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Gauge Field Theories on Riemann Surfaces

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Abstract. In this paper the free gauge field theories on a Riemann surface of any genus are quantized in the covariant gauge. The propagators of the gauge fields are explicitly derived and their properties are analysed in details. As an application, the correlation functions of an Yang–Mills field theory with gauge group $SU(N)$ are computed at the lowest order.

1 Introduction

Recently, the quantized Yang–Mills field theories on Riemann surfaces have been the subject of several investigations. A partial list of the most relevant contributions is given in refs. [1]–[8]. Despite of many important results, for instance the nonperturbative computation of the partition function and of the amplitudes of the Wilson loops, the possibility of performing explicit calculations in the case of gauge fields interacting with matter is confined until now to the simplest topologies, like the cylinder, the disk, the sphere and the torus [9]. On the other side, the perturbative series of Yang–Mills theories can be derived exploiting the powerful heat kernel techniques [10]. For instance it is possible to check in this way the renormalizability of any gauge field theory up to one loop approximation. However, apart from the difficulty of performing calculations at higher order, we are interested here in the explicit dependence of the theory on the geometry of the Riemann surface, which is not so easy to treat with heat kernel methods.

Consequently, in order to extend the investigations of refs. [9] also to the case of Riemann surfaces, we propose here a perturbative approach. One important step in this direction is the construction of the propagator of the gauge fields. To this purpose, we compute here the explicit expression of the propagator in terms of theta functions and prime forms [11]. Once the propagator is known, one can derive for instance the vacuum expectation value (VEV) of the energy−momentum tensor [12] at the lowest order. In this calculation, the dependence on the moduli of the two point function turns out to be crucial in order to ascertain the existence of pseudoparticles in the physical amplitudes. The latter are connected to the presence of a gravitational background in certain local systems of reference, see on this point refs. [13]–[14], where the example of free conformal field theories is discussed. The knowledge of the propagator alone, however, is not sufficient in order to evaluate the radiative corrections of the correlation functions on a Riemann surface because of the presence of zero modes and of topologically nontrivial classical fields. For this reason, we will give here explicit formulas also for the flat connections following refs. [5], [6] and [15]. These connections play the role of external background fields, so that the Yang−Mills theories on Riemann surfaces can be treated within the perturbative approach using the techniques explained in refs. [16]. As a consequence, the final expression of the generating functional of the one-particle irreducible Green functions will be gauge invariant with respect to the background fields.

With the ingredients provided in this paper it is possible to start the perturbative calculations of the $n-$point functions of many two dimensional gauge field theories. Indeed, even if the generating functional considered here involves for simplicity only gauge fields, we are able to treat also interactions with matter fields without problems. Possible models are Yang−Mills field theories interacting with massless fermions or scalars, for which the propagators are already known from string theory. Some of these theories are not exactly integrable, so that the use of perturbative techniques is appropriate in these cases. On the other side, if the theory is integrable on the complex plane, nonperturbative calculations can be achieved also on Riemann surfaces once the free propagators are known. An example concerning the Schwinger model [17] has been given in ref. [18].

In order to quantize the Yang−Mills field theory we choose here the class of covariant gauge fixings $\nabla_\mu A^\mu = 0$, where ∇_μ is the covariant derivative acting on the vector field A^μ. Unfortunately, due to the presence of the metric, the equations of motion satisfied by the Yang−Mills propagator are not easily solvable in this gauge. A possible way out from this problem is to exploit the Lorentz gauge, in which there is the advantage that the coexact components of the gauge fields decouple in the free Lagrangian from the unphysical exact components. The linearized equations of motion become equivalent to biharmonic equations whose solutions is known on every Riemannian manifold [19]. This is the strategy followed in ref. [20] in the abelian case. In the more complicated Yang−Mills field theories, however, the exact components remain in the nonlinear part of the action, so that the perturbative expansion in the Lorenz gauge is very cumbersome. For this reason we will use here another strategy, computing the propagators after choosing on the Riemann surface a general, but conformally flat metric. This is not a limitation, because every metric on a Riemann surface of given genus h is conformally flat modulo global diffeomorphisms. Thus, the expressions given here for the propagators can be extended to any other metric exploiting the invariance under global changes of coordinates of the Yang−Mills functional quantized in the covariant gauge.

Another problem to be solved in order to find the physical propagator of the gauge fields is that the Green functions obtained from the equations of motion with a point source are not unique. The origin of this nonuniqueness is the existence of the flat connections and the residual gauge invariance typical of the covariant gauges. The latter invariance can be related to the presence of a constant zero mode in the free equations of motion [20]. The arbitrariness in the propagator is here removed imposing the condition that the unphysical flat connections should not be propagated inside the amplitudes. As we will see, this requirement is sufficient also to eliminate the constant zero mode. As a proof that our propagator is the physical one, we check that it satisfies the Slavnov–Taylor identities [21] at the free level. We notice that, on the contrary of what happens in string theory, the 2-D Yang–Mills field theories are not conformally invariant. Therefore, the only possible Slavnov-Taylor identities are those associated to the gauge invariance of the theory.

The material contained in this paper is divided as follows. In Section 2 we quantize the Yang–Mills field theories on a Riemann surface in the covariant gauge using the BRST formalism [22]. The equations defining the propagators of the gauge fields are explicitly derived. They are too complicated to be solved for a general metric, so that we limit ourselves to the conformally flat metrics. We show however that the expression of the propagator can be derived for any other metric exploiting the covariance of the theory under general diffeomorphisms. In Section 3 the two point functions of the ghost and gauge fields are constructed. The already mentioned arbitrariness given by the flat connections and by the residual gauge invariance is totally eliminated by introducing three physical requirements. In Section 4 the properties of the gauge propagator are investigated. First of all, we verify that, on any open subset of the Riemann surface, it is equivalent to the standard two point function of $\mathbf{R}^2$. Secondly, it is checked that the flat connections are not propagated in the amplitudes. As a consequence of the physicality of our propagator, we prove that its components fulfil the Slavnov–Taylor identities at the free level. Finally, for future applications in perturbation theory, the structure of the divergent and finite parts of the two point function at short distances is computed. In Section 5 the generating functional of the correlation functions for an $SU(N)$ Yang–Mills theory is considered. The missing ingredient, the flat connections, are explicitly derived in terms of the abelian differentials and of the Lie algebra generators. In Section 6 we present the conclusions and the possible future developments. Finally, the explicit form of the components of the propagator in the short distance limit is calculated in the appendix, pointing out the differences that appear considering Riemann surfaces of different genera.

2 The Covariant Gauge Fixing on a Riemann Surface

In this paper we consider the following Yang–Mills functional:

$$S_{YM} = \int_M d^2x\sqrt{g}\ \mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \tag{2.1}$$

where M is a general closed and orientable Riemann surface of genus h parametrized by the real coordinates x_μ, $\mu = 1, 2$. The metric on M is given by the tensor $g_{\mu\nu}$ with Euclidean signature and determinant denoted by $g \equiv \det[g_{\mu\nu}]$. To fix the ideas, we suppose that the fields A_μ are elements of a $su(N)$ algebra, so that $A_\mu \equiv \sum_a A_\mu^a T^a$, $a = 1, \ldots, N^2 - 1$, where the T^a are the generators of $SU(N)$ in the adjoint representation. The elements of the gauge group connected to the identity are mappings $U(x) : M \to SU(N)$ parametrized as follows: $U(x) = \exp[i\kappa\alpha^a(x)T^a]$. Here the $\alpha^a(x)$ represent real functions on M and κ is a real coupling constant. The field strength $F_{\mu\nu}$ appearing in eq. (2.1) is of the form:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\kappa[A_\mu, A_\nu]$$

In this way it is easy to see that the action (2.1) is invariant under a local $SU(N)$ transformation of the kind:

$$A_\mu(x) \to A_\mu^U(x) = U^{-1}(x)\left[A_\mu(x) - i\kappa^{-1}\partial_\mu\right]U(x) \tag{2.2}$$

To evaluate the trace in eq. (2.1) we will use the following conventions:

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab} \tag{2.3}$$

$$\mathrm{Tr}(T^aT^bT^c) = \frac{1}{4}(d^{abc} + if^{abc}) \tag{2.4}$$

where d^{abc} is a totally symmetric tensor given by $\{T^a, T^b\} = d^{abc}T^c$, with $\{,\}$ denoting the anticommutator, while the f^{abc} are the structure constants of the group $SU(N)$.

The classical action (2.1) is degenerate and in order to perform the quantization we adopt the standard Faddeev and Popov procedure. To this purpose, we introduce the set of nilpotent BRST transformations [22]:

$$\delta A_\mu^a = (D_\mu(A)c)^a \qquad\qquad \delta B^a = 0 \tag{2.5}$$

$$\delta c^a = \frac{1}{2}f^{abc}c^bc^c \qquad\qquad \delta\bar{c}^a = \frac{1}{4}B^a \tag{2.6}$$

where $\bar{c}^a$ and c^a are the ghost fields and the B^a play the role of Lagrange multipliers. The covariant derivative $D_\mu(A)$ appearing in eq. (2.5) is of the form: $D_\mu(A) = \nabla_\mu - i\kappa[A_\mu, \ ]$. Acting on the ghost scalar field $c(x)$, the differential operator ∇_μ is just the usual partial derivative ∂_μ. After choosing a suitable gauge fixing $f^a(A) = 0$, the total BRST invariant action becomes:

$$S = S_{YM} + S_{GF} + S_{FP} \tag{2.7}$$

where

$$S_{GF} + S_{FP} = \delta(\bar{c}^aF^a(A)) \tag{2.8}$$

is a pure BRST variation.

To our purposes, i.e. explicit perturbative calculations of the correlation functions, it is convenient to impose the covariant gauge fixing

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}A^\mu) = 0 \tag{2.9}$$

This preserves the covariance under general diffeomorphisms in the action (2.7). However, the equations satisfied by the propagators of the gauge fields are complicated by the presence of the metric. An exception is provided by the Lorentz gauge already studied in ref. [20], where the coexact components of the A_μ fields are completely decoupled from the exact components at the free level. This fact allows the calculation of the propagator in a relatively easy way and for any two dimensional manifold, also with boundary, once the biharmonic Green function with the proper boundary conditions explained in [20] is known. For this reason, the Lorentz gauge is very useful in treating some models with abelian group of symmetry, like for instance the two dimensional massless electrodynamics, in which the exact components can be simply factored out from the path integral [18]. The situation is however different in the case of nonabelian gauge field theories, because the exact components remain present in the nonlinear interaction Lagrangian, making the perturbative approach in the Lorentz gauge very cumbersome.

To solve the equations satisfied by the propagators in the covariant gauge (2.9) we use the following strategy. First of all, we introduce on M a set of complex coordinates $z = x_1 + ix_2$, $\bar{z} = z^*$. Moreover, we exploit the fact that on a Riemann surface it is always possible to choose a conformally flat metric of the kind:

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \qquad\qquad g_{z\bar{z}} = g_{\bar{z}z} = e^{\phi(z,\bar{z})} \tag{2.10}$$

$\phi(z,\bar{z})$ being a real function. At this point, we impose the gauge fixing (2.9), which, in the particular metric (2.10), reduces to the simpler condition: $\nabla_z A_{\bar{z}} + \nabla_{\bar{z}} A_z = 0$. This is a good gauge fixing apart from Gribov ambiguities [23], which we will not discuss because our treatment is strictly perturbative. Once the gauge invariance is fixed, the component $A_z^a dz$ ($A_{\bar{z}}^a d\bar{z}$) of the gauge connection belongs to $T^{*(1,0)}(M)$ ($T^{*(0,1)}(M)$), which is an holomorphic (antiholomorphic) line bundle admitting holomorphic (antiholomorphic) transition functions. As a consequence, the covariant gauge fixing (2.9) in a conformally flat metric reads:

$$f^a(A) = g^{z\bar{z}}(\partial_z A_{\bar{z}}^a + \partial_{\bar{z}} A_z^a) = 0 \tag{2.11}$$

Starting from the gauge fixed action (2.7) and integrating over the Lagrange multipliers B^a with the functional measure

$$d\mu[B] = \int DB e^{-\frac{\lambda}{32}\int_m d^2x\sqrt{g}\mathrm{Tr}B^2}$$

we obtain the final formula:

$$Z[J] = \int DA_\mu D\bar{c}Dc\exp\left\{-\mathrm{Tr}\int_M d^2x\sqrt{g}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\lambda}f^2(A) + \partial_\mu\bar{c}D^\mu(A)c + J_\mu A^\mu\right]\right\} \tag{2.12}$$

From this generating functional it is possible to derive perturbatively all the correlation functions of the gauge fields in a conformally flat gravitational background. This is not a serious limitation, since the results can be easily extended to any other metric in the following way. For instance, let us suppose that the propagators are known for a general metric $g_{z\bar{z}}$ which is conformally flat. To derive the propagators also in the case of another metric $\tilde{g}_{\mu\nu}(z', \bar{z}')$, obtained from $g_{z\bar{z}}$ after a global diffeomorphism plus a Teichmüller deformation, it is sufficient to exploit the covariance under diffeomorphisms of the action appearing in eq. (2.12). This covariance is assured by the fact that the gauge fixing (2.11) is nothing but the covariant gauge (2.9) written in the conformally flat metric (2.10). As an upshot, the classical equations of motion satisfied by the propagators and the respective solutions turn out to be covariant under global diffeomorphisms. At this point, we notice that the metric $\tilde{g}_{\mu\nu}(z', \bar{z}')$ is equivalent to a conformally flat metric $g_{w\bar{w}}(w, \bar{w})$ up to a change of variables of the kind:

$$w = w(z', \bar{z}') \qquad \bar{w} = \bar{w}(z', \bar{z}') \tag{2.13}$$

(see for example [24] and references therein). In the new metric $g_{w\bar{w}}(w, \bar{w})$ the components of the propagator are known by hypothesis. Therefore, they can be computed also in the old metric $\tilde{g}_{\mu\nu}(z', \bar{z}')$ inverting the diffeomorphism (2.13) and using the covariance of the propagator mentioned above under this transformation of coordinates. Concerning the other correlation functions, they are easily obtained from the propagators exploiting perturbation theory. Finally, let us notice that, in our perturbative framework, the addition in eq. (2.12) of interactions with matter fields is not a problem. For example, one may consider massless scalar or fermionic fields, for which the propagators are already known from string theory.

3 The Propagators in the Covariant Gauge

Following the ideas of the previous section, we construct here the propagators of the Yang–Mills field theory in the conformally flat metrics described by eq. (2.10). To this purpose, it is sufficient to consider only the free part S_0 of the action appearing in eq. (2.12). Using a set of complex coordinates and dropping the color indices we obtain:

$$S_0 = \int_M d^2z g^{z\bar{z}} \left[(\partial_z A_{\bar{z}})^2 \left(\frac{1}{\lambda} - 1 \right) + (\partial_{\bar{z}} A_z)^2 \left(\frac{1}{\lambda} - 1 \right) + 2\partial_z A_{\bar{z}} \partial_{\bar{z}} A_z \left(\frac{1}{\lambda} + 1 \right) + \right.$$
$$\left. 2J_z A_{\bar{z}} + 2J_{\bar{z}} A_z \right] \tag{3.1}$$

The classical equations of motion following from eq. (3.1) are:

$$\left(1 + \frac{1}{\lambda}\right) \partial_{\bar{z}}(g^{z\bar{z}} \partial_z A_{\bar{z}}) - \left(1 - \frac{1}{\lambda}\right) \partial_{\bar{z}}(g^{z\bar{z}} \partial_{\bar{z}} A_z) = J_{\bar{z}}$$

$$\left(1+\frac{1}{\lambda}\right)\partial_z(g^{z\bar z}\partial_{\bar z}A_z)-\left(1-\frac{1}{\lambda}\right)\partial_z(g^{z\bar z}\partial_z A_{\bar z})=J_z$$

As these equations show, the advantage of working with conformally flat metrics is that the covariant derivatives are substituted by partial derivatives, simplifying the calculations. Now we are ready to compute the propagator of the gauge fields:

$$G_{\alpha\beta}(z,w)\equiv\langle A_\alpha(z,\bar z)A_\beta(w,\bar w)\rangle \tag{3.2}$$

In eq. (3.2) and in the following we adopt the conventions: $\alpha=z,\bar z$, $\beta=w,\bar w$. The equations defining the propagator become:

$$\left(1+\frac{1}{\lambda}\right)\partial_{\bar z}(g^{z\bar z}\partial_z G_{\bar z w}(z,w))-\left(1-\frac{1}{\lambda}\right)\partial_{\bar z}(g^{z\bar z}\partial_{\bar z}G_{zw}(z,w))=$$

$$\delta^{(2)}_{\bar z w}(z,w)-\sum_{i,j=1}^{h}\bar\omega_i(\bar z)\left[\mathrm{Im}\ \Omega\right]^{-1}_{ij}\omega_j(w) \tag{3.3}$$

$$\left(1+\frac{1}{\lambda}\right)\partial_z(g^{z\bar z}\partial_{\bar z}G_{z\bar w}(z,w))-\left(1-\frac{1}{\lambda}\right)\partial_z(g^{z\bar z}\partial_z G_{\bar z\bar w}(z,w))=$$

$$\delta^{(2)}_{z\bar w}(z,w)-\sum_{i,j=1}^{h}\omega_i(z)\left[\mathrm{Im}\ \Omega\right]^{-1}_{ij}\bar\omega_j(\bar w) \tag{3.4}$$

$$\left(1-\frac{1}{\lambda}\right)\partial_{\bar z}(g^{z\bar z}\partial_{\bar z}G_{z\bar w}(z,w))-\left(1+\frac{1}{\lambda}\right)\partial_{\bar z}(g^{z\bar z}\partial_z G_{\bar z\bar w}(z,w))=0 \tag{3.5}$$

$$\left(1-\frac{1}{\lambda}\right)\partial_z(g^{z\bar z}\partial_z G_{\bar z w}(z,w))-\left(1+\frac{1}{\lambda}\right)\partial_z(g^{z\bar z}\partial_{\bar z}G_{zw}(z,w))=0 \tag{3.6}$$

In the first two equations written above Ω_{ij}, $i,j=1,\dots,h$, denotes the period matrix and the $\omega_i(z)dz$ form a canonically normalized basis of abelian differentials. Moreover, the term in the right hand side of eqs. (3.3) and (3.4) is a projector onto the space of the zero modes, given in this case by the h abelian differentials $\omega_i(z)$, $i=1,\dots,h$. As shown in ref. [20] for the Lorentz gauge $\lambda=0$, the presence of this projector is necessary because otherwise also the unphysical harmonic components of the fields would be propagated in the amplitudes. The proof that in the flat case eqs. (3.3)-(3.6) are equivalent to the usual equations:

$$\left[\delta_{\mu\nu}\triangle-\partial_\mu\partial_\nu\left(1-\frac{1}{\lambda}\right)\right]G_{\nu\rho}(x-y)=\delta_{\mu\rho}\delta^{(2)}(x-y) \tag{3.7}$$

where $\triangle$ denotes the Laplacian in cartesian coordinates and $\mu,\nu=1,2$, is straightforward.

At this point, we use eq. (3.5) in order to derive the expression of $G_{z\bar w}(z,w)$ in terms of $G_{\bar z\bar w}(z,w)$:

$$\partial_{\bar z}G_{z\bar w}(z,w)=\left(\frac{\lambda+1}{\lambda-1}\right)\partial_z G_{\bar z\bar w}(z,w) \tag{3.8}$$

Substituting (3.8) in (3.4), one obtains an equation containing only $G_{\bar{z}\bar{w}}(z,w)$:

$$\frac{\partial_z(g^{z\bar{z}}\partial_z G_{\bar{z}\bar{w}}(z,w))}{(\lambda-1)} = \frac{1}{4}\left[\delta^{(2)}_{z\bar{w}}(z,w) - \sum_{i,j=1}^{h}\omega_i(z)\,[\text{Im }\Omega]^{-1}_{ij}\,\bar{\omega}_j(\bar{w})\right] \tag{3.9}$$

In the same way, from eqs. (3.6) and (3.3) it follows that:

$$\partial_z G_{\bar{z}w}(z,w) = \left(\frac{\lambda+1}{\lambda-1}\right)\partial_{\bar{z}} G_{zw}(z,w) \tag{3.10}$$

and:

$$\frac{\partial_{\bar{z}}(g^{z\bar{z}}\partial_{\bar{z}} G_{zw}(z,w))}{(\lambda-1)} = \frac{1}{4}\left[\delta^{(2)}_{\bar{z}w}(z,w) - \sum_{i,j=1}^{h}\bar{\omega}_i(\bar{z})\,[\text{Im }\Omega]^{-1}_{ij}\,\omega_j(w)\right] \tag{3.11}$$

Thus we are left only with the two independent equations (3.9) and (3.11) in the components $G_{zw}(z,w)$ and $G_{\bar{z}\bar{w}}(z,w)$. To simplify these equations and to determine uniquely the form of the propagator, the following physical requirements play an important role:

a) The unphysical zero modes should not be propagated.

b) The components of the propagator should be singlevalued. Taking from example their integrals in z, the differential in $w, \bar{w}$ obtained in this way must not be periodic around the homology cycles A_i and B_i, $i = 1, \ldots, h$, of the Riemann surface:

$$\oint_\gamma dz G_{z\beta}(z,w) = \oint_\gamma d\bar{z} G_{\bar{z}\beta}(z,w) = 0 \tag{3.12}$$

where γ is an arbitrary nontrivial homology cycle and $\beta = w, \bar{w}$. Analogous equations are valid integrating in the variables $w, \bar{w}$.

c) The gauge fields A_z and $A_{\bar{z}}$ can be decomposed according to the Hodge decomposition in coexact, exact and harmonic components as follows:

$$A_z = \partial_z\phi + \partial_z\rho + A_z^{\text{har}} \tag{3.13}$$

$$A_{\bar{z}} = -\partial_{\bar{z}}\phi + \partial_{\bar{z}}\rho + A_{\bar{z}}^{\text{har}} \tag{3.14}$$

Here ϕ is a purely complex scalar field, while ρ is a real scalar field. A_z^{har} and $A_{\bar{z}}^{\text{har}}$ take into account the presence of the abelian differentials. The decomposition (3.13) and (3.14) is not invertible unless

$$\int_M d^2z g_{z\bar{z}}\phi(z,\bar{z}) = \int_M d^2z g_{z\bar{z}}\rho(z,\bar{z}) = 0 \tag{3.15}$$

Accordingly, also the propagator $G_{\alpha\beta}(z,w)$, which from point a) propagates only the coexact and exact components, should satisfy analogous relations.

Applying the Hodge decomposition theorem [24] to the propagator (3.2), one obtains that the most general form of this tensor is:

$$G_{\alpha\beta}(z,w) = \partial_\alpha \partial_\beta G(z,w) + \sum_{i=1}^{h} \left(\partial_\alpha f^i A_\beta^{iT} + A_\alpha^{iT} \partial_\beta f^{i\prime} + A_\alpha^{iT} A_\beta^{iT} \right)$$

where $G(z,w) \equiv G(z,\bar{z};w,\bar{w})$, $f^i(z,\bar{z})$ and $f^{i\prime}(w,\bar{w})$ are scalar functions and $A_\alpha^{iT}, A_\beta^{iT}$, $i=1,\ldots,h$, represent a basis for the $2h$ real harmonic differentials in the variables $(z,\bar{z})$ and $(w,\bar{w})$ respectively. Let us notice that in the above Hodge decomposition we ignored possible instantonic contributions. In the Yang-Mills case they are ruled out by the fact that the group $SU(N)$ is simply connected. In the abelian case, instead, the instantonic gauge fields do not play any role because they decouple in the free action S_0 from the exact and coexact components.

Clearly, requirements a) and b) are satisfied only if:

$$G_{\alpha\beta}(z,w) = \partial_\alpha \partial_\beta G(z,w) \tag{3.16}$$

i.e. the harmonic components A_α^{iT} and A_β^{iT} do not appear in the propagator. The function $G(z,w)$ can be now computed exploiting the ansatz (3.16) in eqs. (3.9) and (3.11). As an upshot, we obtain a biharmonic equation which is solvable on any Riemann surface as shown in ref. [20] for the particular case of the Lorentz gauge:

$$G(z,w) = \int_M d^2t \sqrt{g} K(z,t) K(w,t) \tag{3.17}$$

Here $K(z,t)$ is the well known scalar Green function defined by the equations:

$$\partial_z \partial_{\bar{z}} K(z,t) = \delta_{z\bar{z}}^{(2)}(z,t) - \frac{g_{z\bar{z}}}{A} \qquad A = \int_M d^2 z g_{z\bar{z}} \tag{3.18}$$

$$\partial_{\bar{z}} \partial_t K(z,t) = -\delta_{\bar{z}t}^{(2)}(z,t) + \sum_{i,j=1}^{h} \bar{\omega}_i(\bar{z}) \left[\text{Im } \Omega\right]_{ij}^{-1} \omega_j(t) \tag{3.19}$$

$$\int_M d^2 t g_{t\bar{t}} K(z,t) = 0 \tag{3.20}$$

After a straightforward computation we obtain from eqs. (3.8)-(3.11) the following final expressions for the components of the propagator:

$$G_{zw}(z,w) = -\frac{\lambda-1}{4} \int_M d^2 t g_{t\bar{t}} \partial_z K(z,t) \partial_w K(w,t) \tag{3.21}$$

$$G_{\bar{z}w}(z,w) = -\frac{\lambda+1}{4} \int_M d^2 t g_{t\bar{t}} \partial_{\bar{z}} K(z,t) \partial_w K(w,t) \tag{3.22}$$

$$G_{\bar{z}\bar{w}}(z,w) = -\frac{\lambda-1}{4} \int_M d^2 t g_{t\bar{t}} \partial_{\bar{z}} K(z,t) \partial_{\bar{w}} K(w,t) \tag{3.23}$$

$$G_{z\bar{w}}(z,w) = -\frac{\lambda+1}{4}\int_M d^2t g_{t\bar{t}}\partial_z K(z,t)\partial_{\bar{w}}K(w,t) \tag{3.24}$$

It is now easy to check by direct substitution that the tensors (3.21)-(3.24) satisfy the equations of motion (3.3)-(3.6) identically. In the proof, we have to permute the derivatives in z, w or in their complex complex conjugate variables with the integrals in d^2t appearing in eqs. (3.21)-(3.24). This can be done without problems (see for example ref. [19]) because the components of the propagator given above are well defined distributions. As a matter of fact, they are derivatives of the biharmonic Green function $G(z,w)$ which has been extensively studied on any Riemannian manifold.

We notice at this point that the requirements a)-c), together with the free equations of motion (3.3)-(3.6), determine the propagator of the gauge fields uniquely. Indeed, from a) and b) we obtained that the propagator should be of the form (3.16). Moreover, from the equations of motion we were able to determine the biharmonic Green function $G(z,w)$ up to solutions of the homogeneous biharmonic equation $\Delta_g^2\varphi = 0$ in z and w, where $\Delta_g = 2g^{z\bar{z}}\partial_z\partial_{\bar{z}}$. On a closed and orientable Riemann surface this equation is equivalent to the following one:

$$\Delta_g\varphi = \text{constant} \tag{3.25}$$

Now, it is well known that (3.25) does not admit any global solution on M apart from the trivial case in which the right hand side vanishes and $\varphi = \varphi_0$ is constant. This possibility of adding a constant φ_0 to the biharmonic Green function is however ruled out by the conditions (3.15), which require that the physical biharmonic Green function satisfies the relations:

$$\int_M d^2 g_{z\bar{z}}G(z,w) = \int d^2w g_{w\bar{w}}G(z,w) = 0$$

It is in fact easy to see with the help of (3.20) that the function $G(z,w) + \varphi_0$ verifies the above equations only if $\varphi_0 = 0$.

Before concluding this section, two remarks are in order. First of all, we notice that eqs. (3.21)-(3.24) yield the explicit form of the components of the gauge propagators on a Riemann surface of any genus for the class of covariant gauges (2.9). As a matter of fact, the expression of $K(z,t)$ in terms of the prime form and of the abelian differentials is known on every closed and orientable Riemann surface [25] and can be explicitly constructed also on algebraic curves [11], [26]. Moreover the propagator (3.2) computed here is a well defined tensor on M. Exploiting its covariance under diffeomorphisms in the two indices α and β it is possible to extend the calculations performed here also to a general metric as explained in the previous section.

To complete our discussion, we have to derive the propagator $G_{gh}(z,w)$ of the ghost fields. From eq. (2.12) it turns out that this Green function satisfies at the lowest order the following harmonic equation:

$$\Delta_g G_{gh}(z,w) = \delta^{(2)}(z,w) - \frac{g_{z\bar{z}}}{A} \tag{3.26}$$

The term $1/A$ is required by the presence of a constant zero mode. Comparing with eq. (3.18), it is clear that

$$G_{gh}(z,w) = K(z,w) \tag{3.27}$$

4 Further Properties of the Propagator

First of all we verify that, locally, the components of the propagator computed in the previous section coincide with the flat ones. To prove this fact we start with the well known flat propagator, written in real coordinates $x = (x_1, x_2)$ and $y = (y_1, y_2)$:

$$G_{\mu\nu}(x,y) = \frac{\delta_{\mu\nu}}{\triangle} + (\lambda - 1)\frac{\partial_\mu \partial_\nu}{\triangle^2} \tag{4.1}$$

Formally, this propagator satisfies eq. (3.7). We compute now the components of (4.1) in complex coordinates exploiting the conventions:

$$\begin{cases} z = x_1 + ix_2 \\ \bar{z} = x_1 - ix_2 \end{cases} \qquad \begin{cases} \partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \\ \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \end{cases} \tag{4.2}$$

After a few calculations one finds:

$$G_{zw}(z,w) = \frac{1}{4}\left[G_{11} - G_{22} - i(G_{12} + G_{21})\right](x,y) = -(\lambda - 1)\frac{\partial_z \partial_w}{\triangle^2} \tag{4.3}$$

and, analogously:

$$G_{z\bar{w}}(z,w) = -(\lambda + 1)\frac{\partial_z \partial_{\bar{w}}}{\triangle^2} \tag{4.4}$$

$$G_{\bar{z}w}(z,w) = -(\lambda + 1)\frac{\partial_{\bar{z}} \partial_w}{\triangle^2} \tag{4.5}$$

$$G_{\bar{z}\bar{w}}(z,w) = -(\lambda - 1)\frac{\partial_{\bar{z}} \partial_{\bar{w}}}{\triangle^2} \tag{4.6}$$

In deriving the above equations we have used the translational invariance of the flat Green functions, so that $\partial_z \frac{1}{\triangle} = -\partial_w \frac{1}{\triangle}$ and so on for the derivatives $\partial_{\bar{z}}$ and $\partial_{\bar{w}}$.

On the other side, the scalar Green function $K(z,t)$ appearing in eq. (3.18) is proportional to the inverse of the Laplacian $\triangle_g$ defined on the Riemann surface, i.e. $K(z,t) \equiv \frac{2}{\triangle_g}$. This can be seen from eq. (3.18) noting that, in complex coordinates, $g^{z\bar{z}}\partial_z\partial_{\bar{z}} = \frac{\triangle_g}{2}$. Thus it follows that the biharmonic Green function $G(z,w)$ introduced in eq. (3.17) is equal to $\frac{4}{\triangle_g^2}$. At this point it is easy to check that the components (4.3)-(4.6) obtained from the flat propagator (4.1) are equivalent to those of eqs. (3.21)-(3.24) on any open patch U of M. For example, from eq. (3.21) it is possible to rewrite $G_{zw}(z,w)$ in the following way:

$$G_{zw}(z,w) = -(\lambda - 1)\frac{\partial_z \partial_w}{\triangle_g^2}$$

Choosing on U a locally flat metric, we have $\triangle_g = \triangle$ and the above equation coincides with (4.3). Analogous identities arise in the case of the remaining components completing our proof.

Next, we verify the compatibility of the propagator derived in Section 2 with requirements a)-c). The proof of a) is very simple. The components of the propagator are in fact exact or coexact differentials in z and w, so that one can exploit the orthogonality properties of the Hodge decomposition stating that the exact and coexact differentials are always orthogonal with respect to the abelian differentials on M [24]. Therefore, using the standard definition of the scalar product between one−forms, one obtains:

$$\int_M d^2z G_{z\beta}(z,w)\bar{\omega}_i(\bar{z}) = \int_M G_{\bar{z}\beta}(z,w)\omega_i(z) = 0$$

for $i = 1, \ldots, h$ and $\beta = w, \bar{w}$. Analogous equations are valid in the variables w and $\bar{w}$ proving requirement a). Also the singlevaluedness of the propagator, in particular eq. (3.12) of point b), is a direct consequence of the form of the components (3.21)-(3.24), which are total derivatives of the biharmonic function (3.17) with respect to the variables z, w and their complex conjugates. Finally, eq. (3.15) follows from eq. (3.20) as already shown in the previous section.

Since the propagator is uniquely fixed by the equations of motion and by the physical requirements a)-c), it should also satisfy the Slavnov-Taylor identities associated to the BRST invariance of the gauge fixed theory (2.12) under the transformations (2.5) and (2.6). In particular, let us consider the Green function $\langle A^a_\alpha(z,\bar{z})\bar{c}^b(w,\bar{w})\rangle$:

$$0 = \delta\langle A^a_\alpha(z,\bar{z})\bar{c}^b(w,\bar{w})\rangle = \langle\left(\partial_\alpha c^a(z,\bar{z}) - \kappa f^{ade}c^d(z,\bar{z})A^e_\alpha(z,\bar{z})\right)\bar{c}^b(w,\bar{w})\rangle$$

$$-\frac{1}{\lambda}\langle A^a_\alpha(z,\bar{z})\partial_\beta A^{\beta b}(w,\bar{w})\rangle$$

Applying the operator ∂^α to both sides of the above equation and keeping only the zeroth order terms with respect to the coupling constant κ, we obtain the identity:

$$\frac{1}{\lambda}\partial^\alpha\partial^\beta G_{\alpha\beta}(z,w) = -\delta^{(2)}(z,w) + \frac{g_{z\bar{z}}}{A} \tag{4.7}$$

The right hand side has been computed exploiting the equations of motion of the ghost fields (3.26). At this point we substitute in eq. (4.7) the components of the propagator (3.21)-(3.24) derived before. Eqs. (3.8) and (3.10) yield:

$$\partial^\alpha\partial^\beta G_{\alpha\beta}(z,w) = g^{z\bar{z}}g^{w\bar{w}}\frac{2\lambda}{\lambda-1}\left[\partial_z\partial_w G_{\bar{z}\bar{w}}(z,w) + \partial_{\bar{z}}\partial_{\bar{w}}G_{zw}(z,w)\right] \tag{4.8}$$

Using the fact that

$$\partial_w G_{\bar{z}\bar{w}}(z,w) = -\frac{(\lambda-1)}{4}g_{w\bar{w}}\partial_{\bar{z}}K(z,w)$$

$$\partial_{\bar{w}}G_{zw}(z,w) = -\frac{(\lambda-1)}{4}g_{w\bar{w}}\partial_z K(z,w)$$

and with the help of eq. (3.18), it is easy to see that (4.8) is nothing but the Slavnov-Taylor identity (4.7).

To conclude this section, we compute the structure of the singularities in the components of the propagator. In view of perturbative applications, in fact, it is important to know the degree of divergence in the correlation functions. First of all, since the propagator is defined on a compact manifold, infrared divergencies are absent. Choosing the Feynman gauge, $\lambda = 1$, one picks up in eqs. (3.21)-(3.24) the components $G_{\bar{z}w}(z,w)$ and $G_{z\bar{w}}(z,w)$. In analogy with the flat case, we expect that the propagator in the Feynman gauge has a logarithmic singularity at short distances. This implies that the derivatives of the propagator should have a simple pole when $z \longrightarrow w$. Indeed, deriving eq. (3.22) in z and using the property (3.18) of the scalar Green function $K(z,w)$, one obtains:

$$\partial_z G_{\bar{z}w}(z,w) = -\frac{1}{2} g_{z\bar{z}} \partial_w K(z,w) + \frac{1}{2A} g_{z\bar{z}} \int_M d^2t g_{t\bar{t}} \partial_w K(w,t) \tag{4.9}$$

Clearly, the right hand side has a simple pole, since $\partial_w K(z,w) \sim \frac{1}{z-w}$. No other divergencies are present in eq. (4.9) because the second term in the right hand side vanishes due to eq. (3.20). An analogous result holds in the case of $G_{z\bar{w}}(z,w)$.

Now we consider the components $G_{zw}(z,w)$ and $G_{\bar{z}\bar{w}}(z,w)$. They are picked up choosing the gauge $\lambda = -1$. The possible divergencies may arise only in the limit $z \longrightarrow w$. However, a simple look at eqs. (4.3) and (4.6) shows that there are no poles in this limit at least in the flat case. As a matter of fact, the expression of the biharmonic Green function $\frac{1}{\triangle^2}$ at short distances is given by:

$$G_{flat}(z,w) \sim \frac{1}{2}|z-w|^2 \log|z-w| + \dots \tag{4.10}$$

From the above formula, it is clear that $\partial_z \partial_w G_{flat}(z,w)$ and $\partial_{\bar{z}} \partial_{\bar{w}} G_{flat}(z,w)$ do not have any divergence when $z \longrightarrow w$. The finiteness of these components is also clear from the expression of the propagator (4.1) in the Fourier space. This is just an accident, caused by the fact that the logarithmic divergence of $G_{flat}(z,w)$ in $z = w$ is hidden by the factor $|z-w|^2$. Indeed, $G_{zw}(z,w)$ and $G_{\bar{z}\bar{w}}(z,w)$ remain distributions and the singularities emerge after exploiting the equations of motion (3.3)-(3.4). Since the short distance behavior of the correlation functions should not depend on the topology, we expect that the finiteness of the components holds not only in the flat case, but also on a Riemann surface of any genus. To prove this statement, we rewrite the integral in (3.21) as follows:

$$\int_M d^2t g_{t\bar{t}} \partial_z K(z,w) \partial_w K(z,w) =$$

$$\int_D d^2t g_{t\bar{t}} \partial_z K(z,w) \partial_w K(z,w) + \int_{M-D} d^2t g_{t\bar{t}} \partial_z K(z,w) \partial_w K(z,w) \tag{4.11}$$

where D is a small disk of radius ϵ cut in the Riemann surface. D contains both the points z and w, which are supposed to be very close. The second integral in the right hand side of eq. (4.11) is harmless and the potential singularities are present only in the first integral over the disk D, where $K(z,w) \sim \log|z-w|$. As a consequence, taking a locally flat metric on D, the leading divergent term is:

$$\int_D d^2t g_{t\bar{t}} \partial_z K(z,w) \partial_w K(z,w) \sim \int_D d^2t \frac{1}{(t-z)^2} + O(z-w) \tag{4.12}$$

Using a system of polar coordinates r and θ centered at the point z, the above integral becomes (see also ref. [27], pag.375):

$$\int_D \frac{dt d\bar{t}}{(t-z)^2} = -2i \int_0^\epsilon r dr \int_0^{2\pi} e^{-2i\theta} d\theta = 0 \tag{4.13}$$

Therefore, inserting eq. (4.13) in eq. (4.12) and substituting again the latter into eq. (4.11), it turns out that $G_{zw}(z,w)$ remains finite in the limit $z \longrightarrow w$. This result is independent of the fact that we have used the particular topology of a disk. The choice of another simply connected manifold with boundary amounts in fact only to a conformal transformation, which is irrelevant in eq. (4.11), because it is written in a covariant way. An analogous proof can be performed also in the case of $G_{\bar{z}\bar{w}}(z,w)$.

The finite parts of $G_{zw}(z,w)$ and $G_{\bar{z}\bar{w}}(z,w)$ may also play a role in perturbation theory. In order to compute them, one has to evaluate the following integrals:

$$G_{zz}(z,\bar{z}) = \int_M d^2t g_{t\bar{t}} \left[\partial_z K(z,t)\right]^2 \tag{4.14}$$

$$G_{\bar{z}\bar{z}}(z,\bar{z}) = \int_M d^2t g_{t\bar{t}} \left[\partial_{\bar{z}} K(z,t)\right]^2 \tag{4.15}$$

$G_{zz}(z,\bar{z})$ and $G_{\bar{z}\bar{z}}(z,\bar{z})$ should be singlevalued tensors on a Riemann surface without singularities. The strategy exploited in order to solve these integrals is to rewrite the integrand in another form, which reproduces the poles of $\partial_z K(z,t)^2$ at $z=t$ but is linear in $K(z,w)$. For instance, we start with the sphere of genus zero S^2. Choosing the metric $g_{z\bar{z}} dz d\bar{z} = \frac{dz d\bar{z}}{(1+z\bar{z})^2}$ the scalar Green function $K(z,w)$ becomes:

$$K(z,w) = \log\left[\frac{|z-w|^2}{(1+z\bar{z})(1+w\bar{w})}\right] \tag{4.16}$$

and one can apply this formula in eqs. (3.21)-(3.23) in order to obtain the explicit form of the propagator. Moreover, from eq. (4.16) we infer the following nice identity:

$$\left[\partial_z K(z,t)\right]^2 = -\nabla_z \partial_z K(z,t) \tag{4.17}$$

where $\nabla_z = \partial_z + g_{z\bar{z}} \partial_z g^{z\bar{z}}$ is the covariant derivative acting on the $(1,0)-$forms. Substituting eq. (4.17) in eq. (4.14) and exploiting the properties of the scalar Green function $K(z,w)$, in particular eq. (3.20), one obtains:

$$G_{zz}(z,\bar{z}) = \int_{S^2} d^2t g_{t\bar{t}} \nabla_z \partial_z K(z,t) = 0 \tag{4.18}$$

An analogous result holds for $G_{\bar{z}\bar{z}}(z,\bar{z})$.

On the torus the computation of $G_{zz}(z,\bar{z})$ and $G_{\bar{z}\bar{z}}(z,\bar{z})$ is very simple due to the translational invariance of the scalar Green function $K(z,w) \equiv K(z-w)$. As a matter of fact, choosing a flat metric $g_{t\bar{t}} = 1$ in eqs. (4.14) and (4.15), one can perform the substitution $t' = z - t$ and set $\partial_z K(z-t) = -\partial_t K(z-t)$. The upshot is that $G_{zz}(z,\bar{z})$ and $G_{\bar{z}\bar{z}}(z,\bar{z})$ are constants given by:

$$G_{zz}(z,\bar{z}) = \int_M d^2t' \left[\partial_{t'} K(t')\right]^2 \qquad G_{\bar{z}\bar{z}}(z,\bar{z}) = \int_M d^2t' \left[\partial_{\bar{t}'} K(t')\right]^2$$

On the Riemann surfaces of genus $g > 1$, however, there is no translational invariance, so that the tensors $G_{zz}(z,\bar{z})$ and $G_{\bar{z}\bar{z}}(z,\bar{z})$ receive a dependency on z. Their expression will be explicitly computed in appendix A.

5 Yang–Mills Field Theories

In the previous sections the propagators of Yang–Mills field theories quantized in the covariant gauge have been explicitly computed on any Riemann surface of genus h. Adding also the color indices, which play however an irrelevant role in the free equations of motion (3.3)-(3.6), the components of the propagator read:

$$G^{ab}_{zw}(z,w) = -\delta^{ab}\frac{\lambda-1}{4}\int_M d^2t g_{t\bar{t}} \partial_z K(z,t)\partial_w K(w,t) \tag{5.1}$$

$$G^{ab}_{\bar{z}w}(z,w) = -\delta^{ab}\frac{\lambda+1}{4}\int_M d^2t g_{t\bar{t}} \partial_{\bar{z}} K(z,t)\partial_w K(w,t) \tag{5.2}$$

$$G^{ab}_{\bar{z}\bar{w}}(z,w) = -\delta^{ab}\frac{\lambda-1}{4}\int_M d^2t g_{t\bar{t}} \partial_{\bar{z}} K(z,t)\partial_{\bar{w}} K(w,t) \tag{5.3}$$

$$G^{ab}_{z\bar{w}}(z,w) = -\delta^{ab}\frac{\lambda+1}{4}\int_M d^2t g_{t\bar{t}} \partial_z K(z,t)\partial_{\bar{w}} K(w,t) \tag{5.4}$$

Analogously we have for the ghost fields:

$$G^{ab}_{gh}(z,w) = \delta^{ab} K(z,w) \tag{5.5}$$

where $K(x,y)$ is the scalar Green function (3.18).

Unfortunately, the knowledge of the propagators alone is not sufficient on a Riemann surface in order to compute all the other correlation functions perturbatively. The second necessary ingredient is provided by the flat connections [1], [5], [6] and [15]. In complex coordinates, they are given by the independent solutions of the equation:

$$\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + i\kappa[A_z, A_{\bar{z}}] = 0 \tag{5.6}$$

which can be constructed as follows (see also the Appendix of ref. [15]). We consider the $2h(N^2-1)$ independent gauge fields $A_z^{(i),\bar{a}}(z)$ and $A_{\bar{z}}^{(i),\bar{a}}(\bar{z}) \equiv (A_z^{(i),\bar{a}}(z))^*$ defined by:

$$A_z^{(i),\bar{a}}(z) = \omega_i(z)\delta^{\bar{a}b}T^b \qquad\qquad A_{\bar{z}}^{(i),\bar{a}}(\bar{z}) = \bar{\omega}_i(\bar{z})\delta^{\bar{a}b}(T^b)^* \tag{5.7}$$

where $i = 1, \ldots, h$ and $\bar{a} = 1, \ldots, (N^2-1)$. In the usual representation of the connections as $su(N)$ valued vector fields $A_\alpha = A_\alpha^b T^b$, we have that $(A_z^{(i),\bar{a}}(z))^b = \omega_i(z)\delta^{\bar{a}b}$ and $(A_{\bar{z}}^{(i),\bar{a}}(\bar{z}))^b = -\bar{\omega}_i(\bar{z})\delta^{\bar{a}b}$. Thus $\bar{a}$ labels the possible independent solutions of eq. (5.6) and simultaneously is also a color index.

We recall that the T^a are in the adjoint representation, so that we can use here the standard form of the $SU(N)$ generators $(T^a)_{ik} = if^{aik}$. In this way the f^{abc} turn out to be real structure constants from the commutation relations $[T^a, T^b] = if^{abc}T^c$ and the elements of the totally antisymmetric matrices T^a are purely imaginary, i.e. $(T^a)^\dagger = T^a$ and $(T^a)^* = -T^a$. It is now clear that the commutator $[A_z^{(i),\bar{a}}(z), A_{\bar{z}}^{(i),\bar{a}}(\bar{z})]$ vanishes, because $[T^a, (T^a)^*] = -[T^a, T^a] = 0$ and therefore

$$[A_z^{(i),\bar{a}}(z), A_{\bar{z}}^{(i),\bar{a}}(\bar{z})] = -[T^{\bar{a}}, T^{\bar{a}}]\ \omega_i(z)\bar{\omega}_i(\bar{z}) = 0$$

Moreover, since the $\omega_i(z)$ and $\bar{\omega}_i(\bar{z})$ are abelian differentials, the following identity is valid:

$$\partial_z A_{\bar{z}}^{(i),\bar{a}} - \partial_{\bar{z}} A_z^{(i),\bar{a}} = 0 \tag{5.8}$$

Hence, we have shown that the differentials $A_z^{(i),\bar{a}}(z)$ and $A_{\bar{z}}^{(i),\bar{a}}(\bar{z})$ satisfy eq. (5.6). Exploiting the freedom of performing gauge transformations of the kind (2.2), the most general expression of these flat connections will be:

$$\tilde{A}_z^{(i),\bar{a}}(z,\bar{z}) = U^{-1}[A_z^{(i),\bar{a}}(z) - i\kappa^{-1}\partial_z]U \tag{5.9}$$

$$\tilde{A}_{\bar{z}}^{(i),\bar{a}}(z,\bar{z}) = U^{-1}[A_{\bar{z}}^{(i),\bar{a}}(\bar{z}) - i\kappa^{-1}\partial_{\bar{z}}]U \tag{5.10}$$

We notice at this point that the $2h(N^2-1)$ special flat connections given above are apparently independent, but some degrees of freedom can still be eliminated by means of the gauge transformations (5.9) and (5.10). The dimension of the moduli space of flat connections $M_F(M, SU(N)))$ is indeed $(2h-2)(N^2-1)$. A proof of this fact, extended also to the more general self-dual connections, is in ref. [6]. In our particular case the dimensionality of $M_F(M, SU(N)))$ does not play an important role, since we are only interested in the perturbative expansion of the Yang–Mills amplitudes near a classical configuration $A_\alpha^{cl}(z,\bar{z})$ satisfying eq. (5.6). Clearly, $A_\alpha^{cl}(z,\bar{z})$ can be always written as a linear combination of the basis (5.7). Accordingly to our strategy, we expand the gauge fields as follows:

$$A_\alpha(z,\bar{z}) = A_\alpha^{cl}(z,\bar{z}) + A_\alpha^q(z,\bar{z}) \tag{5.11}$$

where $A_\alpha^q(z,\bar{z})$ describes a quantum fluctuation around A_α^{cl}.

To quantize the theory, it is now possible to proceed as in the previous sections, imposing the covariant gauge (2.11) only on the quantum perturbation A_α^q. As an upshot, the ghost action and the gauge fixing term (2.8) do not contain A^{cl}_α and the generating functional is the same of eq. (2.12):

$$Z[J] = \int DA^q_\mu D\bar{c}Dc \; e^{\left\{-\mathrm{Tr}\int_M d^2x\sqrt{g}\left[\frac{1}{4}F^2(A)+\frac{1}{2\lambda}f^2(A^q)+\partial_\mu\bar{c}D^\mu(A^q)c+J^\mu A^q_\mu\right]\right\}} \tag{5.12}$$

apart from the replacement: $F_{\mu\nu}(A) \equiv F_{\mu\nu}(A^{cl} + A^q)$.

In this way, however, the invariance of the amplitudes under gauge transformations of A^{cl}_μ is lost. To remedy, one can apply the techniques of refs. [16], choosing the background gauge fixing:

$$f'(A^q, A^{cl}) = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}A^\mu) + \kappa f^{abc}\tilde{A}^{(i)b}_\mu A^{\mu c} = 0$$

Since this gauge fixing is not affecting the free part of the action (when $\kappa = 0$ it coincides with the covariant gauge (2.9)), the free propagators of the theory can be computed as before and are given again by eqs. (5.1)-(5.5).

6 Conclusions

The main result of this paper is the calculation of the relevant propagators entering in Yang–Mills field theories defined on a Riemann surface of any genus. In particular, we have shown that the requirements a)-c) of Section 3 determine the propagator of the gauge fields uniquely. As a proof of the physicality of our propagators, the Slavnov-Taylor identity (4.7) has been verified. We would like to notice that on a Riemann surface only *exact* and *coexact* forms propagate, while the notion of particles is lost. From our investigations two unexpected results emerge. First of all, in complex coordinates not only the Feynman gauge, but also the gauge $\lambda = -1$ is very suitable for calculations. Moreover, we have used here a covariant gauge fixing, but the analysis of Section 2 indicates that there is also the interesting possibility of quantizing the Yang–Mills theories on a compact two dimensional manifold in a noncovariant gauge. As a matter of fact, starting from a metric which is not conformally flat, we are still allowed to impose the gauge fixing (2.11). The reason is that eq. (2.11) is compatible with the holomorphic transition functions on the Riemann surface and can be globally extended over the entire manifold. Involving only the component $g_{z\bar{z}}$ of the metric, this gauge fixing destroys the covariance of the pure Yang–Mills functional under global diffeomorphisms. We remark that this procedure of choosing gauge has no analogous in the flat space. In particular, more classical noncovariant gauges, like for instance the axial gauge [28], the Coulomb gauge or the light cone gauge [29], are not suitable in our case because they cannot be globally imposed on M.

With the expressions given here for the propagators it is possible to start the computation of the other correlation functions and of their radiative corrections. The contributions coming from the flat connections can be evaluated by means of the explicit formulas (5.7).

Many simplifications are expected to occur in the amplitudes because, due to requirement a), it is easy to see that the gauge propagator (5.1)-(5.4) is orthogonal with respect to the flat connections. Moreover, most of the physically relevant two dimensional models, like Quantum chromodynamics, are superrenormalizable. For instance, in the pure Yang–Mills case, there is only one logarithmically divergent Feynman diagram, corresponding to the one-loop correction of the two point function. Using the fact that on a compact manifold all the possible singularities are ultraviolet, so that they occur at short distances where the topology does not play any particular role, it should not be difficult to subtract suitable counterterms in the Lagrangian in order to achieve a finite theory. This would be an important result, proving the renormalizability of gauge field theories on every closed and orientable Riemann surface in an explicit and direct way. However, the computability of the divergent Feynman integrals should still be improved. This is not a simple problem. Even in the case of string theory, explicit calculations have been performed only representing the Riemann surface as an algebraic curve, i.e. as an n sheeted covering of the complex plane [26], [30], [31], [32]. An important step in this direction would be the construction of the biharmonic Green function on any algebraic curve, which is currently under investigation [33]. Recently the Schwinger model quantized in the Lorentz gauge has been successfully solved on any Riemann surface within our explicit formalism, computing the correlation functions of the fermionic currents in a nonperturbative way [18], [20]. We hope therefore that, with the material presented here, it will be possible to extend these results also to the Yang–Mills field theories.

Appendix A

In this appendix the explicit form of the tensor $G_{zz}(z,\bar{z})$ of eq. (4.14) will be computed. We start from the following formula [34], which generalizes eq. (4.17) to any Riemann surface:

$$[\partial_z K(z,t)]^2 = \partial_z^2 K(z,t) + 2\partial_z K(z,t)\int_M d^2y g_{y\bar{y}}\partial_z K(z,y)\tilde{R}_g(y) -$$
$$2\int_M d^2y \partial_z K(z,y)\partial_y K(t,y) P_{z\bar{y}} + \Psi_{zz}(z,t) +$$
$$\frac{1}{A}\int_M d^2v g_{v\bar{v}}\partial_z K_z^{(+)v}(z,v)\int_M d^2y g_{y\bar{y}}\partial_v K(v,y)\tilde{R}(y) \tag{A.1}$$

where

$$\tilde{R}_g(z) \equiv R_g(z) + \frac{1}{2}g^{z\bar{z}}\sum_{i,j=1}^{h}\omega_i(z)\,[\mathrm{Im}\ \Omega]^{-1}_{ij}\,\bar{\omega}_j(\bar{z})$$

and

$$P_{z\bar{y}} \equiv \frac{1}{2}\sum_{ij=1}^{h}\omega_i(z)\,[\mathrm{Im}\ \Omega]^{-1}_{ij}\,\bar{\omega}_j(\bar{y})$$

Moreover the Green function $G_z^{(+)v}(z,v)$ satisfies the equation (see also [35]):

$$\triangle_1^{(+)} G_z^{(+)v}(z,w) = \delta^{(2)}(z,v)$$

and finally $\Psi_{zz}(z,t)$ is a linear combination of the $3h-3$ holomorphic quadratic differentials with coefficients depending on t. We notice also that our formula is slightly different to that of [34] in order to take into account of the different normalization of the scalar Green function $K(z,w)$ given in eqs. (3.18) and (3.19). Substituting eq. (A.1) in eq. (4.14), and exploiting the property (3.20), one easily proves that

$$G_{zz}(z,\bar{z}) = \Psi_{zz}(z) + \int_M d^2v g_{v\bar{v}} \partial_z K_z^{(+)v}(z,v) \int_M d^2y g_{y\bar{y}} \partial_v K(v,y) \tilde{R}(y) \qquad \text{(A.2)}$$

An analogous formula can be found for $G_{\bar{z}\bar{z}}(z,\bar{z})$. As anticipated in section 3, the lack of translational invariance yields a form of $G_{zz}(z,\bar{z})$ and $G_{\bar{z}\bar{z}}(z,\bar{z})$ which is dependent on the space-time variables.

References

[1] M. Atiyah and R. Bott, *Phil. Trans. R. Soc. Lond.* **A 308** (1982), 523.

[2] E. Witten, *Comm. Math. Phys.* **141** (1991), 153; *J. of Geom. and Phys.* **9** (1992), 3781.

[3] D. J. Gross *Nucl. Phys.* **B400** (1993), 161; D. J. Gross and Washington Taylor IV, *Nucl. Phys.* **B400** (1993), 181; *ibid.* **B403** (1993), 395.

[4] J. Fröhlich, On the Construction of Quantized gauge Fields, in "Field Theoretical Methods in Particle Physics" (W. Rühl Ed.) Plenum, New York, 1980.

[5] M. Blau and G. Thompson, *Int. Jour. Mod. Phys.* **A7** (1992), 3781; Lectures on the 2d Gauge Theories, lectures presented at the 1993 Trieste Summer School in High Energy Physics, Preprint IC/93/356, (hep-th/9310144); G. Thompson, 1992 Trieste Lectures on Topological Gauge Theory and Yang–Mills Theory, Preprint IC/93/356 and references therein.

[6] N. J. Hitchin, *Gauge Theories on Riemann Surfaces*, in Lectures on Riemann Surfaces, M. Cornalba & al. (eds), ICTP Trieste Italy, 9 Nov.-18 Dec. 1987, World Scientific editions; *Topology*, **31** (3) (1992), 449; *Proc. London Math. Soc.* **55** (1987), 59.

[7] B. Rusakov, *Mod. Phys. Lett.* **A5** (1990), 693.

[8] D. S. Fine, *Comm. Math. Phys.* **134** (1990), 273; S. G. Rajeev, *Phys. Lett.* **212B**, 203; A. Sengupta, *Ann. Phys.* **147** (1992), 191.

[9] A. Z. Capri and R. Ferrari, *Nuovo Cimento* **A 62** (1981), 273; *Jour. Math. Phys.* **25** (1983), 141; N. Manton, *Ann. Phys.* **159** (1985), 220; J. E. Hetrick and Y.

Hosotani, *Phys. Rev.* **D38** (1988), 2621; A. K. Raina and G. Wanders, *Ann. of Phys.* **132** (1981), 404; C. Jayewardena, *Helv. Phys. Acta* **61** (1988), 636; A. Bassetto and L. Griguolo, *Anomalous Dimensions and Ghost Decoupling in a Perturbative Approach to the Generalized Chiral Schwinger Model*, Preprint DFPD-94-TH-29, hep-th/9495953; A. Bassetto, F. de Biasio and L. Griguolo, *Phys. Rev. Lett.* **72** (1994), 3141; H. Joos, *Nucl. phys.* **B17** (Proc. Suppl.) (1990), 704; *Helv. Phys. Acta* **63** (1990), 670; H. Dilger and H. Joos, How Well Do Lattice Simulations Reproduce the Different Aspects of the Geometrical Schwinger Model, Contribution to the "XI International Symposium on Lattice Field Theory", Dallas 1993, Preprint DESY 93-144; H. Joos and S. I.Azakov, The Geometric Schwinger Model on the Torus.2, Preprint DESY-94-142; K. S. Gupta, R. J. Henderson. S. G. Rajeev and O. T. Torgut, *Jour. Math. Phys.* **35** (1994), 3845; I. Sachs and A. Wipf, *Helv. Phys. Acta* **65** (1992), 653; E. Abdalla, M.C.B. Abdalla and K. D. Rothe, Nonperturbative Methods in Two Dimensional Quantum Field Theory, World Scientific, Singapore 1991.

[10] B. S. DeWitt, The Spacetime Approach to Quantum Field Theory, in: Relativity, Groups and Topology II, B. S. DeWitt and R. Stora (eds), North Holland, Amsterdam, pp. 381-738; G. Bernard, A. Duncan, *Ann. Phys.* **107** (1977), 201; T. K. Leen, *Ann. Phys.* **147** (1983), 417; I. L. Buchbinder and S. D. Odintsov, *Sov. Phys.* **J26** (1983), 359; D. J. Toms, *Phys. Lett.* **126B** (1983), 37; *Phys. Rev.* **D27** (1983), 1803; L. Parker and D. J. Toms, *Phys. Rev.* **D29** (1984), 1584; M. D. Freeman, *Ann. Phys.* **153** (1984), 339; I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, Effective Action in Quantum gravity, IOP Publishing, Bristol 1992; G. Cognola, L. Vanzo and S. Zerbini, *Phys. Lett* **241B** (1990), 381; G. Cognola, K. Kirsten and S. Zerbini, *Phys. Rev.* **D48** (1993), 790; E. Elizalde and S. D. Odintsov, *Phys. Lett.* **303B** (1993), 240; G. Cognola, *Phys. Rev.* **D50** (1994), 909.

[11] J. D. Fay, *Lect. Notes in Math.* **352**, Springer Verlag, 1973.

[12] N. D. Birrel and P. C. W. Davies, Quantum Fields in Curved Space, Cambridge University Press, Cambridge (1982); S. A. Fulling, Aspects of Quantum Field Theory in Curved Space-Time, Cambridge University Press, Cambridge (1989).

[13] F. Ferrari, *Phys. Lett.* **277B** (1992), 423; *Comm. Math. Phys.* **156** (1993), 179; *Int. Jour. Mod. Phys.* **A9** (3) (1994), 313.

[14] F. Ferrari, J. Sobczyk and W. Urbanik, Operator Formalism on the Z_n Symmetric Algebraic Curves, Preprint LMU-TPW 93-20, ITP UWr 856/93.

[15] J. Sonnenschein, *Phys. Rev.* **D42** (1990), 2080.

[16] L. F. Abbott, *Nucl. Phys.* **B185** (1981), 189; B. S. DeWitt, *Phys. Rev.* **162** (1967), 1195, 1239; G. 't Hooft, *Nucl. Phys.* **B62** (1973), 444; H. Kluberg-Stern and J. -B. Zuber, *Phys. Rev.* **D12** (1975), 467, 482, 3159; S. D. Joglekar and B. W. Lee, *Ann. Phys. (NY)* **97** (1976), 160.

[17] J. Schwinger, *Phys. Rev.* **128** (1962), 2425.

[18] F. Ferrari, On the Schwinger Model on Riemann Surfaces, Preprint LMU-TPW 93-26,

MPI-Ph/93-71.

[19] L. Sario, M. Nakai, C. Wang and L. O. Chung, Lecture Notes in Mathematics **605**, Springer Verlag 1977.

[20] F. Ferrari, *Class. Q. Grav.* **10** (1993), 1065.

[21] A. A. Slavnov, *Theor. Math. Phys.* **10** (1972), 99; J. C. Taylor, *Nucl. Phys.* **B33** (1971), 436.

[22] C. Becchi, A. Rouet and R. Stora, *Comm. Math. Phys.* **42** (1975), 127; *Ann. Phys.* **98** (1976), 287.

[23] V. N. Gribov, *Nucl. Phys.* **B139** (1978), 1.

[24] M. Nakahara, Geometry, Topology and Physics, Adam Hilger, Bristol, Philadelphia and New York, 1990.

[25] E. Verlinde and H. Verlinde, *Nucl. Phys.* **B288** (1987), 357.

[26] F. Ferrari, *Int. Jour. Mod. Phys.* **A5** (1990), 2799.

[27] I. M. Gelfand and G. E. Shilov, Generalized Functions I, Academic Press, New York and London 1964.

[28] R. L. Arnowitt and S. I. Flicker, *Phys. Rev.* **127** (1962), 1821.

[29] G. 't Hooft, *Nucl. Phys.* **B75** (1974), 461.

[30] V. G. Knizhnik, *Sov. Phys. Usp.* **32** (11) (1989), 945; M. A. Bershadsky and A. O. Radul, *Int. Jour. Mod. Phys.* **A2** (1987), 165; J. Sobczyk, *Mod. Phys. Lett.* **A6** (1991), 1103.

[31] D. Lebedev and A. Morozov, *Nucl. Phys.* **B302** (1988), 63; A. A. Belavin, V. G. Knizhnik, A. Yu. Morozov and A. M. Perelomov, *JETP Lett.* **43** (1986), 411; D. J. Gross and P. F. Mende, *Nucl. Phys.* **B303** (1988), 407; R. Iengo, *Nucl. Phys.* **B15** (Proc. Suppl.) (1990), 67; E. Gava, R. Iengo and G. Sotkov, *Phys. Lett.* **207B** (1988), 283; D. Montano, *Nucl. Phys.* **B297** (1988), 125.

[32] F. Ferrari and J. Sobczyk, Operator Formalism on General Algebraic Curves, Preprint U.T.F. 333, IFT UWr 879/94; F. Ferrari, *Int. Jour. Mod. Phys.* **A7** (1992), 5131; J. Sobczyk and W. Urbanik, *Lett. Math. Phys.* **21** (1991), 1; J. Sobczyk, *Mod. Phys. Lett.* **A6** (1991), 1103.

[33] F. Ferrari and J. Sobczyk, work in progress.

[34] S. M. Kuzenko and O. A. Solov'ev, JETP 51 (1990), 265.

[35] E. D'Hoker and D. H. Phong, *Rev. Mod. Phys.* **60** (1988), 917.