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The Geometric Schwinger Model on the Torus II

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Abstract. In the Schwinger model with iso-spin on the torus the regularized effective action, the fermion propagator in a background field, and the correlation functions of the fermion bilinears are calculated. These calculations illustrate in particular the dynamics of the $U_L(2) \times U_R(2)$ symmetry breaking by an $U_A(1)$ axial anomaly, and the role of the topological zero modes in this scheme.

1 INTRODUCTION

The geometric Schwinger Model (gSM) is the theory of a $U(1)$-gauge field in two dimensions coupled to a massless Dirac Kähler field. It is equivalent to a Schwinger model with Dirac fields $\phi_a^b(x)$ carrying iso-spin 1/2. We consider this model on the Euclidean space time of a torus. In Part I we discussed in detail the zero mode structure of this model [1]. The main aim of this Part is the calculation of the correlation functions of currents and densities. Since it turned out that the gSM illustrates the generally interesting structure of anomalous chiral symmetry breaking in a very transparent manner, we shall present our results in the more familiar language of Dirac fields. In the introduction to the first part of our investigations we mentioned as motivation for the study of the gSM on the torus the possibility of a systematic lattice approximation of this model [2]. In the meanwhile this project was realized to a large extend [3, 4]. Here we give the details of the discussion of the different quantities in the continuum to which we applied the lattice approximation. For these we formulate the 'geometric' description by differential forms of quantities which we consider interesting in this context. Some comments on the success of the lattice approximation can be found in our conclusions.

In terms of Dirac fields the action of the gSM is expressed as (I,Eq.(1)),

$$S = \int_{\mathcal{T}_2} dx \left(\frac{1}{2} F(x)^2 + \sum_b \bar{\phi}^{(b)}(x) \gamma^\mu (\partial_\mu - ieA_\mu) \phi^{(b)}(x) \right). \tag{1.1}$$

The topology of gauge fields on the torus is contained in the following representation of the gauge potential

$$A_\mu(x) = C_\mu^{(k)}(x) + t_\mu + \epsilon_\mu^\nu \partial_\nu b(x) + \partial_\mu a(x), \tag{1.2}$$

(compare I,Eq.(9)). $C_\mu^{(k)}(x) = -\frac{\pi k}{eL_1L_2}\epsilon_{\mu\nu}x^\nu$, is the gauge potential of a representative with constant field strength $F_{\mu\nu}^{(k)}(x) = \frac{2\pi k}{eL_1L_2}\epsilon_{\mu\nu}$ of the Chern class $\mathcal{CH}^k$. The Chern index $k = (e/2\pi)\int_{\mathcal{T}} F_{12}dx^{12}$ takes the values $k = 0, \pm 1, \pm 2, \ldots$. $\partial_\mu a(x)$ is a pure gauge, $\epsilon_\mu^\nu \partial_\nu b(x)$ the transverse physical component. $a(x)$, $b(x)$ are continuous on the torus $\mathcal{T}$, $0 \leq x_\mu \leq L_\mu$, and orthogonal on the constant. $C_\mu^{(k)}(x)$ is only periodic up to a gauge transformation: $A_\mu(x + \hat{L}_\nu) = A_\mu(x) - \frac{i}{e}\Lambda_\nu^{-1}(x)\partial_\mu \Lambda_\nu(x)$ (I,Eq.(3)). The harmonic (constant) gauge potential t_μ, $\Box t_\mu = 0$, is called toron. It is gauge invariant up to large gauge transformations:

$$t_\mu \xrightarrow{\Lambda} t_\mu + \frac{2\pi}{eL_\mu} m_\mu, \quad \Lambda(x) = \exp 2\pi i (m_1 \frac{x^1}{L_1} + m_2 \frac{x^2}{L_2}), \quad m_i \text{ integer}. \tag{1.3}$$

In I we studied in detail the spectrum of the Dirac operator $D_A = \gamma^\mu(\partial_\mu - ieA_\mu)$, in particular its topological zero mode solutions (I,Eq.(26)):

$$\gamma^\mu(\partial_\mu - ieA_\mu)\phi^b(x) = 0 \quad \text{with} \quad \phi^b(x + \hat{L}_\nu) = \Lambda_\nu(x)\phi^b(x). \tag{1.4}$$

Quantum mechanical expectation values are calculated by the path integral formula. In particular vacuum expectation values (VEV) of Dirac fields are expressed after 'fermion integration' by a functional integration over gauge fields

$$\langle \Phi(x_1)\bar{\Phi}(y_1)\ldots\Phi(x_n)\bar{\Phi}(y_n)\rangle =$$
$$\tfrac{1}{Z}\sum_k \int \mathcal{D}^{(k)}[A] e^{-S_F[A]} \det{}'[D_A] \times \sum_{perm} \left\{ \begin{array}{c} (\pm 1)\phi_1(x_{i_1})\ldots\phi_l(x_{i_l})\bar{\phi}_1(y_{j_1})\ldots\bar{\phi}_l(y_{j_l}) \\ \mathcal{S}^{(k)}(x_{i_{l+1}}, y_{j_{l+1}}; A)\ldots\mathcal{S}^{(k)}(x_{i_n}, y_{j_n}; A) \end{array} \right\}. \tag{1.5}$$

$\sum_k$ denotes summation over the different topological sectors, $\mathcal{D}^{(k)}[A]$ stands for the formal functional measure $\mathcal{D}[a]\mathcal{D}[b]dt$ in these sectors. The ingredients of this formula are the zero mode wave functions $\phi_i(x)$, $\bar{\phi}_j(y)$, $i, j = 1, \ldots, l$, discussed in I.2, the effective action $\Gamma[A]$, with $\exp\frac{1}{2}\Gamma[A] = \det{}'[D_A]$, and the Green's function $\mathcal{S}^{(k)}(x, y; A)$ of the Dirac operator in the space orthogonal to the zero modes. In the next Section 2 we calculate this effective action and the Green's functions. In Section 3 we shall apply formula Eq. (1.5) for the calculation of the correlation function of the field strength $F_{\mu\nu}(x)$, the screened static potential including finite size corrections, and last but not least the correlation functions of fermion currents and densities.

2 EFFECTIVE ACTION AND PROPAGATOR

2.1. In the following we calculate the regularized effective action by standard methods. First we give a short **sketch of the general procedure** [5]. In the application to the gSM we follow reference [6]. The fermion integration of the path integral formula introduces in Eq. (1.5) an effective action $\Gamma^{(k)}(A)$ for $A_\mu(x) \in \mathcal{CH}^k$. We have to regularize this expression. For Pauli-Villars regularization we introduce a sufficient number of regulator fields of the fermion type $e_i = +1$, and of the boson type $e_i = -1$ with regulator masses M_i, $i = 1, \dots, r$. They have to satisfy the regulator conditions: $\sum_{i=0}^r e_i = 0$, $\sum_{i=0}^r e_i M_i^{2p} = 0$, $p = 1, \dots, r$, $(e_0 = 1,\ M_0^2 = 0)$. The expression for the regularized effective action is then:

$$\begin{aligned}\exp \frac{1}{2}\Gamma^{(k)}_{reg}(A) &= \det{}' D_A \textstyle\prod_{i=1}^r \det(D_A - M_i)^{e_i} \\ &= \det{}'(D_A D_A^\dagger)^{\frac{1}{2}} \textstyle\prod_{i=1}^r \det(D_A D_A^\dagger + M_i^2)^{e_i/2}, \end{aligned} \tag{2.1}$$

$\det{}'$ indicates the omission of the $2k$ eigenvalues zero in the calculation of the determinant. The second line results from the first by the fact that the non-zero eigenvalues of D_A appear in pairs $\pm i|E_n|$, a consequence of the relation $\gamma^5 D_A = -D_A\gamma^5$.

The regularization allows for the following mathematical operations leading to an evaluation of the effective action $\Gamma^{(k)}_{reg}$ by the ζ-function of the operator $D_A D_A^\dagger$:

$$\Gamma^{(k)}_{reg}(A) = \mathrm{Tr}'(\sum_{i=0}^r e_i \log(D_A D_A^\dagger + M_i^2)) + 2|k| \sum_{i=1}^r e_i \log M_i^2 \tag{2.2}$$

$$= -\int_0^\infty \frac{dt}{t} (\sum_{i=0}^r e_i e^{-tM_i^2} \mathrm{Tr}(e^{-tD_A D_A^\dagger} - \mathcal{P}_0)) + 2|k| \sum_{i=1}^r e_i \log M_i^2 \tag{2.3}$$

$$\begin{aligned}&= \sum_{i=1}^r e_i(-r_1(D_A D_A^\dagger) M_i^2 \log M_i^2 + \zeta(0|D_A D_A^\dagger) \log M_i^2) + 2|k| \sum_{i=1}^r e_i \log M_i^2 \\ &\quad -\zeta'(0|D_A D_A^\dagger).\end{aligned} \tag{2.4}$$

The equality of the first with the second line results from a straightforward calculation permitted by the existence of the integral $\int_0^\infty \frac{dt}{t} \sum_{i,n} e_i \exp -t(M_i^2 + |E_n^2|)$ as a consequence of the regularization condition $\sum_{i=1}^r e_i = 0$. It uses the expansion of the logarithmic integral $\int_x^\infty \frac{dt}{t} e^{-t} \sim -\log(x\gamma)$ for small x, $\gamma = 0.577\dots$. The omission of the zero eigenvalues in the trace Tr' corresponds to the subtraction of the projection operator $\mathcal{P}_0$ on the zero modes. The ζ-function

$$\zeta(s|D_A D_A^\dagger) = \sum_n{}' d_n |E_n|^{-2s}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{Tr}(e^{-tD_A D_A^\dagger} - \mathcal{P}_0) = \frac{r_1(D_A D_A^\dagger)}{(s-1)} + \ldots \tag{2.5}$$

is defined for sufficiently large real part $Re\, s$ as a sum over all non-zero eigenvalues $|E_n|^2$ of $D_A D_A^\dagger$ with multiplicity d_n. $\Gamma(s)\zeta(s)$ is the Mellin transformation of $\mathrm{Tr}'(\exp -tD_A D_A^\dagger)$. Thus introducing for this expression the inverse Mellin transformation of $\Gamma(s)\zeta(s)$ in Eq. (2.3) allows for an evaluation of the r.h.s. with help of the residuum formula. For this it is important to know that the ζ-function of the modified 2-dimensional Laplacian $D_A D_A^\dagger$ has a pole with residuum $r_1(D_A D_A^\dagger)$ for $s = 1$. This follows from the heat kernel expansion [7] of $\exp(-tD_A D_A^\dagger)$ for small t and $|x - y|$:

$$< x|e^{-tD_A D_A^\dagger}|y >= \frac{e^{-(x-y)^2/4t}}{4\pi t}\{1 + ieA_\mu(x)(x^\mu - y^\mu) + e\gamma_5 F_{12}(x)t + \ldots\}. \tag{2.6}$$

For a detailed description of the well-known relation between heat kernel expansion and ζ-function we refer f.i. to Lemma 1.10.1 of reference [7]. The result of this calculation up to order $O(M_i^{-2}\log M_i)$ is Eq. (2.4). (In order not to overload the formulas we assume an universal length scale which drops out in the final result.) The main content of Eq. (2.4) is the well-known expression of the 'ζ-function regularized' determinant $\log\det(D_A D_A^\dagger) \sim -\zeta'(0|D_A D_A^\dagger)$. As we shall see later, the explicit inclusion of the regulator terms allows for interesting consistency checks of the regulation procedure in the different topological sectors.

A calculation of $\Gamma^{(k)}_{reg}[A]$ by Eq. (2.4) would require the determination of the dependence of the eigenvalues $|E_n|^2$ on $A_\mu(x)$. One can circumvent this problem by considering the functional derivative of $\Gamma_{reg}(A)$. Formally the propagator determines the change of the effective action under an infinitesimal deformation of the gauge potential

$$\bar{\delta}\Gamma = 2\bar{\delta}\log\det D = 2Tr(\bar{\delta}D \cdot D^{-1}). \tag{2.7}$$

Since for the gSM we have explicit expresssions for the Green's function, this functional differential equation becomes in our case very explicit and solvable. However, the trace involves the value of $< x|D^{-1}|y >$ and of its derivative at $x = y$. For a calculation including regularization we calculate the change of $\Gamma^{(k)}_{reg}(A)$ as expressed in Eq. (2.3) under a variation $\delta b(x)$ in the expression of the potential $A_\mu(x)$, Eq. (1.2). With help of the following sequence of formulas: $\gamma_5\gamma^\mu = i\epsilon^\mu_\nu\gamma^\nu$ (in 2 dim.), which implies $\delta D_A = e\gamma_5[D_A, \delta b(x)]$,
leading to $\delta(D_A D_A^\dagger) = -\delta(D_A^2) = -e\gamma_5\{2D_A\delta b(x)D_A + D_A D_A^\dagger\delta b(x) + \delta b(x) D_A D_A^\dagger\}$,
from which follows $\delta(\mathrm{Tr}\exp(-tD_A D_A^\dagger)) = -4et\frac{d}{dt}\mathrm{Tr}(\gamma_5\delta b(x)\exp(-tD_A D_A^\dagger))$,
we get after partial integration

$$\begin{aligned}\delta\Gamma^{(k)}_{reg} &= 4e\int_0^\infty dt \sum_i e_i M_i^2 e^{-tM_i^2}\mathrm{Tr}\left(\gamma_5\delta b(x)e^{-tD_A D_A^\dagger}\right) \\ &\quad +4e\,\mathrm{Tr}\left(\gamma_5\delta b(x)e^{-tD_A D_A^\dagger}\right)\left(1 + \sum_i e_i e^{-tM_i^2}\right)\Bigg|_{t=0}^{t=\infty} \\ &= \delta\left(\frac{2e^2}{\pi}\int_T dx b(x)\Box b(x)\right) + 2\delta\left(\log\det\mathcal{N}_A^{(k)}\right). \end{aligned} \tag{2.8}$$

One should have in mind that Tr denotes a trace in function space and in the four dimensional space of spin and iso-spin. For the evaluation of the trace in the first line we used the heat kernel expansion Eq. (2.6) of $< x|e^{-DD^\dagger}|y >$. It leads to the first term in the final result. In the second line, only $4e\mathrm{Tr}(\gamma_5\delta b(x)e^{-tD_AD_A^\dagger})|^{t=\infty} = 4e\mathrm{Tr}(\gamma_5\delta b(x)\mathcal{P}_0)$ contributes. Expressing the projection operator on the zero modes $\mathcal{P}_0$ by the zero mode wave functions I, Eq.(26) leads to the term with $\mathcal{N}_A^{(k)}$, the matrix of the zero mode wave functions of $\mathcal{N}_{A\,mm'} = \int(\bar{\Phi}^m, \Phi^{m'})_0$, $m = (i,k)$. The variational equation Eq. (2.8) has the solution

$$\Gamma_{reg}^{(k)}(A) = 2\log\det\mathcal{N}_A^{(k)} + \frac{2e^2}{\pi}\int_T dx b(x)\Box b(x) + C_0(k, M_i), \tag{2.9}$$

where $C_0(k, M_i)$ is the integration constant.

2.2 The particular problem of the calculation of **the effective action of the gSM on the torus** is the determination of the integration constant $C_0(k, M_i)$. For this we calculate $\Gamma_{reg}^{(k)}$ according to Eq. (2.4) for the special gauge potentials $C_\mu^{(k)}$, with toron field, which are the representatives of the different topological sectors.

In the free case, i.e. for $k = 0$, the spectrum of $D_C D_C^\dagger$ is given by the fourfold degenerate Eigenvalues $|E_n|^2 = 4\pi^2[(\frac{n_1}{L_1} - et_1)^2 + (\frac{n_2}{L_2} - et_2)^2]$ (see I,Eq.(13)). From this we get the expression for the ζ-function

$$\zeta(s|D_t D_t^\dagger) = 4(\frac{4\pi^2}{L_1^2})^{-s}\sum_{n_1,n_2}\left[(n_1 - \tilde{t}_1)^2 + \frac{1}{|\tau|^2}(n_2 - \tilde{t}_2)^2\right]^{-s}, \quad \tilde{t}_\mu = \frac{eL_\mu}{2\pi}t_\mu. \tag{2.10}$$

This is a special case of Kronecker's double series, and can be discussed by the appropriate methods [8, 9]. By a simple generalization of Kronecker's limit formula we get

$$r_1(D_t D_t^\dagger) = \frac{L_1L_2}{\pi}, \quad \zeta(0|D_tD_t^\dagger) = 0,$$

$$\zeta'(0|D_tD_t^\dagger) = -4\log|e^{-\frac{\pi}{|\tau|}\tilde{t}_2^2}\theta_1(\hat{t}_+|\tau)\eta^{-1}(\tau)|^2, \tag{2.11}$$

with $\hat{t}_\pm = \hat{t}_1 \pm i\hat{t}_2 = (eL_2/2\pi)(t_2 \pm i\,t_1)$, Jacobi's θ-function $\theta(z|\tau)$, $\tau = iL_2/L_1$, (see also the Appendix) and Dedekind's function $\eta(\tau) = e^{i\pi\tau/12}\prod_{n=1}^\infty(1 - e^{2ni\pi\tau})$. Inserting these results in Eq. (2.4) we get the value of the effective action for the representative gauge field of the trivial sector:

$$\begin{aligned}\Gamma_{reg}^{(0)}[t] &= 4\log|e^{-\frac{\pi}{|\tau|}\tilde{t}_2^2}\theta_1(\hat{t}_+|\tau)\eta^{-1}(\tau)|^2 - \sum_j e_j(\frac{M_j^2L_1L_2}{\pi}\log M_j^2) \\ &\equiv \tilde{\Gamma}(t) - \sum_j e_j(\frac{M_j^2L_1L_2}{\pi}\log M_j^2).\end{aligned} \tag{2.12}$$

The ζ-function of $D_C D_C^\dagger$ for the representative gauge fields of the non-trivial sectors $k \neq 0$ follows from the spectrum :

$$|E_n|^2 = \frac{4\pi|k|}{L_1L_2}n, \quad n = 0, 1, \ldots, \quad \text{with multiplicity } 2|k| \text{ for } n = 0,\ 4|k| \text{ for } n > 0,$$

(see I,Eq.(33)). It is

$$\zeta(s|D_kD_k^\dagger) = 4|k|\left(\frac{4\pi|k|}{L_1L_2}\right)^{-s}\sum_{n=1}^{\infty} n^{-s} = 4|k|\left(\frac{4\pi|k|}{L_1L_2}\right)^{-s}\zeta_R(s). \tag{2.13}$$

Here $\zeta_R(s)$ is Riemann's ζ-function. From the particular values [10]: $r_1 = 1$, $\zeta_R(0) = -1/2$, $\zeta_R'(0) = -\frac{1}{2}\log 2\pi$ we get the relevant factors in Eq. (2.4):

$$r_1(D_kD_k^\dagger) = \frac{L_1L_2}{\pi}, \quad \zeta(0|D_kD_k^\dagger) = -2|k|, \quad \zeta'(0|D_kD_k^\dagger) = 2|k|\log\frac{2|k|}{L_1L_2}. \tag{2.14}$$

The explicit values of the effective action for the representative potentials $C^{(k)}(x)$ follows from Eq. (2.4) as above

$$\Gamma_{reg}^{(k)}[C^k] = -2|k|\log\left(\frac{2|k|}{L_1L_2}\right) - \sum_j e_j\left(\frac{M_j^2L_1L_2}{\pi}\log M_j^2\right). \tag{2.15}$$

These special values of the effective action in the different sectors determine the integration constant of the variational differential equation with respect to variations of $b(x)$. Thus $C_0(k, M_i)$ in Eq. (2.9) is determined. We can combine our final result in the following formula:

$$\begin{aligned}\Gamma_{reg}^{(k)}[A] \; = \; & \frac{2e^2}{\pi}\int_T dx b(x)\Box b(x) \; + \; 4\delta_{0,k}\log|e^{-\frac{\pi}{|\tau|}\hat{t}_2^2}\theta_1(\hat{t}_+|\tau)\eta^{-1}(\tau)|^2 \\ & +\{2\log\det\mathcal{N}_A^{(k)}/\mathcal{N}_C^{(k)} - 2|k|\log(2|k|/L_1L_2)\} \\ & -\sum_j e_j(\frac{M_j^2L_1L_2}{\pi}\log M_j^2).\end{aligned} \tag{2.16}$$

We want to discuss shortly the implications of this result for the path integral formula after fermion integration, Eq. (1.5). The normalization factor Z in the path integral formula follows from the condition $< 1 >= 1$, and therefore is determined by integration over the trivial sector $k = 0$ only.

The k-independent parts of $\Gamma_{reg}^{(k)}[A]$ can be factorized out of the sum over the gauge sectors. These are the regulator mass dependent term and the term with $b(x)$. Therefore the regulator mass term in $\Gamma_{reg}^{(k)}[A]$ cancels against that of Z, and the normalized result is independent of these masses as required for a consistent renormalization scheme. Having in mind the rather different determination of the regularization terms for the different sectors, Eq. (2.11) and Eq. (2.14), this is a remarkable consistency check. Also the integration over the pure gauge component $\partial_\mu a(x)$ factorizes from all other integrations. Thus we let this contribution cancel with the corresponding one of Z without going into further details of gauge fixing. The term depending on $b(x)$ corresponds to the familiar anomalous mass term of the Schwinger Model [11, 12].

Particularly interesting are the terms depending on k. Only in the trivial sector we have a term $\tilde{\Gamma}(t)$ depending on the toron field. It is induced by the fermions via the spectrum flow of the Dirac operator [13]. The term $2|k|\log(L_1L_2)$ compensates the length scale dependence of the normalization of the zero mode wave functions. It determines the relative weights of the contributions from the different topological sectors. Similarly the term with the zero mode matrix $\mathcal{N}_A$ compensates for the use of a non-orthonormal base for the separation of the zero modes. This will become more clear with the examples in the parts 4.4 -4.7. The effective action should be invariant under the exchange of L_1 and L_2, and also under large gauge transformations, Eq. (1.3). We have checked this, in particular for the non obvious case of the toron part $\tilde{\Gamma}(t)$ with help of well-known formulas on the θ-functions (see Appendix).

2.3 After fermion integration the gauge field dependent part of the action in the trivial sector is

$$\begin{aligned} S'[b(x)] &= \frac{1}{2}\int_T dx^{12}\left(F_{12}(x)F_{12}(x) - \frac{2e^2}{\pi}b(x)\Box b(x)\right) \\ &= \frac{1}{2}\int_T dx^{12} b(x)\Box(\Box - m^2)b(x), \end{aligned} \tag{2.17}$$

$m^2 = 2e^2/\pi$. It is bilinear in $b(x)$, and hence describes free particles. The generating functional of its correlation functions can be calculated by Gaussian integration

$$\frac{1}{Z_0}\int \mathcal{D}[b(x)]e^{\int dx J(x)b(x)}e^{-S'[b]} = e^{\frac{1}{2}\int dxdy J(x)G(x-y)J(y)}, \quad Z_0 = \int \mathcal{D}[b]e^{-S'[b]}, \tag{2.18}$$

with **the propagator $G(x-y)$ related to $S'[b]$**:

$$\Box(\Box - m^2)G(x-y) = \delta^{(\prime)}(x-y) = \delta(x-y) - \frac{1}{L_1L_2}. \tag{2.19}$$

The propagator $G(x-y)$ plays an important role in the future calculations. We may express $G(x)$ by the eigenfunctions and eigenvalues of the Laplacian $\Box$ on $\mathcal{T}$:

$$\begin{aligned} G(x) &= \frac{1}{m^2L_1L_2}\sum_{n_i}{}'\left\{\frac{e^{\frac{2\pi i}{L_1}(n_1x^1+|\tau|^{-1}n_2x^2)}}{(\frac{2\pi}{L_1})^2(n_1^2+|\tau|^{-2}n_2^2)} - \frac{e^{\frac{2\pi i}{L_1}(n_1x^1+|\tau|^{-1}n_2x^2)}}{m^2+(\frac{2\pi}{L_1})^2(n_1^2+|\tau|^{-2}n_2^2)}\right\} \\ &\equiv \frac{1}{m^2}\left\{G_0(x) - G_m(x)\right\}. \end{aligned} \tag{2.20}$$

The summation $\sum'_{n_i}$ excludes $n_1 = n_2 = 0$. The first part $G_0(x)$ is therefore the Green's function of the Laplacian on $\mathcal{T}$ which as an integral operator transforms a constant function into zero: $G_0 * (const) = 0$. It has the representation [8]:

$$\begin{aligned} G_0(x) &= \frac{1}{L_1L_2}\sum_{n_i}{}'\frac{e^{\frac{2\pi i}{L_1}(n_1x^1+|\tau|^{-1}n_2x^2)}}{(\frac{2\pi}{L_1})^2(n_1^2+|\tau|^{-2}n_2^2)} = -\frac{1}{2\pi}\log\frac{|\theta_1(z|\tau)|}{|\theta_1'(0|\tau)|} + \frac{(x_2)^2}{2L_1L_2} + C \\ &= -\frac{1}{2\pi}\log\left(2\pi\eta^2(\tau)e^{-\pi\frac{(x_2)^2}{L_1L_2}}\frac{|\theta_1(z|\tau)|}{|\theta_1'(0|\tau)|}\right). \end{aligned} \tag{2.21}$$

It is a Green's function $-\Box(\frac{-1}{2\pi}\log(|\theta_1(z|\tau)|/|\theta_1'(0|\tau)|)) = \delta(x)$, because $\theta_1(z|\tau)$ is analytic in $z = \frac{1}{L_1}(x_1 + ix_2)$ and has a single zero on the torus. Therefore it has the correct singularity at $x = 0$: $G_0(x) \approx -\frac{1}{2\pi}\log(|x|/L_1)$. The term $\sim (x_2)^2$ follows from the periodicity of the Green's function. The constant term $\mathcal{C}$ is determined by the condition $G_0 * (const) = 0$. One determines this constant term by explicit calculation using the well-known expansion [14] of $\log(\theta_1(z)/\theta_1'(0))$ with the result:

$$\mathcal{C} = \frac{|\tau|}{12} - \frac{1}{2\pi}\left(\log 2\pi - 2\sum_{m=1}^{\infty}\frac{e^{-2m\pi|\tau|}}{(1-e^{-2m\pi|\tau|})}\cdot\frac{1}{m}\right) = -\frac{1}{2\pi}\log 2\pi\eta^2(\tau). \tag{2.22}$$

For the massive propagator we do not know such an explicit formula. We consider the complete sum Eq. (2.20) including the constant term of the propagator $\overline{G}_m(x) = G_m(x) + 1/(m^2L_1L_2)$. With the help of the formula

$$\sum_{n=-\infty}^{\infty}\frac{e^{2\pi inx}}{a^2+n^2} = \frac{\pi}{a}\frac{\cosh(\pi a[1-2|x|])}{\sinh(\pi a)}, \quad a \neq 0, |x| \leq 1,$$

we can perform one summation in Eq. (2.20):

$$\begin{aligned}\overline{G}_m(x) &= \frac{1}{2L_1}\sum_n\frac{\cosh[E(n)(L_2/2-|x_2|)]e^{\frac{2\pi i}{L_1}nx_1}}{E(n)\sinh[L_2E(n)/2]}, \quad E(n) = \left[\frac{4\pi^2n^2}{L_1^2}+m^2\right]^{1/2} \\ &\rightsquigarrow \frac{1}{2\pi}K_0(m|x|) = \frac{1}{4\pi^2}\int\frac{dp^2e^{ipx}}{p^2+m^2} \quad \text{for } L_1,\ L_2 \to \infty,\end{aligned} \tag{2.23}$$

$\frac{1}{2\pi}K_0(m|x-y|)$ is the 2-point function of free particles with mass m in R_2. (For the limit $L_1 \to \infty$ we used Euler's summation formula for x in the region of convergence).

2.4. After fermion integration the Green's functions of the Dirac fields are expressed by the propagator of the fermion field $\phi^{(b)}(x)$ in a background gauge potential $A_\mu(x)$. It follows from the well-known local solution of the 2 dim. Dirac equation with external gauge potential (see I,Eq.(10)), that we can write this propagator in the trivial sector as

$$\mathcal{S}(x,y;A) = e^{ie\alpha(x)}D_tG_t(x-y)e^{-ie\alpha'(y)} \equiv e^{ie\alpha(x)}\mathcal{S}_t(x-y)e^{-ie\alpha'(y)}, \tag{2.24}$$

with $\alpha(x) = a(x) - i\gamma^5 b(x)$, $\alpha'(x) = a(x) + i\gamma^5 b(x)$, $D_t = \gamma^\mu(\partial_\mu - iet_\mu) = -D_t^\dagger$.
$G_t(x)$ is **the Green's function with a toron background field** only: $D_tD_t^\dagger G_t(x) = \delta(x)$. We have to consider the properties of this Green's function in some detail. $G_t(x)$ is periodic, and can be represented by the plane wave solutions in the usual way:

$$G_t(x) = \frac{1}{L_1L_2}\sum_{n_i}\frac{e^{\frac{2\pi i}{L_1}(n_1x^1+|\tau|^{-1}n_2x^2)}}{(\frac{2\pi}{L_1})^2[(n_1-\tilde{t}_1)^2+|\tau|^{-2}(n_2-\tilde{t}_2)^2]}. \tag{2.25}$$

This double sum is not absolutely convergent. In order to make its evaluation precise, we may use summation procedures described in Weil's book [8]. It starts from a ζ-function like regularization of the Green's function:

$$G_t^s(x) = \frac{1}{L_1L_2}\sum_{n_i}\frac{e^{\frac{2\pi i}{L_1}(n_1x^1+|\tau|^{-1}n_2x^2)}}{(\frac{2\pi}{L_1})^2[(n_1-\tilde{t}_1)^2+|\tau|^{-2}(n_2-\tilde{t}_2)^2]^s}. \tag{2.26}$$

For large enough $Re\ s$ this sum is absolutely convergent. It can be transformed into a absolute convergent series which allows analytic continuation to $s = 1$ for nearly all arguments. The final result for $G_t(x)$ is not very transparent. However, A.Weil derived analyticy properties for such functions suggesting an expression for $S_t(x)$ which can be justified directly. This expression is:

$$(\mathcal{S}_{t;aa'}(x)) = \frac{e^{iet_\mu x^\mu}}{2\pi L_1}\begin{pmatrix} 0 & \frac{\theta_1'(0)\cdot\theta_1(z+\hat{t}_-)}{\theta_1(\hat{t}_-)\cdot\theta_1(z)}e^{-ieL_1t_1z} \\ -\frac{\theta_1'(0)\cdot\theta_1(z^*-\hat{t}_+)}{\theta_1(\hat{t}_+)\cdot\theta_1(z^*)}e^{-ieL_1t_1z^*} & 0 \end{pmatrix} \equiv \mathcal{S}_{t,\mu}\gamma^\mu. \quad (2.27)$$

That this expression satisfies

$$D_t\mathcal{S}_t(x) = 2L_1\begin{pmatrix} 0 & \partial_z - \frac{ie}{2L_1}(t_1 - it_2) \\ \partial_{z^*} - \frac{ie}{2L_1}(t_1 + it_2) & 0 \end{pmatrix}\mathcal{S}_t(x) = \delta(x)$$

follows by direct calculation. Simply regard $e^{iet_\mu x^\mu}$ as formal gauge factor and use analyticity and periodicity of the θ-functions, as well as the singularity of the matrix elements at $z = 0$. We remark

- $\mathcal{S}_t(x)$ is periodic, i.e. invariant under the transformation $x_1 \to x_1 + L_1,\ \ x_2 \to x_2 + L_2$.
- Under large gauge transformations Eq. (1.3), i.e : $\hat{t}_1 \to \hat{t}_1 + m_1,\ \ \hat{t}_2 \to \hat{t}_2 + m_2\cdot\tau$, the Green's function $\mathcal{S}_t(x)$ transforms covariantly:

$$\mathcal{S}_{t+2\pi m_i/eL_i}(x) = e^{2\pi i(m_1\frac{x_1}{L_1}+m_2\frac{x_2}{L_2})}\mathcal{S}_t(x). \quad (2.28)$$

- The short distance behaviour of $\mathcal{S}_t(x) = \gamma^\mu(\partial_\mu - iet_\mu)G_t(x)$ is

$$\mathcal{S}_t(x) = \frac{e^{iex^\mu t_\mu}}{2\pi}\gamma^\nu(\frac{x_\nu}{|x|^2} + K_\nu(t) + O(|x|)), \quad (2.29)$$

with

$$K_1 = \frac{1}{2L_1}\left(\frac{\theta_1'(\hat{t}_-)}{\theta_1(\hat{t}_-} - \frac{\theta_1'(\hat{t}_+)}{\theta_1(\hat{t}_+)}\right) - iet_1, \quad K_2 = \frac{i}{2L_1}\left(\frac{\theta_1'(\hat{t}_-)}{\theta_1(\hat{t}_-)} + \frac{\theta_1'(\hat{t}_+)}{\theta_1(\hat{t}_+)}\right).$$

3 APPLICATIONS

We shall illustrate the special physical features of the gSM by calculating: (1) the correlation function of the field strength, (2) the VEV of the Wilson loop for the determination of the screening of the potential with inclusion of the finite volume effects, (3) the correlation functions of currents and densities, (4) the four fermion vacuum condensate. The VEV of the fermion fields exhibit the mechanism of chiral symmetry breaking by an anomaly. These examples are also particularly suited for a comparison with lattice approximations.

3.1. In the calculation of **the correlation function of the field strength** $F_{12}(x) = -\Box b(x)$ according to the general path integral formula Eq. (1.5), only the trivial sector $\mathcal{CH}^{(0)}$ contributes. According to the remarks above several factors like the regularization terms cancel. We get simply

$$\begin{aligned}
< F_{12}(x)F_{12}(y) > &= \frac{1}{Z}\int \mathcal{D}[A]F_{12}(x)F_{12}(y)e^{\frac{1}{2}\Gamma^{(0)}_{reg}-\frac{1}{2}\int_T F^2_{12}(x)dx^{12}} \\
&= \frac{1}{Z_0}\int \mathcal{D}[b(x)]\Box b(x)\Box b(y)e^{-S'[b(x)]} = \Box_x\Box_y G(x-y) \\
&= \frac{1}{m^2}\Box_x\Box_y(G_0(x-y)-G_m(x-y)) \\
&= -m^2 G_m(x-y)+\delta^{(\prime)}(x-y). \qquad (3.1)
\end{aligned}$$

In the last line we used: $-\Box G_0(x-y) = (-\Box + m^2)G_m(x-y) = \delta^{(\prime)}(x-y)$. Thus we get in the large volume limit, Eq. (2.23):

$$< F_{12}(x)F_{12}(y) >= -\frac{e^2}{\pi^2}K_0(m|x-y|) + \dots \quad . \qquad (3.2)$$

Thus we have the well-known result that in the Schwinger Model the effect of the virtual fermion pairs described by Γ_{reg} is a mass term for the free gauge field quanta. m^2 defers from the result of the conventional SM, $m_s^2 = e^2/\pi$ by a factor 2. This is a consequence of the isospin degree of freedom of the gSM. The deviation of $G_m(x-y)$ from $-1/2\pi K_0(m|x-y|)$, Eq. (2.23), describes the finite volume effect.

3.2. Now we determine the **gauge invariant static potential** as defined by the VEV of the Wilson loop $W(\mathcal{C})$. By arguments similar to those above, we get an expression for $W(\mathcal{C})$ which can be evaluated by Gaussean integration:

$$W(\mathcal{C}) = \frac{1}{Z_0}\int \mathcal{D}[b(x)]e^{-ie\int_{\mathcal{C}} ds^\mu A_\mu(s)}e^{-S'[b(x)]} = \exp\left(-\frac{e^2}{2}\int_{\mathcal{C}} ds^\mu \int_{\mathcal{C}} dt^\nu S_{\mu\nu}(s-t)\right). \qquad (3.3)$$

Here $S_{\mu\nu}(x-y))$ denotes the 2-point function of the gauge potential in the special gauge $A_\mu(x) = -\epsilon^\nu_\mu \partial_\nu b(x)$. From Eq. (2.20) follows

$$S_{\mu\nu}(x-y) = (\partial_\mu\partial_\nu - \Box\delta_{\mu\nu})G(x-y) = \partial_\mu\partial_\nu G(x-y) - G_m(x-y)\delta_{\mu\nu}. \qquad (3.4)$$

$\mathcal{C}$ is a rectangular path with side length R and T. The formula for the potential $V(R) = -\lim_{T\to\infty}\frac{1}{T}\log W(\mathcal{C})$ leads to the result

$$\begin{aligned}
V(R) &= -\frac{e^2}{T}\int_{-T/2}^{T/2}dt'\int_{-T/2}^{T/2}dt''\{G_m((t'-t'',R)) - G_m(t'-t'',0))\} \\
&= \frac{e^2}{2m}(1-e^{-mR}) - \frac{e^2(\cosh mR - 1)}{m(e^{mL_2}-1)}. \qquad (3.5)
\end{aligned}$$

The well-known fact that the 1+1-dimensional propagator of a massive particle leads to a 1 dim. Yukawa potential follows immediately from the integration of the integral representation of $\frac{1}{2\pi}K_0(m|x-y|)$ in Eq. (2.23). The second term in Eq. (3.5) describes the finite volume effect in the calculation of the potential. Physically the potential $V(R)$ shows the screening of the linear rising 1-dim. Coulomb potential by virtual fermion pair creations [15].

3.3. In the following we consider VEV of fermion fields. In particular we shall calculate the 2-point functions of **bilinear composite fields:** $M(x) = \bar{\Phi}(x)\Phi(x)$. In 'physical' terms these are the iso-scalar current $j^0_\mu(x)$, the iso-vector current $j^k_\mu(x)$, $k = 1,2,3$, the iso-scalar, and iso-vector, scalar densities $s^0(x)$, $s^k(x)$, and pseudo-scalar densities $s^0_5(x)$, $s^k_5(x)$ defined in the following way:

$$\begin{array}{ll} j^0_\mu(x) = i\bar{\phi}(x)\gamma_\mu\phi(x), & j^k_\mu(x) = i\bar{\phi}(x)\tau^k\gamma_\mu\phi(x), \\ s^0(x) = \bar{\phi}(x)\phi(x), & s^k(x) = \bar{\phi}(x)\tau^k\phi(x), \\ s^0_5(x) = -i\bar{\phi}(x)\gamma_5\phi(x), & s^k_5(x) = -i\bar{\phi}(x)\tau^k\gamma_5\phi(x). \end{array} \tag{3.6}$$

$\tau^0 = 1$, τ^k are the conventional 2-dim. iso-spin Pauli matrices. In 2-dimensions the axial currents $j^a_{5,\mu}(x) = i\bar{\phi}(x)\tau^a\gamma_\mu\gamma_5\phi(x)$, $a = 0,..,3$ are dual to the vector currents: $j^a_{\mu;5}(x) = \epsilon^\nu_\mu j^a_\nu(x)$. These expressions are chosen in such a way that their Euclidean 2-point functions are the analytic continuations of the VEV of the products of hermitean observables. For the later discussion of our main result, it is of interest to consider the transformation properties under the $U(2) \times U(2)$- symmetry of the classical model. The infinitesimal generators of $U(2) \times U(2)$, i.e. the chiral charges $Q^a_\pm$, (or $Q^a = Q^a_+ + Q^a_-$, $Q^a_5 = Q^a_+ - Q^a_-$) transform the operators of the currents and densities in the following way:
Iso-vector currents:

$$[Q^0, j^k_\mu] = [Q^0_5, j^k_\mu] = 0, \quad [Q^r, j^k_\mu] = +i\,\epsilon^{rkl}\,j^l_\mu, \quad [Q^r, j^k_{5,\mu}] = +i\,\epsilon^{rkl}\,j^l_{5,\mu},$$

$$[Q^r_5, j^k_\mu] = +i\,\epsilon^{rkl}\,j^l_{5,\mu}, \quad [Q^r_5, j^k_{5,\mu}] = +i\,\epsilon^{rkl}\,j^l_\mu, \qquad r,k,l = 1,2,3, \tag{3.7}$$

iso-scalar currents:

$$[Q^a, j^0_\mu(x)] = 0, \quad [Q^a_5, j^0_\mu(x)] = 0, \tag{3.8}$$

scalar and pseudoscalar densities ($s^a_0 = s^a$):

$$[Q^0, s^a_{0,5}] = 0, \quad [Q^0_5, s^a_0] = -i\,s^a_5, \quad [Q^0_5, s^a_5] = +i\,s^a_0, \quad a = 0,1,2,3,$$

$$[Q^r, s^0_{0,5}] = 0, \quad [Q^r, s^k_{0,5}] = +i\,\epsilon^{rkl}\,s^l_{0,5}, \quad [Q^r_5, s^0_0] = -i\,s^r_5, \quad [Q^r_5, s^0_5] = +i\,s^r_0,$$

$$[Q^r_5, s^k_0] = -i\,s^0_5\delta^{rk} \quad [Q^r_5, s^k_5] = +i\,s^0_0\delta^{r,k}. \tag{3.9}$$

We may consider the extension of this group by space reflections π: $(U(2) \times U(2))_\pi$ defined by the relation $\pi Q^a_\pm \pi = Q^a_\mp$, $\pi^2 = 1$. The currents and densities are transformed in the usual manner. It follows from the above equations, that the iso-vector current components, iso-scalar current components, and the scalar and pseudo-scalar densities span irreducible representations of this group extended by (Lorentz-) rotations of dimensions 6, 2, 8, respectively. Because of the quantum mechanical anomaly Q_5 is not conserved. This restricts the symmetry group $(U(2) \times U(2))_\pi$ to $(SU(2) \times SU(2) \times U(1))_\pi$. Under this restriction the irreducible representations of the currents remain irreducible. However, the 8-dim. representation of the densities split into two 4-dim. irreducible representations. The infinitesimal transformations $\delta s^a_{0,5} = -i[Q^a_\pm s^a_{0,5}]$ etc. can be analytically continued to the Euclidean fields. They leave the Euclidean Green's function invariant.

For applications for which the DK formulation is preferred, e.g. the study of lattice approximation , we give a transcription of the different $M(x)$:

$$\begin{array}{ll}\bar\phi\phi = (\bar\Phi,\Phi), & \bar\phi\gamma^5\phi = -i(\bar\Phi, dx^{12}\vee\Phi),\\ \bar\phi\tau^i\phi = (\bar\Phi,\Phi\vee dx^i), & \bar\phi\tau^3\phi = -i(\bar\Phi,\Phi\vee dx^{12}),\\ \bar\phi\gamma^5\tau^i\phi = -i(\bar\Phi, dx^{12}\vee\Phi\vee dx^i), & \bar\phi\gamma^5\tau^3\phi = -(\bar\Phi, dx^{12}\vee\Phi),\\ \bar\phi\gamma^\mu(\tau^i)^T\phi = (\bar\Phi, dx^\mu\vee\Phi\vee dx^i), & \bar\phi\gamma^\mu\tau^3\phi = -i(\bar\Phi, dx^\mu\vee\Phi\vee dx^{12}),\\ \bar\phi\gamma^\mu\phi = (\Phi, dx^\mu\vee\Phi). & \end{array} \quad (3.10)$$

The isospin matrices τ^a act on the isospin index (b). (Φ,Φ') denotes the scalar product between DK forms which we have introduced in the Introduction to Part I.

3.4. Calculating correlations $< M(x)M(y) >=< \bar\phi(x)\Gamma\tau\phi(x)\cdot\bar\phi(y)\Gamma'\tau'\phi(y) >$ with help of the fermion integrated path integral formula, Eq. (1.5), we get contributions from the gauge sectors $\mathcal{CH}^k$, $k = 0,\pm1$. The contribution from the trivial sector $k = 0$ become formally:

$$\begin{array}{lr}< M(x)M(y) >_0= & \\ -\frac{1}{Z}\int\mathcal{D}[A(x)]e^{-S'[b(x)]+\frac{1}{2}\tilde\Gamma(t)}\mathrm{Tr}(\Gamma\tau\mathcal{S}(x,y;A)\Gamma'\tau'\mathcal{S}(y,x;A)) & (c)\\ +\frac{1}{Z}\int\mathcal{D}[A(x)]e^{-S'[b(x)]+\frac{1}{2}\tilde\Gamma(t)}\mathrm{Tr}(\Gamma\tau\mathcal{S}(x,x';A)|_{x=x'})\mathrm{Tr}(\Gamma'\tau'\mathcal{S}(y,y';A)|_{y=y'}) & (d).\end{array} \quad (3.11)$$

In the topological sector $\mathcal{CH}^{(1)}$ we have two zero modes which contribute to $< M(x)M(y) >$. Their Dirac components are (I,Eq.(22,26))

$$\begin{aligned} {}^{odd}\phi^{(b)}_a(x) &= \mathcal{F}(x)\delta^1_a\delta^b_2, \quad {}^{ev}\phi^{(b)}_a(x) = \mathcal{F}(x)\delta^1_a\delta^b_1,\\ \mathcal{F}(x) &= \frac{1}{L_1}\left(\frac{2}{|\tau|}\right)^{1/4}e^{ie(a(x)+ib(x)+\frac{1}{2}t^\mu x_\mu)}e^{\frac{\pi}{2|\tau|}(z'^2-|z'|^2)}\theta_3(z'), \quad z' = z+\hat t_-. \end{aligned} \quad (3.12)$$

Inserted in Eq. (1.5) their contribution become

$$< \bar\phi^{(b_1')}_{a_1'}(x)\phi^{(b_1)}_{a_1}(x)\bar\phi^{(b_2')}_{a_2'}(y)\phi^{(b_2)}_{a_2}(y) >= \frac{1}{Z}\int\mathcal{D}[b]e^{-S[b]}e^{-2e(b(x)+b(y))} \quad (3.13)$$

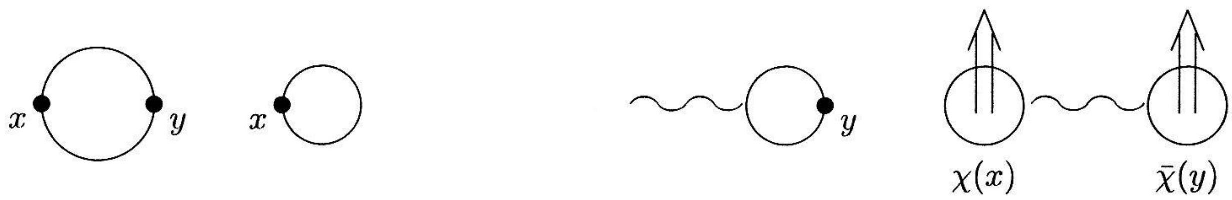
$$\times\int dte^{-\frac{2\pi}{|\tau|}(x_2'^2+y_2'^2)}|\theta_3(z_x')|^2|\theta_3(z_y')|^2\frac{1}{2}(1+\gamma_5)_{a_1a_1'}\frac{1}{2}(1+\gamma_5)_{a_2a_2'}\epsilon^{b_1b_2}\epsilon^{b_1'b_2'} \quad (zm).$$

Similarly we get a contribution from the topological sector $\mathcal{CH}^{(-1)}$

$$< \bar\phi^{(b_1')}_{a_1'}(x)\phi^{(b_1)}_{a_1}(x)\bar\phi^{(b_2')}_{a_2'}(y)\phi^{(b_2)}_{a_2}(y) >= \frac{1}{Z}\int\mathcal{D}[b]e^{-S[b]}e^{-2e(b(x)+b(y))} \quad (3.14)$$

$$\times\int dte^{-\frac{2\pi}{|\tau|}(x_2'^2+y_2'^2)}|\theta_3(z_x')|^2|\theta_3(z_y')|^2\frac{1}{2}(1-\gamma_5)_{a_1a_1'}\frac{1}{2}(1-\gamma_5)_{a_2a_2'}\epsilon^{b_1b_2}\epsilon^{b_1'b_2'} \quad (zm).$$

We call the contribution in Eq. (3.11c) 'connected' (c), the contribution (3.11d) 'discon-



c d zm

Figure 1: Contributions to correlation functions of fermion densities, c: connected, d: disconnected, zm: zeromode contributions.

nected' (d), and the parts Eqs. (3.13),(3.14) 'zero mode contributions' (zm). Fig. 1 symbolizes these different parts. The full line represents fermion propagation in the gauge field background. The unsystematic use of wavy lines reminds gauge field integration.

Using Eq. (2.24) for $\mathcal{S}(x,y;A)$, the evaluation of the traces with respect to the γ- and τ-matrices leads to the following result:

- the iso-vector current gets only contributions from the connected part,
- the iso-scalar current gets contributions from the connected and disconnected part,
- the densities get contributions from the connected part and from the zero modes.

The definition of currents and densities by a product of fermion fields needs some care. We apply a gauge invariant point splitting procedure [12, 16]. A complete analysis shows that this procedure is only relevant in the case of the iso-scalar current, as it is indicated by the appearence of the singular fermion propagator at $x = y$ in the disconnected part. More details are discussed together with the correlation function of this current.

3.5. The evaluation of the **connected part** is 'straightforward'. It allows for a complete discussion of the **iso-vector current correlation function**. We mention the main steps of the calculation. Inserting the expression for $\mathcal{S}(x,y;A)$, Eq. (2.24), into Eq. (3.11) we get

$$
\begin{aligned}
< j^a_\mu(x) j^b_\nu(y) >_c &= \frac{1}{Z}\int \mathcal{D}[A(x)] e^{-S'[b(x)]+\frac{1}{2}\bar{\Gamma}(t)} \mathcal{S}_{t,\rho}(x-y)\mathcal{S}_{t,\sigma}(y-x) \\
&\quad \times \mathrm{Tr}\left(\gamma_\mu \tau^a e^{ie\alpha(x)}\gamma^\rho e^{-ie\alpha'(y)}\right) \mathrm{Tr}\left(\gamma_\nu \tau^b e^{ie\alpha(y)}\gamma^\sigma e^{-ie\alpha'(x)}\right) \qquad (3.15)
\end{aligned}
$$

$$
= \frac{4\delta^{ab}}{Z_t}\int\int_0^{Ti} dt_1 dt_2 e^{\frac{1}{2}\bar{\Gamma}(t)} \left(\begin{array}{c} \mathcal{S}_{t,\mu}(x-y)\mathcal{S}_{t,\nu}(y-x) + (\mu \leftrightarrow \nu) \\ -\delta_{\mu\nu}\mathcal{S}_{t,\rho}(x-y)\mathcal{S}_{t,\rho}(y-x)\} \end{array} \right). \qquad (3.16)
$$

Because of $\gamma_\mu \exp(i\alpha(x)) = \exp(i\alpha'(x)\gamma_\mu$, the dependence on the gauge field drops out. The traces with respect to the iso-spin index leads to the factor δ^{ab}, $a, b = 0, 1, 2, 3$. The evaluation of the second line gets simplified by considering the 'light cone components' $j^a_\pm(x) = j^a_1(x) \mp i\, j^a_2(x),\ \ \mathcal{S}_\pm(x;A) = \mathcal{S}_1(x;A) \mp i\,\mathcal{S}_2(x;A)$ etc.

$$< j^a_\pm(x) j^b_\pm(0) >= 8\delta^{ab} < \mathcal{S}_{t,\pm}(x)\mathcal{S}_{t,\pm}(-x) >_t, \ \ < j^a_+(x) j^b_-(0) >= 0. \tag{3.17}$$

Now we may apply an 'addition theorem' of the θ-functions:

$$\theta_1(z+w)\theta_1(z-w) = \frac{1}{\theta_4^2(0)}(\theta_1^2(z)\theta_4^2(w) - \theta_4^2(z)\theta_1^2(w)), \tag{3.18}$$

which leads to a simple result in terms of the Weierstrass functions $\wp(z)$:

$$< j^a_+(x) j^b_+(0) >=< j^a_-(x) j^b_-(0) >^*= \frac{-2}{\pi^2 L_1^2}\{\wp(z) - < \wp(\hat{t}_-) >_t\}\delta^{ab}. \tag{3.19}$$

The toron dependence appears in an additive term. The averaging over the toron configurations results in:

$$\begin{aligned}
< \wp(\hat{t}_-) >_t &= \frac{1}{Z_t}\int\!\!\int_0^{T_i} dt_1 dt_2 e^{\frac{1}{2}\tilde{\Gamma}(t)}\wp(\hat{t}_-) \qquad (T_i = 2\pi/(eL_i)) \\
&= 2\theta_3^{-2}(0)|\tau|^{-\frac{1}{2}}\int_0^1 d\hat{t}_1 \int_0^{|\tau|} d\hat{t}_2\, \wp(\hat{t}_-)|e^{-\frac{\pi}{|\tau|}\hat{t}_2^2}\theta_1(\hat{t}_-|\tau)|^4 \\
&= \frac{\pi^2}{3}(\theta_2^4(0) - \theta_4^4(0)) \\
Z_t &= \int\!\!\int_0^{T_i} dt_1 dt_2 e^{\frac{1}{2}\tilde{\Gamma}(t)} = \frac{2\pi^2}{e^2 L_1 L_2}\frac{\theta_3^2(0|\tau)}{\eta^4(\tau)|\tau|^{1/2}},
\end{aligned} \tag{3.20}$$

(compare with Eq. (A.5). It is interesting to bring this result into an other form. Using the expression of the Weierstrass function:

$$\wp(z) = -\frac{d^2}{dz^2}\log\theta_1(z) + \frac{1}{3}\frac{\theta_1'''(0)}{\theta_1'(0)}$$

and returning to Cartesian components, we get with $z = \frac{1}{L_1}[(x_1 - y_1) + i(x_2 - y_2)]$

$$\begin{aligned}
< j^a_\mu(x) j^b_\nu(y) >_c &= \delta_{ab}\left(-\frac{1}{\pi^2}\epsilon_{\mu\rho}\epsilon_{\nu\sigma}\partial_\rho\partial_\sigma \log\frac{|\theta_1(z)|}{|\theta_1'(0)|} + \frac{1}{\pi^2 L_1 L_2}\mathcal{C}_{\mu\nu}(\tau)\right) \\
&= \delta_{ab}\left(\frac{2}{\pi}\epsilon_{\mu\rho}\epsilon_{\nu\sigma}\partial_\rho\partial_\sigma\, G_0(x-y) + \frac{1}{\pi^2 L_1 L_2}R_{\mu\nu}(\tau)\right).
\end{aligned} \tag{3.21}$$

The additive constants are determined by the integral Eq. (3.20), and by the expression Eq. (2.21) for the massless Green's function on $\mathcal{T}$:

$$\begin{aligned}
\mathcal{C}_{11} &= -\mathcal{C}_{22} = R_{11} - 2\pi = -R_{22} = \frac{|\tau|}{3}\left(-\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} + \pi^2(\theta_2^4(0|\tau) - \theta_4^4(0|\tau))\right) \\
\mathcal{C}_{\mu\nu} &= R_{\mu\nu} = 0 \ \text{ for } \mu \neq \nu.
\end{aligned} \tag{3.22}$$

For the iso-vector current this is our final result to which we want to add some comments:

- Since $G_0(x-y)$ is the 'massless' propagator, only massless one-particle intermediate states of isospin 1 contribute to the iso-vector two point function.
- The bosonization procedure [17] describes the currents by derivatives of scalar fields. This leads to formulas like Eq. (3.21). Thus this expression represents a generalization of a bosonization formula to the torus.
- Because the axial vector current is dual to the vector current: $j^k_{5,\mu}(x) = \epsilon^\nu_\mu j^k_\nu(x)$, it follows from $-\Box G_0(x-y) = \delta^{(\prime)}(x-y)$ that the iso-vector current correlations, Eq. (3.21), satisfy the Ward identities for conserved vector and axial vector currents.
- Like the whole correlation function, also $R(\tau)$ is invariant under the exchange of the (1) and (2) direction: $R_{11}(\tau) = R_{22}(-1/\tau)$
- Before t-averaging, the correlation function, Eq. (3.19) is singular at $t = 0$. This 'infrared' singularity is compensated in the integral Eq. (3.20) by a zero of the toron Boltzmann factor $\exp\tilde{\Gamma}(t)/2$.

3.6. In order to determine the correlation function of the **iso-scalar current**, we have to add to the connected part, Eq. (3.21), the contribution from the disconnected part. For this we have to base our calculation on a more precise definition of the current as a composite operator [16, 18]. As usual in the discussion of the Schwinger model, we choose a definition of $j^a_\mu(x) = i : \bar\phi(x)\tau^a\gamma_\mu\phi(x) :$ based on the gauge invariant short distance expansion (sde):

$$\bar\phi(x+\zeta)\gamma_\mu\tau^a e^{ie\int_{x-\zeta}^{x+\zeta} A_\alpha(y)dy^\alpha}\phi(x-\zeta) = G^a_\mu(\zeta)\cdot\mathbf{1} + G^a_0(\zeta) : \bar\phi(x)\tau^a\gamma_\mu\phi(x) : . \tag{3.23}$$

The coefficient $G^a_\mu(\zeta)$ can be determined by calculations of the VEV like above:

$$< \bar\phi(x+\zeta)\gamma_\mu\tau^0 e^{ie\int_{x-\zeta}^{x+\zeta} A_\alpha(y)dy^\alpha}\phi(x-\zeta) >= \frac{2}{\pi}\left(\frac{\zeta_\mu}{2|\zeta|^2} - < K_\mu(t) >_t\right) + o(|\zeta|). \tag{3.24}$$

$K_\mu(t)$ is the toron part in the sde of the fermion propagator $S_t(x)$, Eq. (2.29). By comparison with Eq. (2.12) we find the familiar relation $K_\mu(t) = \frac{i\pi}{4eL_1L_2}\frac{\partial}{\partial t_\mu}\tilde\Gamma(t)$. It has the consequence $< K_\mu(t) >_t = 0$.

Now we evaluate the contribution of the disconnected part to the point split current up to order $o(|\zeta|)$ by a straightforward calculation:

$$\left\langle \bar\phi(x+\zeta)\gamma_\mu e^{ie\int_{x-\zeta}^{x+\zeta} A_\alpha(\bar x)d\bar x^\alpha}\phi(x-\zeta)\bar\phi(y+\zeta')\gamma_\mu e^{ie\int_{y-\zeta'}^{y+\zeta'} A_\alpha(\bar y)d\bar y^\alpha}\phi(y-\zeta')\right\rangle_d$$

$$= \frac{1}{Z}\int \mathcal{D}[A(x)]e^{-S'[b(x)]+\frac{1}{2}\tilde\Gamma(t)}\{S_{t,\alpha}(-2\zeta)S_{t,\beta}(-2\zeta')e^{ie\int_{x-\zeta}^{x+\zeta} A_\rho(\bar x)d\bar x^\rho + ie\int_{y-\zeta'}^{y+\zeta'} A_\rho(\bar y)d\bar y^\rho}\}$$

$$\times 4\mathrm{Tr}[\gamma^\mu\gamma^\alpha e^{ie(\alpha'(x-\zeta)-\alpha(x+\zeta))}]\mathrm{Tr}[\gamma^\nu\gamma^\beta e^{ie(\alpha'(y-\zeta')-\alpha(y+\zeta'))}]$$

$$= \frac{1}{Z_t}\iint_0^{T_i} dt_1 dt_2 e^{\frac{1}{2}\tilde\Gamma(t)}$$

$$\times\left\{\frac{4}{\pi^2}\left(\frac{\zeta_\mu}{2|\zeta|^2}-K_\mu(t)\right)\left(\frac{\zeta'_\nu}{2|\zeta'|^2}-K_\nu(t)\right)+\frac{4e^2}{\pi^2}\epsilon_{\mu\rho}\epsilon_{\nu\sigma}\partial_\rho\partial_\sigma G(x-y)\right\}$$
$$=\frac{1}{\pi^2}\frac{\zeta_\mu}{|\zeta|^2}\frac{\zeta'_\nu}{|\zeta'|^2}+\frac{4e^2}{\pi^2}\epsilon_{\mu\rho}\epsilon_{\nu\sigma}\partial_\rho\partial_\sigma G(x-y)+\frac{4}{\pi^2}<K_\mu(t)K_\nu(t)>_t . \tag{3.25}$$

A calculation of $< K_\mu(t)K_\nu(t) >_t$ similar to Eq. (3.20), applying the manipulations of θ-functions described in the Appendix, leads to the following result:

$$< K_\mu(t)K_\nu(t) >_t=\frac{1}{4L_1L_2}R_{\mu\nu}. \tag{3.26}$$

$R_{\mu\nu}$ is defined in Eq. (3.22). Adding the connected and disconnected part, we get for the correlation function of the iso-scalar current : $j^0_\mu(x)=i:\bar\phi(x)\gamma_\mu\phi(x):$

$$\begin{aligned}< j^0_\mu(x)j^0_\nu(y) > &= \frac{2}{\pi}\epsilon_{\mu\rho}\epsilon_{\nu\sigma}\partial_\rho\partial_\sigma G_0(x-y) - \frac{4e^2}{\pi^2}\epsilon_{\mu\rho}\epsilon_{\nu\sigma}\partial_\rho\partial_\sigma G(x-y) \\ &\quad +\frac{1}{\pi^2L_1L_2}R_{\mu\nu}(\tau)-\frac{4}{\pi^2}< K_\mu(t)K_\nu(t) >_t \\ &= \frac{2}{\pi}\,\epsilon_{\mu\rho}\epsilon_{\nu\sigma}\partial_\rho\partial_\sigma G_m(x-y).\end{aligned} \tag{3.27}$$

Here we add the following remarks:

- In the above expression the first term comes from the connected part, (compare Eq. (3.21)), the second from the disconnected part. It follows from Eq. (2.20) that the connected part is compensated by the zero mass contribution of the disconnected part. The constant terms like $R_{\mu\nu}(\tau)$ get compensated similarly. Thus only a contribution from a free massive particle ($m^2=2e^2/\pi$) remains, as expressed by the third line.
- The above result is a bosonization formula on the torus with a massive pseudo-scalar field.
- The vector current is conserved. Because of $(-\Box+m^2)G_m(x-y)=\delta^{(\prime)}(x-y)$ the axial current becomes anomalous.

3.7. The correlation functions of the **scalar and pseudo-scalar densities** get contributions from the connected part and from the zero modes. We consider first the connected part. Inserting the expression for the Fermion propagator, Eq. (2.24), and performing the traces results in

$$< s^a(x)s^b(y) >_c=< s^a_5(x)s^b_5(y) >_c$$
$$=-4\delta^{ab}< \mathcal{S}_{t,\mu}(x-y)\mathcal{S}_{t,\mu}(y-x) >_t\frac{1}{Z}\int\mathcal{D}[b(x')]e^{-S'[b(x')]-2e(b(x)+b(y))}. \tag{3.28}$$

The functional integral is calculated by Gaussian integration, Eq. (2.18), leading to a factor

$$e^{4e^2(G(0)-G(x-y))}=2\pi\eta^2(\tau)\left|\frac{\theta_1(z)}{\theta'_1(0)}\right|e^{4e^2G(0)}e^{-\frac{\pi(x-y)_2^2}{L_1L_2}}e^{2\pi G_m(x-y)}. \tag{3.29}$$

It remains the evaluation of the toron average $< S_{t,\mu}(x-y)S_{t,\mu}(y-x) >_t$, an integral of type Eq. (3.20). After involved calculations with θ-functions, described generally in the Appendix we get the result:

$$< S_{t,\mu}(x)S_{t,\mu}(-x) >_t = -\frac{e^{\pi x_2^2/L_1L_2}|\theta_1'(0)|^2}{(2\pi L_1)^2|\theta_1(z)|^2}\left\{\frac{|\theta_1(z/2)|^4+|\theta_3(z/2)|^4}{|\theta_3(0)|^4}\right\}. \qquad (3.30)$$

Now we discuss the zero mode contribution described in Eqs. (3.13,3.14). For the traces we get for the sum of the contributions from the sectors $k=\pm1$ for scalar and pseudo scalar:

$$\frac{1}{4}(\mathrm{Tr}(1+\gamma_5)(1+\gamma_5)+\mathrm{Tr}(1-\gamma_5)(1-\gamma_5))\epsilon_{c_1c_2}\epsilon_{\bar{c}_1\bar{c}_2}T^a_{\bar{b}_1b_1}T^b_{\bar{b}_2b_2} = \begin{cases} 4\delta^{ab} & \text{for } a=0 \\ -4\delta^{ab} & \text{for } a\neq 0 \end{cases} \qquad (3.31)$$

The averaging over the toron field requires the calculation of the following integral

$$\begin{aligned}&\tfrac{-1}{Z_t}\textstyle\iint_0^{Ti} dt_1dt_2 e^{-\frac{2\pi}{|\tau|}(x_2'^2+y_2'^2)}|\theta_3(z_x')|^2|\theta_3(z_y')|^2 \\ &= \tfrac{2|\tau|^{1/2}\pi^2}{e^2L_2^2}\cdot\tfrac{|\theta_1(z/2)|^4+|\theta_3(z/2)|^4}{|\theta_3(0)|^4}e^{-\frac{\pi}{L_1L_2}(x_2-y_2)^2}\ .\end{aligned} \qquad (3.32)$$

The averaging over the $b(x)$-configurations may be performed as in Eq. (2.18) by Gaussian integration.

Adding up the different pieces from the connected part and the zero mode contributions, we get the following final result:

$$\begin{aligned}< s^a_\pm(x)s^b_\pm(y) > &= Const.\ \delta_{ab}\frac{|\theta_1(\frac{z}{2}|\tau)|^4\ +\ |\theta_3(\frac{z}{2}|\tau)|^4}{|\theta_1(z|\tau)|} \\ &\times\ \left[\exp(2\pi\,\overline{G}_m(x-y))\ \pm\ \exp(-2\pi\,\overline{G}_m(x-y))\right]\ .\end{aligned} \qquad (3.33)$$

with

$$Const. = 2\frac{\eta^2(\tau)|\theta_1'(0)|}{\pi L_1^2|\theta_3(0)|^4}e^{4e^2G(0)-\pi^2/(e^2L_1L_2)}.$$

($s^i_+(x)=s^i(x)$, $s^i_-(x)=s^i_5(x)$ for $i=0$, and $s^i_+(x)=s^i_5(x)$, $s^i_-(x)=s^i(x)$ for $i=1,2,3$). The first term in the bracket is the contribution from the topologically trivial sector, the second from the $k=\pm1$ sectors. We see

- The $\pm$ sign splits the correlations of the 8 (pseudo) scalar densities in those of $s^a_+(x)$ and $s^b_-(x)$. This is the splitting of the $U_L(2)\times U_R(2)$ octet into two $U_V(1)\times SU_L(2)\times SU_R(2)$ quadruplets, which we have discussed in Sect. 3.3. Therefore the anomalous $U_A(1)$-symmetry breaking in the space of fermion densities is realized by the zero mode contributions from the different topological sectors.

- There are massive and massless intermediate particle states in the infinite volume limit, as expressed by the complicated space dependence of the correlation function of the scalar densities.

3.8. It is well-known that the anomaly of the conventional Schwinger model is related to a non-vanishing **fermion condensate** $\langle\bar{\phi}(x)\phi(x)\rangle$. On the torus it results from the zero mode contribution. It was calculated by I. Sachs and A. Wipf, [19] as:

$$\langle(\bar{\phi}(x)\phi(x)\rangle = \frac{\eta^2(\tau)}{L_1^2} e^{2e^2 G(0) - 2\pi^2/e^2 L_1 L_2}. \tag{3.34}$$

where $G(0)$ defined in Eq. (2.20) refers to the mass $m^2 = e^2/\pi$ of the conventional Schwinger model. They discussed several limiting cases of the size of the torus. In particular the case $L_1 \to \infty$ with L_2 varying from 0 to ∞. In the spirit of finite temperature field theory, one may interpret $1/L_2$ as temperature. Thus Eq. (3.34) describes the transition of the condensate between zero temperature and the high temperature region.

In the case of the geometric Schwinger Model the situation is slightly different. Isospin symmetry prohibits a non-vanishing simple condensate. The existence of two linearly independent zero modes in the lowest non-trivial topogical sectors allows only, according to Eq. (1.5), a zero mode contribution to a $(\bar{\phi}\phi)^2$ condensate. It follows immediately from the zero mode contributions, and the calculation in the Section above

$$\langle(\bar{\phi}(x)\phi(x))(\bar{\phi}(x)\phi(x))\rangle = 2\frac{\eta^4(\tau)}{L_1^2} e^{8e^2 G(0) - 2\pi^2/e^2 L_1 L_2}. \tag{3.35}$$

This is, up to a missing additional instanton weight factor $\exp(-2\pi/e^2 L_1 L_2)$ $= \exp(-S_f(C_\mu^{(1)}))$, and the different mass in $G(0)$, close to the square of the simple condensate Eq. (3.34), in particular it is equal to the square in the infinite volume limit.

4 CONCLUSION

Which are the interesting features of our treatment of the geometric Schwinger model on the torus? Most of these are related to the fact that on the torus one can separate in a simple manner the zero modes from the other degrees of freedom. In our treatment the role of the topological zero modes in the symmetry breaking by an anomaly of the Schwinger model became particularly transparent. The zero mode contribution causes the symmetry splitting in the $U(2) \times U(2)$ octet of the correlation functions of the densities. The dynamics of the zero modes of the gauge potential, the torons, is determined by a toron action $\tilde{\Gamma}(t)$. This is induced by the effect of the torons on the fermion fields. It controls infrared singularities. Toron averaging assures a translational invariant distribution of the symmetry breaking zero modes in the topological non-trivial sectors.

We believe that our studies further the intuition into some problems of QCD. The Schwinger model always provided an example for the screening of a confining static potential by fermion pairs [21]. Here we calculated the finite volume correction to this effect. The mass splitting between the massless iso-vector pseudo scalar pions and the massive iso-scalar η-particle, discussed as $U(1)$-problem in massless QCD, finds in the particle structure of the gSM a close analogy.

It is a challenge for lattice approximation to reproduce such intricate features of gauge theories with massless fermions. Of course, the gSM on the torus is particularly suited for such a study. The restriction to a 2-dim. finite volume together with super-renormalizability allows for numerical simulations to reproduce continuum physics in the scaling region. With a suitable hybrid MC method it was possible to reproduce [3, 4, 20]:

- the toron action $\tilde{\Gamma}(t)$, Eq.(2.12, 2.16);
- the static potential, clearly separating screening from the finite volume effect, Eq.(3.5);
- the mass spectrum inherent in the correlation functions of the field strength, Eq.(3.1), and of the iso-scalar and iso-vector currents, Eq.(3.27), and Eq.(3.21);
- the anomalous $U(2) \times U(2)$ symmetry breaking of the scalar-pseudo scalar densities, Eq.(3.33).

For this it was crucial that the systematic lattice approximation of geometric fermions allows for a control of the relation between the symmetry in the continuum and on the lattice [22].

It might be worthwhile to add some short remarks on the comparison between the geometric Schwinger model and the conventional Schwinger model on the torus [19]. Of course many expressions are very similar. We mention the explicit expression for the fermion propagator in a background field. The effective action differs by a factor 2 which implies a factor $\sqrt{2}$ in the mass of the iso-scalar particle. The appearence of an iso-spin multiplet of massless particles is a particularity of Schwinger models with internal degree of freedom [23]. We discussed the effect of the doubling of topological zero modes in the gSM on the appearence of condensates. The factor 2 in the toron action changes the character of the integrations over the torons considerably, and hence the dynamical role of the torons in these cases.

The Schwinger models seem to provide an infinite source of theoretical problems. There are quite a few interesting problems which we did not treat. We restricted the discussion to Green's functions with a few arguments. Contributions to higher Green's functions from topologically non-trivial sectors contain fermion propagators orthogonal to zero modes. These we have not yet studied. A systematic discussion of the different limits $L_1, L_2 \to \infty$ is still missing. It is known [6] that zero mode contributions are essential for the cluster properties of higher Green's function in this limit. Systematic non-abelian bosonization [24] might provide a starting point for an operator solution of the gSM. Some consequences of such an approach are found in our investigations. As a last interesting topic we want to mention the study of the relation of the Hamiltonean approach to the Schwinger model on the circle [25] to the limits of the Euclidean Schwinger model on the torus.

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APPENDIX

In this Appendix we give a selection of formulas on θ-functions on which we base our different explicit calculations in the text. For more details and proofs we refer to text books [14, 26, 27]. As already mentioned in Part I we use the
definition

$$\theta_{ab}(z|\tau) = \sum_{n=-\infty}^{+\infty} \exp[\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)], \tag{A.1}$$

$z = x_1 + ix_2$, $\tau = i|\tau|$; as usual the parameter τ is sometimes suppressed, ($a, b = 0, \frac{1}{2}$; in the notation of [14]: $\theta_1 = -\theta_{\frac{1}{2}\frac{1}{2}}$, $\theta_2 = \theta_{\frac{1}{2}0}$, $\theta_3 = \theta_{00}$, $\theta_4 = \theta_{0\frac{1}{2}}$). These functions are analytic and satisfy the
'periodicity conditions'

$$\theta_{a,b}(z + m\tau + n|\tau) = \exp[-2\pi i(mz + \frac{m^2\tau}{2} + bm - an)]\theta_{ab}(z|\tau). \tag{A.2}$$

It follows the strict periodicity of the 'density' $\rho_a(z) = |\theta_a(z|\tau)\exp(-\pi x_2^2/|\tau|)|$

$$\rho_a(z+1) = \rho_a(z+\tau) = \rho_a(z). \tag{A.3}$$

Using the definition Eq.(A.1) we get the
normalization type integrals

$$\int_0^1 dx_1 \int_0^{|\tau|} dx_2 \rho_a^2 = \frac{|\tau|^{1/2}}{\sqrt{2}}. \tag{A.4}$$

A similar type of integral

$$\int_0^1 dx_1 \int_0^{|\tau|} dx_2 \rho_a^4 = \frac{|\tau|^{1/2}}{2}\theta_3^2(0|\tau), \tag{A.5}$$

we may evaluate by the use of so-called
'addition theoremes' [26, 27] for $\theta_a(z+w)\theta_a(z-w)\theta_4^2(0)$. From the many different formulas we gave an example in Eq.(3.18). These allow to evaluate such 2 dim. integrals over the torus as products of simple intergrals of monomials of θ-functions.

For checking of the invariance under the exchange of the 1-2 axis of the torus the following formulas for **'transformations of the first order'** are relevant:

$$\theta_a(\frac{iz}{|\tau|}|\frac{i}{|\tau|}) = \iota|\tau|^{1/2}e^{\frac{\pi z^2}{|\tau|}}\theta_{a'}(z|\tau), \tag{A.6}$$

with $\iota = -i$ for $a = 1$, $\iota = 1$ for $a = 2, 3, 4$, and $1' = 1,\ 2' = 4,\ 3' = 3,\ 4' = 2$. Similarly we have for Dedekind's function $\eta(\tau)$, Eq.(2.11):

$$\eta(-\frac{1}{\tau}) = |\tau|^{1/2}\eta(\tau). \tag{A.7}$$

Landen's second order transformation formulas, for example [14]

$$\theta_a^2(z|\tau) = \theta_b(0|2\tau)\theta_3(2z|2\tau) \pm \theta_c(0|2\tau)\theta_2(2z|2\tau), \tag{A.8}$$

with the following combinations of signs and indices:
$(a; b, \pm, c) = (1; 2, -, 3),\ (2 : 2, +, 3),\ (3; 3, +, 2),\ (4 : 3, -, 2)$, allow to decrease the powers of θ-functions in integrals of the type:

$$\int_0^1 dx_1 \int_0^{|\tau|} dx_2 \theta_3^2(z|\tau)(\theta_2^2(z|\tau))^* = \frac{|\tau|^{1/2}}{2}\theta_2^2(0|\tau). \tag{A.9}$$

References

[1] H. Joos, Helv. Phys. Acta 63 (1990) 670.

[2] H. Joos, Nucl. Phys. (Proc. Suppl.) B17 (1990) 704.

[3] H. Dilger, PhD Thesis, University Hamburg, DESY 93-181; Chiral symmetry breaking on the lattice - a study in the two-flavor Schwinger model, to be published in Nucl. Phys. B

[4] H. Dilger, H. Joos, Nucl. Phys.B (Proc.Suppl.) 34 (1994) 195, DESY 93-144.

[5] G. 't Hooft, Phys. Rev. D14 (1976) 3432.
H. Leutwyler, Phys. Lett. 152B (1985) 78, 153B (1985) 65.
M. Lüscher, privat instruction, Ann. Phys.(N.Y.) 142 (1982) 359.

[6] C. Jayewardena, Helv. Phys. Acta 61 (1988) 636.

[7] P.B. Gilkey, 'Invariance Theory, The Heat Equation, And the Atiyah-Singer Index Theorem', Publish or Perish Inc. 1984.

[8] A. Weil, 'Elliptic Functions according to Eisenstein and Kronecker', Springer-Verlag 1976.

[9] S.W. Hawking, Comm. Math. Phys. 55 (1977) 133.
C. Itzykson, and J.B. Zuber, Nucl. Phys. B275 (1986) 580.

[10] A. Erdélyi (Director),Higher Transcedental Functions, Vol.3, Chapter 17, McGraw-Hill, 1953.

[11] J. Schwinger, Phys. Rev. 128 (1962) 2425.

[12] J.H. Lowenstein, and J.A. Swieca, Ann. Phys. (N.Y.) 68 (1971) 172

[13] M.F.Atiyah, W.K.Patodi, and I.M.Singer, Math. Proc. Camb. Philos. Soc., 799 (1976) 71.

[14] A. Erdélyi (Director), Higher Transcedental Functions, Vol.2, Chapter 13, McGraw-Hill, 1953.

[15] S. Coleman, Ann. Phys. (N.Y.), 101 (1976) 239.
H.J. Rothe, K.D. Rothe, and J.A.Swieca, Phys.Rev.D19 (1979) 3020.

[16] J.C. Collins, 'Renormalization', Cambridge University Press 1984.

[17] S. Coleman, Phys. Rev. D11 (1975) 2088.

[18] W. Zimmermann, Ann. Phys. (N.Y.) 77 (1972) 536,570.

[19] I. Sachs, A. Wipf, Helv. Phys. Acta 65 (1992) 652. see also S. Azakov, 'The Schwinger model on the torus', Proceedings of the Course and the Conference on Path Integration. Trieste 26 August-10 September 1991, Would Scientific,1993.

[20] H. Dilger, Phys. Lett. B 294 (1992) 263.

[21] H. Joos, and I. Montvay, Nucl. Phys. B 225 (1983) 565.

[22] H. Joos, M. Schäfer, Z.Phys. C-Particles and Fields, 34 (1987) 465.

[23] L.V. Belvedere, J.A. Swieca, K.D. Rothe, and B. Schroer, Nucl. Phys. B 153 (1979) 112.
Chr. Gattringer and E.Seiler, 'Functional approach to the N-flavour Schwinger Model', MPI-PHT-93-56.

[24] E. Witten, Comm. Math. Phys. 92 (1984) 455.

[25] N.S. Manton, Ann. Phys. (N.Y.) 159 (1985)220.
J.E. Hetrick and Y.Hosotany, Phys. Rev. D 38 (1988) 2621.
S. Iso and H. Murayama, Progr. Theor. Phys. 84 (1990) 142.

[26] D .Mumford, Tata Lectures on Theta, Birkhäuser Verlag, 1983.

[27] E.T. Whittaker, G.N. Watson, Modern Analysis, p.487, Cambridge University Press.