , \quad \chi_k = 2 \\ \mathrm{sh}(\mathrm{N}_k\Phi_k)/\mathrm{sh}\,\Phi_k, & \Phi_k = \mathrm{arcch}(\chi_k/2) \quad \chi_k > 2 \end{cases}$$

and $\chi_k = \mathrm{Tr}(\mathscr{T}(-i\kappa, 2a_k)R^{(k)}) = 2(\cos\phi_k\,\mathrm{ch}\,2a_k\kappa + \frac{1}{2}\omega_+\sin\phi_k\,\mathrm{sh}\,2a_k\kappa)$.

From (6) and (9a) for the tunneling coefficient J with respect to our potential $V(x)$ we get

$$J = 4\gamma_1\gamma_3^{-1}\left\{\left[\sum_{i=1}^{4} F_i\right]^2 + \left[\sum_{i=1}^{4} G_i\right]^2\right\}^{-1}, \tag{10}$$

where

$$\begin{aligned} F_1 = {} & C_{\mathrm{N}_1}^{(1)}C_{\mathrm{N}_2}^{(2)}\{\Gamma_+[\cos\phi_1\cos\phi_2\,\mathrm{ch}\,2\kappa(b_1+b_2+a) + \tfrac{1}{2}\omega_+\sin(\phi_1+\phi_2)\mathrm{sh}\,2\kappa(b_1+b_2+a) \\ & - \tfrac{1}{4}\omega_-^2\sin\phi_1\sin\phi_2\,\mathrm{ch}\,2\kappa(b_1+b_2-a)] - \tfrac{1}{2}\Gamma_-\omega_-[\sin\phi_1 g_2^-(b_1-b_2-a) \\ & + \sin\phi_2 g_1^+(b_1-b_2+a)] + \tfrac{1}{4}\Gamma_+\omega_+^2\sin\phi_1\sin\phi_2\,\mathrm{ch}\,2\kappa(b_1+b_2+a)\} \end{aligned}$$

$$F_2 = -C_{\mathrm{N}_1}^{(1)}C_{\mathrm{N}_2-1}^{(2)}\{\Gamma_+f_1^-(b_1+b_2+a-a_2) - \tfrac{1}{2}\Gamma_-\omega_-\sin\phi_1\,\mathrm{sh}\,2\kappa(b_1-b_2-a+a_2)\}$$

$$F_3 = -C_{\mathrm{N}_1-1}^{(1)}C_{\mathrm{N}_2}^{(2)}\{\Gamma_+f_2^-(b_1+b_2+a-a_1) - \tfrac{1}{2}\Gamma_-\omega_-\sin\phi_2\,\mathrm{sh}\,2\kappa(b_1-b_2+a-a_1)\}$$

$$F_4 = C_{\mathrm{N}_1-1}^{(1)}C_{\mathrm{N}_2-1}^{(2)}\Gamma_+\,\mathrm{ch}\,2\kappa(b_1+b_2+a-a_1-a_2)$$

$$\begin{aligned} G_1 = {} & C_{\mathrm{N}_1}^{(1)}C_{\mathrm{N}_2}^{(2)}\{\nabla_+[\cos\phi_1\cos\phi_2\,\mathrm{sh}\,2\kappa(b_1+b_2+a) + \tfrac{1}{2}\omega_+\sin(\phi_1+\phi_2)\mathrm{ch}\,2\kappa(b_1+b_2+a) \\ & - \tfrac{1}{4}\omega_-^2\sin\phi_1\sin\phi_2\,\mathrm{sh}\,2\kappa(b_1+b_2-a)] - \tfrac{1}{2}\nabla_-\omega_-[\sin\phi_2 f_1^-(b_1-b_2+a) \\ & + \sin\phi_1 f_2^+(b_1-b_2-a)] + \tfrac{1}{4}\nabla_+\omega_+^2\sin\phi_1\sin\phi_2\,\mathrm{sh}\,2\kappa(b_1+b_2+a)\} \end{aligned}$$

$$G_2 = -C^{(1)}_{N_1} C^{(2)}_{N_2-1} \{\nabla_+ g_1^+ (b_1 + b_2 + a - a_2) - \tfrac{1}{2} \nabla_- \omega_- \sin \phi_1 \operatorname{ch} 2\kappa (b_1 - b_2 - a + a_2)\}$$

$$G_3 = -C^{(1)}_{N_1-1} C^{(2)}_{N_2} \{\nabla_+ g_2^+ (b_1 + b_2 + a - a_1) - \tfrac{1}{2} \nabla_- \omega_- \sin \phi_2 \operatorname{ch} 2\kappa (b_1 - b_2 + a - a_1)\}$$

$$G_4 = C^{(1)}_{N_1-1} C^{(2)}_{N_2-1} \nabla_+ \operatorname{sh} 2\kappa (b_1 + b_2 + a - a_1 - a_2)$$

and the following abbreviations are used

$$\Gamma_\pm = 1 \pm \Gamma_{12}\Gamma_{23}; \quad \Gamma_{12} = \gamma_1 (i\gamma)^{-1}; \quad \Gamma_{23} = i\gamma\gamma_3^{-1};$$

$$\nabla_\pm = (\gamma_1 \gamma^{-1} \mp \gamma\gamma_3^{-1}); \quad \omega_\pm = (\gamma^{-1} \mp \gamma);$$

$$f_k^\pm (z) = \cos \phi_k \operatorname{ch} 2\kappa z \mp \tfrac{1}{2} \omega_+ \sin \phi_k \operatorname{sh} 2\kappa z;$$

$$g_k^\pm (z) = \cos \phi_k \operatorname{sh} 2\kappa z \pm \tfrac{1}{2} \omega_+ \sin \phi_k \operatorname{ch} 2\kappa z.$$

The expressions obtained are general enough and their further specialization is determined by the exactified structure and parameters of the barrier. Here are some interesting examples from our point of view:

1. $2b_2 = 0$, $N_1 \neq N_2 \neq 0$, $2a_1 \neq 2a_2 \neq 0$, $2a \neq 0$, $2b_1 \neq 0$, $\xi_1 \neq \xi_2 \neq 0$, $V_1 \neq V_3 = 0$, $L = 2b_1 + (N_1 - 1)2a_1 + 2a + (N_2 - 1)2a_2$.

The special feature of this example is that the $(N_1 + N)$-th δ-well and the potential step are placed at the same point $Z = L$. (The case $2b_1 = 0$, $2b_2 \neq 0$ can be studied in a similar way.) Here D and J are obtained immediately from the corresponding expressions in [12] and the above expressions (10) with $2b_2 = 0 (2b_1 = 0)$.

2. For an asymmetric barrier, homogeneous (i.e. with N δ-like potential wells of the same kind) in $\Omega^2 = (0, L)$,

$$N_1 + N_2 = N, \quad 2a = 2a_1 = 2a_2 = 2d, \quad \xi_1 = \xi_2 = \xi, \quad 2b_1 \neq 2b_2 \neq 0$$

$$V_1 \neq V_3, \quad L = 2[b_1 + b_2 + (N-1)d], \quad \phi = 2 \operatorname{arctg}(\xi d / \hbar c),$$

using the general expression for D in [12] we obtain the following for $D^{\text{as}}_{\text{hom}}$:

$$D^{\text{as}}_{\text{hom}} = 4\gamma_1\gamma_3^{-1} \left\{ \left[\sum_{i=1}^{2} Q_i^{\text{hom}} \right]^2 + \left[\sum_{i=1}^{2} S_i^{\text{hom}} \right]^2 \right\}^{-1},$$

where

$$Q_1^{\text{hom}} = \mathscr{C}_N [\Gamma_+ q^- (b_1 + b_2) - \tfrac{1}{2} \Gamma_- \gamma_- \sin \phi \cos 2\kappa_2 (b_1 - b_2)]$$

$$Q_2^{\text{hom}} = \mathscr{C}_{N-1} \Gamma_+ \cos 2\kappa_2 (b_1 + b_2 - d)$$

$$S_1^{\text{hom}} = \mathscr{C}_N [\Delta^+ s^+ (b_1 + b_2) - \tfrac{1}{2} \Delta^- \gamma_- \sin \phi \cos 2\kappa_2 (b_1 - b_2)]$$

$$S_2^{\text{hom}} = -\mathscr{C}_{N-1} \Delta^+ \sin 2\kappa_2 (b_1 + b_2 - d)$$

$$q^-(z) = \cos \phi \cos 2\kappa_2 z - \tfrac{1}{2} \gamma_+ \sin \phi \sin 2\kappa_2 z$$

$$s^+(z) = \cos \phi \sin 2\kappa_2 z + \tfrac{1}{2} \gamma_+ \sin \phi \cos 2\kappa_2 z$$

$$\Gamma_{\pm} = 1 \pm \Gamma_{12}\Gamma_{23}, \quad \Delta^{\pm} = \gamma_1/\gamma_2 \pm \gamma_2/\gamma_3, \quad \gamma_{\pm} = \gamma_2 \pm \gamma_2^{-1}$$

$$\mathscr{C}_{\mathrm{N}} = \begin{cases} \sin(\mathrm{N}\,\varphi)/\sin\varphi, & \varphi = \arccos(\lambda/2), \quad \lambda < 2 \\ \mathrm{N}, & \lambda = 2 \\ \mathrm{sh}(\mathrm{N}\,\varphi)/\mathrm{sh}\,\varphi, & \varphi = \mathrm{arcch}(\lambda/2), \quad \lambda > 2 \end{cases}$$

$$\lambda = \mathrm{Tr}(\mathscr{T}(\kappa_2, 2d)R) = 2(\cos\phi\,\cos 2\kappa_2 d - \tfrac{1}{2}(\gamma_2^{-1} + \gamma_2)\sin\phi\,\sin 2\kappa_2 d).$$

Now for J, making use of (10) we get

$$J^{\mathrm{as}}_{\mathrm{hom}} = 4\gamma_1\gamma_3^{-1}\left\{\left[\sum_{i=1}^{2} F_i^{\mathrm{hom}}\right]^2 + \left[\sum_{i=1}^{2} G_i^{\mathrm{hom}}\right]^2\right\}^{-1},$$

where

$$F_1^{\mathrm{hom}} = C_{\mathrm{N}}[\Gamma_+ f^-(b_1 + b_2) - \tfrac{1}{2}\Gamma_-\omega_-\,\sin\phi\,\mathrm{sh}\,2\kappa(b_1 - b_2)],$$

$$F_2^{\mathrm{hom}} = -C_{\mathrm{N}-1}\Gamma_+\,\mathrm{ch}\,2\kappa(b_1 + b_2 - d),$$

$$G_1^{\mathrm{hom}} = C_N(\nabla_+ g^+(b_1 + b_2) - \tfrac{1}{2}\nabla_-\omega_-\,\sin\phi\,\mathrm{ch}\,2\kappa(b_1 - b_2)),$$

$$G_2^{\mathrm{hom}} = -C_{\mathrm{N}-1}\nabla_+\,\mathrm{sh}\,2\kappa(b_1 + b_2 - d),$$

$$g^+ = \cos\phi\,\mathrm{sh}\,2\kappa(b_1 + b_2) + \tfrac{1}{2}(\gamma^{-1} - \gamma)\sin\phi\,\mathrm{ch}\,2\kappa(b_1 + b_2),$$

$$f^- = \cos\phi\,\mathrm{ch}\,2\kappa(b_1 + b_2) + \tfrac{1}{2}(\gamma^{-1} - \gamma)\sin\phi\,\mathrm{sh}\,2\kappa(b_1 + b_2)$$

and $\chi = 2(\cos\phi\,\mathrm{ch}\,2d\kappa + \tfrac{1}{2}\omega_+\,\sin\phi\,\mathrm{sh}\,2d\kappa)$.

3. For asymmetric (symmetric) rectangular barriers

$$\mathrm{N}_k = 0, \quad \xi_k = 0, \quad 2a_k = 2a = 0 \quad 2b_2 = 0$$

$$V_1 \neq V_3 \quad (V_1 = V_3), \quad L = 2b_1$$

the coefficients are:

$$D^{\mathrm{as}} = 4\gamma_1\gamma_3^{-1}[(1 + \gamma_1\gamma_3^{-1})^2\cos^2\kappa_2 L + (\gamma_1\gamma_2^{-1} + \gamma_2\gamma_3^{-1})^2\sin^2\kappa_2 L]^{-1}$$

$$D^{\mathrm{s}} = 4[4\cos^2\kappa_2 L + (\gamma_1\gamma_2^{-1} + \gamma_2\gamma_1^{-1})^2\sin^2\kappa_2 L]^{-1}$$

$$J^{\mathrm{as}} = 4\gamma_1\gamma_3^{-1}[(1 + \gamma_1\gamma_3^{-1})^2\,\mathrm{ch}^2\,\kappa L + (\gamma\gamma_3^{-3} - \gamma_1\gamma^{-1})^2\,\mathrm{sh}^2\,\kappa L]^{-1}$$

$$J^{\mathrm{s}} = 4[4\,\mathrm{ch}^2\,\kappa L + \tfrac{1}{4}(\gamma_1\gamma^{-1} - \gamma\gamma_1^{-1})^2\,\mathrm{sh}^2\,\kappa L]^{-1}.$$

If $\kappa L \gg 1$ for J^{s}, J^{as} and $J^{\mathrm{as}}_{\mathrm{hom}}$ we have

$$\tilde{J}^{\mathrm{s}} = 16\gamma^2\gamma_1^2[\gamma^2 + \gamma_1^2]^{-2}\exp(-2\kappa L).$$

$$\tilde{J}^{\mathrm{as}} = 16\gamma_1\gamma_3\gamma^2[(\gamma_1^2 + \gamma^2)(\gamma^2 + \gamma_3^2)]^{-1}\exp(-2\kappa L)$$

$$\tilde{J}^{\mathrm{as}}_{\mathrm{hom}} = 16\gamma_1\gamma_3^{-1}\left[\left(\sum_{i=1}^{2}\tilde{F}_i\right)^2 + \left(\sum_{i=1}^{2}\tilde{G}_i\right)^2\right]^{-1}\exp(4\kappa\mathrm{N}d)\exp(-2\kappa L),$$

where

$$\tilde{F}_1 = C_N \exp(2\kappa d)[\Gamma_+ \tilde{f} - \tfrac{1}{2}\Gamma_- \omega_- \sin\phi \exp(-4\kappa b_2)]$$

$$\tilde{F}_2 = -C_{N-1}\Gamma_+$$

$$\tilde{G}_1 = C_N \exp(2\kappa d)[\nabla_+ \tilde{f} - \tfrac{1}{2}\nabla_- \omega_- \sin\phi \exp(-4\kappa b_2)]$$

$$\tilde{G}_2 = -C_{N-1}\nabla_+$$

$$\tilde{f} = \cos\phi + \tfrac{1}{2}\sin\phi.$$

We note that the last expressions for $\tilde{J}^{\text{as}}_{\text{hom}}$, $\tilde{J}^{\text{as}}$ and $\tilde{J}^{\text{s}}$ are close to the corresponding nonrelativistic expressions of Kane [20] with respect to their structure, therefore we call "$\exp(-2\kappa L)$" *relativistic barrier penetration factor*. It is seen that the relativistic corrections appear in the before-exponential factors as well as in the relativistic barrier penetration factors.

From the obvious relations

$$\tilde{J}^{\text{as}} = \gamma_1^{-1}\gamma_3(\gamma_1^2+\gamma^2)(\gamma^2+\gamma_3^2)^{-1}\tilde{J}^{\text{s}},$$

$$\tilde{J}^{\text{as}}_{\text{hom}} = \gamma^{-2}\gamma_3^{-2}(\gamma_1^2+\gamma^2)(\gamma^2+\gamma_3^2)\left[\left(\sum_{i=1}^{2}\tilde{F}_i\right)^2+\left(\sum_{i=1}^{2}\tilde{G}_i\right)^2\right]^{-1}\exp(4\kappa N d)\tilde{J}^{\text{as}}$$

$$= \gamma_1^{-1}\gamma_3^{-1}\gamma^{-2}(\gamma_1^2+\gamma^2)^2\left[\left(\sum_{i=1}^{2}\tilde{F}_i\right)^2+\left(\sum_{i=1}^{2}\tilde{G}_i\right)^2\right]^{-1}\exp(4\kappa N d)\tilde{J}^{\text{s}}$$

we see that the barrier structure demonstrates itself through some multiplication factors that are uniquely determined by the structure; clearly these factors tend to 1 if the structure vanishes.

Now we give some examples of transmission without reflection. In the case of an asymmetric ($V_1 \neq V_3$) rectangular barrier the condition for transmission without reflection $D^{\text{as}} = 1$ (or equivalently $|M^{\text{as}}_{22}|^2 = \gamma_1\gamma_3^{-1}$) leads to transcendental equations for the energy E, from which we get

$$E_n = \pm\left\{\left[\arcsin\left(\pm\frac{(1-\gamma_1\gamma_3^{-1})^2}{(1+\gamma_1\gamma_3^{-1})^2-(\gamma_1\gamma_2^{-1}+\gamma_2\gamma_3^{-1})^2}\right)^{1/2}+n\pi\right]^2 c^2\hbar^2L^{-2}+m_0^2c^4\right\}^{1/2}+V_2,$$

$$n = 0, 1, 2, \ldots$$

for the above-barrier case and

$$E = \pm\left\{m_0^2c^4 - c^2\hbar^2L^{-2}\,\text{arcsh}\left(\pm\frac{(1-\gamma_1\gamma_3^{-1})^2}{(1+\gamma_1\gamma_3^{-1})^2+(\gamma_1\gamma^{-1}-\gamma\gamma_3^{-1})^2}\right)^{1/2}\right\}^{1/2}+V_2$$

for tunneling.

For the symmetric case ($V_1 = V_3$) we have $|M^S_{22}| = 1$ and respectively

$$E_n = \pm(\hbar^2c^2\pi^2n^2L^{-2}+m_0^2c^4)^{1/2}+V_2 \quad \text{and} \quad E = V_2 \pm m_0c^2.$$

3 Conclusion

In conclusion we can say that for an important (for condensed matter physics) class of structured barriers the relativistic one-dimensional scattering problem can be fully described in transfer-matrix terms and the corresponding tunneling and above barrier coefficients can be explicitly obtained.

In the non-relativistic limit $v^2/c^2 \ll 1$, the corresponding non-relativistic expressions for the tunneling coefficients for the Kronig–Penney model of a thin film [13], for asymmetric and symmetric barriers [11, 20, 25] and for the structured barrier with $(N_1 + N_2)$ δ-wells [14] can be easily obtained from our relativistic ones. Because of the complicated dependence of the coefficients D and J on the barrier's parameters an immediate analysis of the influence of these parameters on D and J is difficult to be made. It seems more appropriate to carry out a numerical analysis in every special case as it was done e.g. in [16] for the case of non-relativistic tunneling.

References

[1] J. Callaway, Phys. Rev., 102, 919 (1956).
[2] R. D. Woods and J. Callaway, Bull, Am. Phys. Soc., 2, 18 (1957).
[3] M. L. Glasser and S. G. Davison, Int. J. Quantum Chem. Symp., 3, 867 (1970).
[4] S. G. Davison and J. D. Levine, in *Solid State Physics*, H. Ehrenreih, F. Seitz and D. Turnbull, eds., vol. 25, Academic Press, New York (1970).
[5] R. Subramanian and K. Bhagwat, phys. stat. sol. (b) 48, 399 (1971).
[6] M. Steslicka, Prog. Surf. Sci., 5, 157 (1975).
[7] V. Detcheva, M. Metodieva, B. Kandilarov and M. Primatarova, Helv. Phys. Acta, 63, 908 (1990).
[8] V. Detcheva, M. Metodieva, T. Khristov and B. Kandilarov, phys. stat. sol. (b) 152, 163 (1989).
[9] C. L. Roy, in *Molecular Electronics*, M. Borisov, ed., World Scientific Publishing, p. 130 (1987).
[10] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, Springer, New York (1988).
[11] B. Kandilarov and V. Detcheva, Physica, 68, 579 (1973).
[12] M. Tashkova and I. Mladenov, Comp. rend. Acad. bulg. Sci., 40, 55 (1987).
[13] P. Schnupp, phys. stat. sol., 21, 567 (1967).
[14] M. Tashkova, I. Mladenov, B. Atanasov and B. Kandilarov, Compt. rend Acad bulg. Sci., 37, 1019 (1984).
[15] B. Atanasov, G. Postnicova, U. Sadikov and M. Vokenstein, Mol. Biologia, 11, 573 (1977).

P. Christova, Ph.D.Thesis, Ins. Org. Chem., Bulg. Acad. Sci., Sofia (1980).
G. Postnikova, Biofizika, 31, 163 (1986).

[16] M. Tashkova, I. Mladenov, M. Primatarova, B. Atanasov and B. Kandilarov, Compt. rend. Acad. bulg. Sci., 38, 1149 (1985).

[17] L. I. Shiff, *Quantum Mechanics*, McGraw-Hill, New York (1955).

[18] P. Erdös and R. C. Herndon, Adv. Phys., 31, 65 (1982).

[19] B. Kandilarov and V. Detcheva, *Transfer Matrix Approach and Basic Concepts of Energy Band Theory*, Publ. House of Bulg. Acad. Sci., Sofia (1988), in Bulgarian.

[20] E. O. Kane, in *Tunneling Phenomena in Solids*, E. Burstein and S. Lundgvist, eds., Plenum Press, New York (1969), ch. 1.

[21] P. V. Eliutin and V. D. Krivchenkov, *Quantum Mechanics*, Nauka, Moscow (1976), ch. 3, in Russian.

[22] D. Saxon and R. Hutner, Philips Res. Rep., 4, 81 (1949).

[23] C. L. Roy, Phys. Lett. A 130, 203 (1988).

[24] I. M. Madenov, Phys. Lett. A 137, 313 (1989).

[25] D. Bohm, *Quantum Theory*, Prentice-Hall, Inc. New York (1955).