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B.R.S. Renormalization of Some On-Shell Closed Algebras of Symmetry Transformations: 2) N=2 and 4 Supersymmetric Non-Linear σ Models

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Abstract. We analyse with the algebraic, regularization independent, cohomological B.R.S. methods, the renormalizability of torsionless N=2 and N= 4 supersymmetric non-linear σ models built on Kähler spaces. Surprisingly enough with respect to the common wisdom, in the case of N=2 supersymmetry, we obtain an anomaly candidate, at least in the compact Kähler Ricci-flat case. If its coefficient does differ from zero, such anomaly would imply the breaking of global N=2 supersymmetry and the disruption of some schemes of superstring compactification as such non-linear σ models offer candidates for the superstring vacuum state.

In the compact homogeneous Kähler case, as expected, the anomaly candidate disappears.

The same phenomenon occurs when one enforces N=4 supersymmetry : in that case, we obtain the first rigorous proof of the expected all-order renormalizability -" in the space of metrics"- of the corresponding non-linear σ models.

1 Introduction

Supersymmetric non-linear σ models in two space time dimensions have been considered for many years to describe the vacuum state of superstrings [1],[2]. In particular Calabi-Yau spaces, *i.e.* 6 dimensional compact Kähler Ricci-flat Riemanian manifolds [3], appear as good candidates in the compactification of the 10 dimensional superstring to 4 dimensional flat Minkowski space; indeed, the conformal invariance of the 2.d, N = 2 supersymmetric non-linear σ model (the fields of which are coordinates on this compact manifold) is expected to hold to all orders of perturbation theory [4].

However explicit calculations to 4 or 5 loops [5] and, afterwards, general arguments [6] show that the β functions might be different from zero. But, as argued in my recent review [7], at least two problems obscure these analyses : firstly, the fact that the quantum theory is not sufficiently defined by the Kähler Ricci-flatness requirement ; secondly, the use of "dimensional reduction" [8] or of harmonic superspace [9] [1] in actual explicit calculations and general arguments. Then, we prefer to analyse these models using the B.R.S., algebraic, regularization free cohomological methods.

Moreover, the quantization of extended supersymmetry raises the difficulty of an "on-shell" formalism. Indeed, if one leaves aside harmonic superspace where firm rules for quantization [2] are not at hand, contrary to ordinary superspace [11], one has to deal with (super)symmetry transformations that are non-linear and close only on-shell. This problem was addressed in ref. [12] by O. Piguet and K. Sibold for the Wess-Zumino model as a "toy-model" and, in a still uncomplete way, by P. Breitenlohner and D. Maison [13] for supersymmetric Yang-Mills in the Wess-Zumino gauge ; in the first paper of this series [14], hereafter referred to as (I), we analysed the d=2, N=1 supersymmetric non-linear σ model without auxiliary fields.

In the second paper of this series, we address the question of the all-orders renormalizability of extended supersymmetric (N = 2, 4) non-linear σ models in two space time dimensions. Of course, we are only interested here in the renormalization of the supersymmetry transformations : as discussed by Friedan [15], the action of a non-linear σ model may be identified with a distance on a Riemannian manifold $\mathcal{M}$, the metric depending *a priori* on an infinite number of parameters. One then speaks of "renormalizability in the space of metrics" or "*à la* Friedan". When there exist extra isometries, for example in the case of the non-linear σ models on coset spaces (homogeneous manifolds), the number of such physical parameters becomes finite and we have proved the U.V. renormalizability of these isometries in the purely bosonic case in [16], as well as in the N=1 supersymmetric extension in (I). The present work gives the necessary extended-supersymmetric generalisations. On the other hand, in the generalised non-linear σ models *à la* Friedan, our aim is the proof that no extra difficulty occurs in their supersymmetric extension.

As in (I), we are here interested only in the ultraviolet renormalizability of the d=2 extended supersymmetric non-linear σ models : of course, one has also to deal with infrared divergences. This would require the addition of an infrared regulator which of course breaks the symmetries, but only softly, and then does not affect our results on "hard" divergences

[1] The regularization through dimensional reduction suffers from algebraic unconsistencies and the quantization in harmonic superspace does not rely on firm basis, due to the presence of non-local singularities (in the harmonic superspace)[10].

[2] *i.e.* a subtraction algorithm insuring the locality of the counterterms [10].

and possible anomalies for the supersymmetry in 2 dimensions.

We shall use N=1 superfields, which is allowed by the general superspace quantization methods established by Piguet and Rouet who in particular demonstrated the Quantum Action Principle in that context [11], and the very results of (I), proving that N=1 supersymmetry is all-orders renormalizable. The classical theory was defined in (I), so here we only recall in subsection 2.1 the results needed for the following. Due to the non-linearity of the supersymmetry transformations in a general field parametrisation (*i.e.* coordinate system on the manifold), we shall use a grading (according to the spectral sequences method [17]) in the number of fields, ghosts and their derivatives. As a matter of fact, we find it convenient to use two successive gradings, one in the number of extra supersymmetries, the second one with respect to the number of fields. The "filtrations" , as well as the lowest order nilpotent Slavnov operators: S_L^0 - corresponding in fact to N=2 supersymmetry -, and S_L^{00} - corresponding to the zero field approximation of S_L^0 -, are defined in subsection 2.2. As in [16] and (I), the cohomology of S_L^{00} will give the main information. In Section 3, we analyse the cohomology of S_L^{00} and in Section 4 the one of S_L^0, *i.e.* at that point we are concerned with the special case of N=2 supersymmetric non-linear σ models, and we find a non trivial cohomology in the anomaly sector. Subsection 4.4 is then devoted to a discussion of this N=2 case and our main result is that, surprisingly enough with respect to the common wisdom [18][3], there exists a possible anomaly for <u>global</u> supersymmetry in 2 space-time dimensions [21], at least for <u>torsionless</u> compact Kähler Ricci-flat manifolds (*i.e.* special N=2 supersymmetric models). We also prove that this anomaly disappears when the manifold $\mathcal{M}$ is an homogeneous one, *i.e.* when one deals with N=2 supersymmetric non-linear σ models on coset spaces. Section 5 then constructs the cohomology space of the complete S_L operator, with the essential result of the all orders renormalizability of N=4 supersymmetric non-linear σ models. A discussion of our results is presented in the concluding Section.

2 The classical theory and the Slavnov operator

In (I) we obtained the classical action and the linearised Slavnov operator that describes N=4 supersymmetry and hereafter we summarize the essential results.

2.1 The classical theory and the Slavnov identity

We consider d=2, N=4 supersymmetric non-linear σ models in N=1 superfields $\Phi^i(x,\theta)$ (i, j,.. = 1,2,..4n). In light-cone coordinates and <u>in the absence of torsion</u>, the non-linear N=4 supersymmetry transformations write :

$$\delta\Phi^i = J^i_{A\,j}(\Phi)[\epsilon^+_A D_+\Phi^j + \epsilon^-_A D_-\Phi^j] \quad , \quad A = 1,2,3. \tag{2.1}$$

[3] Notice also that recent works of Brandt [19] and Dixon [20] show the existence of new non-trivial cohomologies in supersymmetric theories.

where the covariant derivatives are

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + i\theta^\pm\partial_\pm$$

and satisfy

$$\{D_\pm, D_\pm\} = 2i\partial_\pm \quad \{D_+, D_-\} = 0 \tag{2.2}$$

As is well known (see for example ref.[22]), N=4 supersymmetry needs the $J^i_{A\,j}(\Phi)$ to be a set [4] of anticommuting integrable complex structures according to :

$$J^i_{A\,j}(\Phi)J^j_{B\,k}(\Phi) = -\delta_{AB}\delta^i_k + \epsilon_{ABC}J^i_{C\,k}(\Phi) \tag{2.3}$$

and the invariance of the action $A^{inv.} = \int d^2xd^2\theta g_{ij}[\Phi]D_+\Phi^iD_-\Phi^j$ needs the target space to be hyperkähler :

* the metric g_{ij} is hermitian with respect to each complex structure

$$J^{ij}_A \equiv J^i_{A\,k}g^{kj} = -J^{ji}_A \quad ; \quad J_{A\,ij} \equiv J^k_{A\,i}g_{kj} = -J_{A\,ji}$$

* the $J^i_{A\,j}$ are covariantly constant

$$D_kJ^i_{A\,j} \equiv \partial_kJ^i_{A\,j} + \Gamma^i_{kl}J^l_{A\,j} - \Gamma^l_{kj}J^i_{A\,l} = 0$$

where Γ^i_{kl} is the (symmetric) connexion with respect to the metric g_{ij}. In the B.R.S. approach [23], the supersymmetry parameters $\epsilon^\pm_A$ are promoted to constant, commuting Faddeev-Popov parameters $d^\pm_A$ [5] and an anticommuting classical source $\eta_i(x)$ for the non-linear field transformation (2.1) is introduced in the classical action [6]. Then, the total effective action [7] is :

$$\Gamma^{class.} = A^{inv.} + \int d^2xd^2\theta\{\eta_iJ^i_{A\,j}(\Phi)[d^+_AD_+\Phi^j + d^-_AD_-\Phi^j] - \frac{1}{2}\epsilon_{ABC}\eta_i\eta_jJ^{ij}_C(\Phi)d^+_Ad^-_B\} \tag{2.4}$$

The terms quadratic in the sources are needed as a consequence of the only on-shell closedness of the N=4 supersymmetry algebra [24](I).

The Slavnov identity writes :

$$S\Gamma^{class.} \equiv \int d^2xd^2\theta\frac{\delta\Gamma^{tot.}}{\delta\eta_i(x,\theta)}\frac{\delta\Gamma^{tot.}}{\delta\Phi^i(x,\theta)} = \int d^2xd^2\theta[(d^+_A)^2(\eta_ki\partial_+\Phi^k) + (d^-_A)^2(\eta_ki\partial_-\Phi^k)]\ . \tag{2.5}$$

---

[4] As a matter of fact, it is sufficient to have 2 anticommuting integrable complex structures : then, the product $J^i_{3\,k} \equiv J^i_{1\,j}J^j_{2\,k}$ offers a third complex structure.

[5] As one is only concerned by integrated local functionals - *i.e.* trivially translation invariant ones -, we forget about the linear translation operators $P_\pm \equiv i\partial_\pm$, to which anticommuting Faddeev-Popov parameters $p^\pm$ should be associated, and do not add in $\Gamma^{class.}$ of equ.(2.4) the effect of translations on the fields Φ^i .

[6] In the absence of torsion, there is a parity invariance

$$+ \to -, d^2x \to d^2x, d^2\theta \to -d^2\theta, \Phi^i \to \Phi^i, \eta_i \to -\eta_i\ .$$

Moreover, the canonical dimensions of $[d^2xd^2\theta]$, $[\Phi^i]$, $[d^\pm_A]$, $[D_\pm]$, $[\eta_i]$ are -1, 0, -1/2, +1/2, +1,respectively and the Faddeev-Popov assignments + 1 for $d^\pm_A$, -1 for η_i, 0 for the other quantities.

[7] For simplicity, no mass term has been added here as we are here only interested in U.V. properties .

This is a non trivial result as in that N = 4 case, no finite set of auxiliary fields does exist.

As is by now well known (for example see [7] or [16]), in the absence of a consistent regularization that respects all the symmetries of the theory, the quantum analysis directly depends on the cohomology of the nilpotent linearised Slavnov operator :

$$\begin{aligned} S_L &= \int d^2x d^2\theta \left[\frac{\delta\Gamma^{class.}}{\delta\eta_i(x,\theta)}\frac{\delta}{\delta\Phi^i(x,\theta)} + \frac{\delta\Gamma^{class.}}{\delta\Phi^i(x,\theta)}\frac{\delta}{\delta\eta_i(x,\theta)}\right] \\ S_L^2 &= 0 \end{aligned} \tag{2.6}$$

in the Faddeev-Popov charge +1 sector [absence of anomalies for the supersymmetry] and 0 sector [number of physical parameters and stability of the classical action through radiative corrections]. Notice that the Slavnov operator (2.6) is unchanged under the following field and source reparametrisations :

$$\Phi^i \to \Phi^i + \lambda W^i[\Phi] \quad , \quad \eta_i \to \eta_i - \lambda \eta_k W^k_{,i}[\Phi] \ ,$$

where $W^i[\Phi]$ is an arbitrary function of the fields $\Phi(x,\theta)$ and a comma indicates a derivative with respect to the field Φ^i. Under this change, the classical action (2.4) is modified :

$$\Gamma^{class.} \to \Gamma^{class.} + \lambda S_L \int d^2x d^2\theta \eta_i W^i[\Phi] \tag{2.7}$$

but the Slavnov identity is left unchanged as

$$S[\Gamma^{class.} + \lambda S_L\Delta] \equiv S\Gamma^{class.} + \lambda S_L[S_L\Delta] = S\Gamma^{class.} \ . \tag{2.8}$$

The quantization of this theory will be studied in the next Sections, using the same algebraic cohomological methods as in the first paper of this series (I). It will be convenient to separate the 3 extra supersymmetries into the one [8] corresponding to J_3 and the 2 others to J_α, α=1, 2, *i.e.* to separate the N=2 supersymmetric case from the N=4 one. In the same way, one splits the linearised Slavnov operator into 3 parts according to their number of ghosts $d^\pm_\alpha$:

$$S_L = S_L^0 + S_L^1 + S_L^2$$

$$(S_L^0)^2 = S_L^0 S_L^1 + S_L^1 S_L^0 = S_L^0 S_L^2 + S_L^1 S_L^1 + S_L^2 S_L^0 = S_L^1 S_L^2 + S_L^2 S_L^1 = (S_L^2)^2 = 0 \ .$$

$$\begin{aligned} S_L^0 &= \int d^2x d^2\theta \{ J^i_j (d^+ D_+ \Phi^j + d^- D_- \Phi^j) \frac{\delta}{\delta\Phi^i} \\ &+ [\frac{\delta A^{inv.}}{\delta\Phi^i} + \eta_k (J^k_{j,i} - J^k_{i,j})(d^+ D_+\Phi^j + d^- D_-\Phi^j) + J^j_i (d^+ D_+ \eta_j + d^- D_- \eta_j)] \frac{\delta}{\delta\eta_i} \} \end{aligned} \tag{2.9}$$

which does not change the number of ghosts $d^\pm_\alpha$, will play a special role. Moreover, notice that the cohomology of S_L^0 corresponds to the special N=2 supersymmetric case.

[8] In the following, we omit the index 3 of the complex struture J_3 as well as the one of the ghost $d^\pm_3$.

2.2 The filtration and the operator S_L^0

In the presence of highly non-linear Slavnov operators such as in (2.9), as recalled in (I), it is technically useful to "approximate" the complete S_L^0 operator by a simpler one S_L^{00} through a suitably chosen "filtration"(ghost number preserving counting operation)[17]. As it does not change this number, S_L^{00}, the nilpotent lowest order part of S_L^0, will play a special role. Here, we take as counting operator the total number of fields $\Phi^i(x,\theta)$ and their derivatives. Then :

$$\begin{aligned} S_L^0 &= S_L^{00} + S_L^{01} + S_L^{02} + ... \equiv S_L^{00} + S_L^r \ , \quad (S_L^{00})^2 = 0 \\ S_L^{00} &= J_j^i(0) \int d^2x d^2\theta \left\{ (d^+ D_+ \Phi^j + d^- D_- \Phi^j) \frac{\delta}{\delta \Phi^i} + (d^+ D_+ \eta_i + d^- D_- \eta_i) \frac{\delta}{\delta \eta_j} \right\} \end{aligned} \tag{2.10}$$

As explained in refs.([17],[16]), when S_L^{00} has no cohomology in the Faddeev-Popov charged sectors, the cohomology of the complete S_L^0 operator in the Faddeev-Popov sectors of charge 0 and +1 is isomorphic to the one of S_L^{00} in the same sectors. The extension to the case where S_L^{00} has some non-trivial cohomology was discussed in the appendix of (I)[9] (see also the original papers [17],[16] and [25]).

Then, in the next Section, we shall determine the cohomology spaces of S_L^{00} in the Fadeev-Popov sectors of charge -1, 0 and +1.

3 The cohomology of S_L^{00}

The most general functional (in the fields, sources, ghosts and their derivatives) of a given Faddeev-Popov charge is built using Lorentz and parity invariance and power counting (see footnote 6).

3.1 The Faddeev-Popov negatively charged sectors

Due to dimensions and Faddeev-Popov charge assignments, dimension zero integrated local polynomials in the Faddeev-Popov parameters, fields, sources and their derivatives have at least a Faddeev-Popov charge -1 :

$$\Delta_{[-1]} = \int d^2x d^2\theta \eta_i V^i[\Phi] \ . \tag{3.1}$$

Then there is no Faddeev-Popov charge -1 coboundaries, so the cohomology of S_L^{00} in that sector is given by the cocycle condition :

$$S_L^{00} \Delta_{[-1]} = 0 \quad \Leftrightarrow \quad J_j^i(0) V_{,i}^k = J_i^k(0) V_{,j}^i \tag{3.2}$$

[9] In particular, the cohomology of S_L^{00} in the Faddeev-Popov -1 sector restricts the dimension of the cohomology of S_L^0 in the 0 charge sector when compared to the one of S_L^{00}.

This condition, when expressed in a coordinate system adapted to the complex structure $J^i_j[\Phi]$ (i $\equiv (a,\ \bar{a})$, $\Phi^i \equiv (\phi^a,\ \bar{\phi}^{\bar{a}})$: $J^a_b = i\delta^a_b, J^{\bar{a}}_{\bar{b}} = -i\delta^a_b, J^{\bar{a}}_b = J^a_{\bar{b}} = 0$), means that $V^i[\Phi]$ is a contravariant analytic vector : $V^a = V^a[\phi^d],\ V^{\bar{a}} = V^{\bar{a}}[\bar{\phi}^{\bar{d}}]$.

Let us now turn to the Faddeev-Popov neutral charge sector.

3.2 The Faddeev-Popov 0 charge sector

Here, one decomposes the set of integrated local polynomials in the Faddeev-Popov parameters, fields, sources and their derivatives with respect to their number of ghosts $d^{\pm}_{\alpha}$, N_{d_α} .

$$\begin{aligned}
\Delta^0_{[0]} &= \int d^2x d^2\theta \left\{ t_{ij}[\Phi] D_+\Phi^i D_-\Phi^j + \eta_i U^i_j[\Phi](d^+D_+\Phi^j + d^-D_-\Phi^j) \right\} \\
\Delta^1_{[0]} &= \int d^2x d^2\theta \left\{ \eta_i U^i_{\alpha\, j}[\Phi](d^+_\alpha D_+\Phi^j + d^-_\alpha D_-\Phi^j) + \eta_i\eta_j(d^+_\alpha d^- - d^-_\alpha d^+) S^{ij}_\alpha[\Phi] \right\} \\
\Delta^2_{[0]} &= d^+_\alpha d^-_\beta \int d^2x d^2\theta \eta_i \eta_j S^{ij}_{[\alpha\beta]}[\Phi]
\end{aligned} \tag{3.3}$$

where, due to parity invariance (footnote 6), t_{ij} (resp. S^{ij}_α, $S^{ij}_{[\alpha\beta]}$) are symmetric (resp. skew-symmetric) in (i,j). Coboundaries being given by $S^{00}_L\Delta_{[-1]}$[arbitrary $V^i(\Phi)$], the analysis of the cocycle condition $S^{00}_L\Delta_{[0]} = 0$ successively gives :

3.2.1 $N_{d_\alpha} = 0$

$$\Delta^0_{[0]} = \Delta^{an.}_{[0]}[t_{ij}(\Phi)] + S^{00}_L\Delta_{[-1]}[V^i(\Phi)] \ ; \ \Delta^{an.}_{[0]}[t_{ij}] = \int d^2x d^2\theta t_{ij}[\Phi] D_+\Phi^i D_-\Phi^j \tag{3.4}$$

where the tensor t_{ij} which occurs in the anomalous part is constrained by :

$$\begin{aligned}
a) &\qquad J^i_j(0)t_{ik} + t_{ji}J^i_k(0) &= 0, \\
b) &\qquad J^i_j(0)[t_{kl,i} - t_{il,k}] - (j \leftrightarrow k) &= 0.
\end{aligned} \tag{3.5}$$

The absence of source dependent non-trivial cohomology means that, up to a field redefinition (see (2.7,2.8)), the complex structure J^i_j is left unchanged through radiative corrections. Moreover, condition (3.5a) means that the metric $g_{ij} + \hbar t_{ij}$ remains hermitian with respect to the complex structure J^i_j, whereas (3.5b) expresses the covariant constancy of J^i_j with respect to the covariant derivative with a connexion corresponding to the metric $g_{ij} + \hbar t_{ij}$. These are precisely the expected conditions for the stability of N=2 supersymmetry.

3.2.2 $N_{d_\alpha} = 1$

$S_L^{00}\Delta^1_{[0]} = 0$ gives (no coboundaries exist in that sector) :

$$U^i_{\alpha j} = 0 \quad , \quad \Delta^1_{[0]} = \Delta^{an.}_{[0]}[S^{ij}_\alpha(\Phi)] = \int d^2x d^2\theta \eta_i\eta_j(d^+_\alpha d^- - d^-_\alpha d^+)S^{ij}_\alpha[\Phi] \quad , \tag{3.6}$$

where the tensor S^{ij}_α which occurs in the anomalous part is constrained by :

$$\begin{array}{lll} a) & J^i_k(0)S^{kj}_\alpha + S^{ki}_\alpha J^j_k(0) & = 0, \\ b) & J^n_k(0)S^{ij}_{\alpha,n} - J^i_n(0)S^{nj}_{\alpha,k} & = 0\ , \end{array} \tag{3.7}$$

i.e., using the same adapted coordinate system as above, is a pure contravariant analytic skew-symmetric tensor (*i.e.* $S^{[ab]}_\alpha = S^{[ab]}_\alpha(\phi^c)$, $S^{[\bar a\bar b]}_\alpha = S^{[\bar a\bar b]}_\alpha(\bar\phi^{\bar c})$, the other components vanish).

3.2.3 $N_{d_\alpha} = 2$

$S_L^{00}\Delta^2_{[0]} = 0$ gives (no coboundaries exist in that sector) :

$$\Delta^2_{[0]} = \Delta^{an.}_{[0]}[S^{ij}_{[\alpha\beta]}(\Phi)] = d^+_\alpha d^-_\beta \int d^2x d^2\theta \eta_i\eta_j S^{ij}_{[\alpha\beta]}[\Phi] \tag{3.8}$$

where the tensor $S^{ij}_{[\alpha\beta]}$ which occurs in the anomalous part is constrained by :

$$\begin{array}{lll} a) & J^i_k(0)S^{kj}_{[\alpha\beta]} + S^{ki}_{[\alpha\beta]} J^j_k(0) & = 0, \\ b) & J^n_k(0)S^{ij}_{[\alpha\beta],n} - J^i_n(0)S^{nj}_{[\alpha\beta],k} & = 0\ , \end{array} \tag{3.9}$$

i.e., using the same adapted coordinate system as above, is a pure contravariant analytic skew-symmetric tensor.

Finally, let us consider the Faddeev-Popov charge +1 sector.

3.3 The Faddeev-Popov +1 charge sector

Here also, one decomposes the set of integrated local polynomials in the Faddeev-Popov parameters, fields, sources and their derivatives with respect to their number of ghosts $d^\pm_\alpha$, N_{d_α} .

3.3.1 $N_{d_\alpha} = 0$

$\Delta^0_{[+1]}$ depends on 8 tensors :

$$\Delta^0_{[+1]} \quad = \quad \int d^2x d^2\theta \{(d^+)^2(d^-)^2\eta_i\eta_j\eta_k t^{[ijk]}$$

$$
\begin{aligned}
&+ \quad d^+d^-[\eta_i\eta_j t_{1\ n}^{[ij]}(d^+D_+\Phi^n - d^-D_-\Phi^n) + \eta_i s_1^{(ij)}(d^+D_+\eta_j - d^-D_-\eta_j)] \\
&+ \quad d^+d^-\eta_k t^k_{2\ [ij]} D_+\Phi^i D_-\Phi^j \\
&+ \quad (d^+)^2\eta_k(t^k_{4\ [ij]} D_+\Phi^i D_+\Phi^j + t^k_{5\ j}(D_+)^2\Phi^j) \\
&+ \quad (d^-)^2\eta_k(t^k_{4\ [ij]} D_-\Phi^i D_-\Phi^j + t^k_{5\ j}(D_-)^2\Phi^j) \\
&+ \quad d^+(\tilde{t}_{[ij]n} D_+\Phi^i D_+\Phi^j D_-\Phi^n + s_{2\ (ij)} D_-D_+\Phi^i D_+\Phi^j) \\
&- \quad d^-(\tilde{t}_{[ij]n} D_-\Phi^i D_-\Phi^j D_+\Phi^n + s_{2\ (ij)} D_+D_-\Phi^i D_-\Phi^j)\} \qquad (3.10)
\end{aligned}
$$

where, due to the anticommuting properties of η_i and $D_\pm\Phi^i$ and to the integration by parts freedom, the tensors $t^{[ijk]}$, $t_{1\ n}^{[ij]}$, $t^n_{2\ [ij]}$, $t^n_{4\ [ij]}$, $\tilde{t}_{[ij]n}$ are skew-symmetric in i, j, k, and $s_1^{(ij)}$, $s_{2\ (ij)}$ symmetric in i, j. Here and in the following, the symmetry (resp. antisymmetry) properties of the involved tensors in the exchange i to j are indicated by parenthesis (ij) (resp. brackets [ij]).

Coboundaries being given by $S_L^{00}\Delta^0_{[0]}$[arbitrary $(t_{ij}[\Phi], U^i_j[\Phi])$], the analysis of the cocycle condition $S_L^{00}\Delta^0_{[+1]} = 0$ leads to:

$$
\Delta^{an.(0)}_{[+1]} = \int d^2x d^2\theta t^{[ijk]}(\Phi)(d^+)^2(d^-)^2\eta_i\eta_j\eta_k \qquad (3.11)
$$

where the skew-symmetric tensor $t^{[ijk]}(\Phi)$ which occurs in the anomalous part is constrained by:

$$
\begin{aligned}
&a) \quad J^i_n(0)t^{[njk]} \quad \text{is i, j, k skew} - \text{symmetric}, \\
&b) \quad J^i_n(0)t^{[njk]}_{,m} = J^n_m(0)t^{[ijk]}_{,n} \qquad (3.12)
\end{aligned}
$$

Using the same adapted coordinate system as above, condition (3.12a) means that the tensor $t^{[ijk]}$ is a pure contravariant skew-symmetric tensor (*i.e.* $t^{[abc]}, t^{[\bar{a}\bar{b}\bar{c}]} \neq 0$, the other components vanish) whereas (3.12b) means that it is analytic (*i.e.* $t^{[abc]} = t^{[abc]}(\phi^d)$, $t^{[\bar{a}\bar{b}\bar{c}]} = t^{[\bar{a}\bar{b}\bar{c}]}(\bar{\phi}^{\bar{d}})$). In particular, due to the vanishing of $t^{[ab\bar{c}]}$, such tensor cannot be a candidate for a torsion tensor on a Kähler manifold [26].

As a first result, this proves that if the manifold $\mathcal{M}$ has a complex dimension smaller than 3, there is no $N_{d_\alpha} = 0$ anomaly candidate.

3.3.2 $N_{d_\alpha} = 1$

With an expansion similar to the one of $\Delta^0_{[+1]}$, $\Delta^1_{[+1]}$ now depends on 11 tensors :

$$
\begin{aligned}
\Delta^1_{[+1]} \quad = \quad & d^+_\alpha \int d^2x d^2\theta \{d^+(d^-)^2\eta_i\eta_j\eta_k t^{[ijk]}_\alpha \\
&+ \quad d^+d^-[\eta_i\eta_j t^{[ij]}_{\alpha 1\ n} D_+\Phi^n + \eta_i D_+\eta_j s^{(ij)}_{\alpha 1}] + d^-d^-[\eta_i\eta_j t'^{[ij]}_{\alpha 1\ n} D_-\Phi^n + \eta_i D_-\eta_j s'^{(ij)}_{\alpha 1}] \\
&+ \quad d^-\eta_k[t^k_{\alpha 2\ [ij]} D_+\Phi^i D_-\Phi^j + t^k_{\alpha 3\ j} D_+D_-\Phi^j] \\
&+ \quad d^+\eta_k[t^k_{\alpha 4\ [ij]} D_+\Phi^i D_+\Phi^j + t^k_{\alpha 5\ j}(D_+)^2\Phi^j]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad [\tilde{t}_{\alpha[ij]n} D_+\Phi^i D_+\Phi^j D_-\Phi^n + s_{\alpha 2\,(ij)} D_- D_+\Phi^i D_+\Phi^j]\} \\
&+ \quad \text{parity exchanged (according to footnote 6)} \qquad (3.13)
\end{aligned}
$$

Coboundaries being given by $S_L^{00}\Delta^1_{[0]}$[arbitrary $(U^i_{\alpha\,j}[\Phi], S^{ij}_\alpha[\Phi])$], the analysis of the cocycle condition $S_L^{00}\Delta^1_{[+1]} = 0$ leads to :

$$
\begin{aligned}
\Delta^{an.(1)}_{[+1]} &= \int d^2x d^2\theta t^{[ijk]}_\alpha(\Phi)[d^+_\alpha d^+ (d^-)^2 + d^-_\alpha d^- (d^+)^2]\eta_i\eta_j\eta_k + \\
&+ \int d^2x d^2\theta \tilde{t}_{\alpha\,[ij]k}(\Phi)[d^+_\alpha D_+\Phi^i D_+\Phi^j D_-\Phi^k - d^-_\alpha D_-\Phi^i D_-\Phi^j D_+\Phi^k] \qquad (3.14)
\end{aligned}
$$

where the constraints on the skew-symmetric tensors $t^{[ijk]}_\alpha(\Phi)$ and $\tilde{t}_{\alpha\,[ij]k}(\Phi)$ which occur in the anomalous part are easily solved in the same adapted coordinate system as above :

- the tensor $t^{[ijk]}_\alpha$ is a pure contravariant analytic skew-symmetric tensor,

- the tensor $\tilde{t}_{\alpha\,[ab]\bar{c}} \equiv \partial_{\bar{c}}[\partial_a t_{\alpha\, b}(\phi,\bar{\phi}) - \partial_b t_{\alpha\, a}(\phi,\bar{\phi})]$ (and the complex conjugate relation), the other components vanish.

3.3.3 $N_{d_\alpha} = 2$

Here, we separate in $\Delta^2_{[+1]}$ the terms symmetric in the exchange of the indices α and β of the 2 ghosts $d^\pm_\alpha$ and $d^\pm_\beta$ from the skew-symmetric ones :

$$
\begin{aligned}
\Delta^2_{[+1]}|_{(\alpha\beta)} &= d^+_\alpha d^+_\beta \int d^2x d^2\theta\{(d^-)^2\eta_i\eta_j\eta_k t^{[ijk]}_{(\alpha\beta)} + d^-[\eta_i\eta_j D_+\Phi^k t^{[ij]}_{(\alpha\beta)1\;k} + \eta_i D_+\eta_j s^{(ij)}_{(\alpha\beta)1}] \\
&+ \eta_k[D_+\Phi^i D_+\Phi^j t^k_{(\alpha\beta)4\,[ij]} + (D_+)^2\Phi^j t^k_{(\alpha\beta)5\,j}]\} + \text{parity exchanged} + \\
&+ d^+_\alpha d^-_\beta \int d^2x d^2\theta\{d^+ d^- \eta_i\eta_j\eta_k t'^{[ijk]}_{(\alpha\beta)} + \\
&+ [\eta_i\eta_j(d^+ D_+\Phi^k - d^- D_-\Phi^k) t'^{[ij]}_{(\alpha\beta)1\;k} + \eta_i(d^+ D_+\eta_j - d^- D_-\eta_j) s'^{(ij)}_{(\alpha\beta)1}] + \\
&+ \eta_k D_+\Phi^i D_-\Phi^j t'^k_{(\alpha\beta)4\,[ij]}\} \quad ; \\
\Delta^2_{[+1]}|_{[\alpha\beta]} &= d^+_\alpha d^-_\beta \int d^2x d^2\theta\{[\eta_i\eta_j(d^+ D_+\Phi^n + d^- D_-\Phi^n) t^{[ij]}_{[\alpha\beta]1\;n} \\
&+ \eta_i(d^+ D_+\eta_j + d^- D_-\eta_j) s^{(ij)}_{[\alpha\beta]1}] \\
&+ \eta_k[D_+\Phi^i D_-\Phi^j t^k_{[\alpha\beta]4\,(ij)} + D_+ D_-\Phi^j t^k_{[\alpha\beta]5\,j}]\} \qquad (3.15)
\end{aligned}
$$

Then, coboundaries being given by $S_L^{00}\Delta^2_{[0]}$[arbitrary $S^{ij}_{[\alpha\beta]}(\Phi)$], the analysis of the cocycle condition $S_L^{00}\Delta^2_{[+1]} = 0$ leads to :

$$
\begin{aligned}
\Delta^{an.(2)}_{[+1]} &= d^+_\alpha d^+_\beta \int d^2x d^2\theta\eta_i \left[(d^-)^2\eta_j\eta_k t^{[ijk]}_{(\alpha\beta)}(\Phi) + (D_+)^2\Phi^j t^i_{(\alpha\beta)\,j}(\Phi)\right] + \text{parity exchanged} + \\
&+ d^+_\alpha d^-_\beta d^+ d^- \int d^2x d^2\theta \eta_i\eta_j\eta_k t'^{[ijk]}_{(\alpha\beta)}(\Phi) + \\
&+ d^+_\alpha d^-_\beta \int d^2x d^2\theta\eta_i \left[D_+\Phi^j D_-\Phi^k t^i_{[\alpha\beta]\,(jk)}(\Phi) + D_+ D_-\Phi^j t^i_{[\alpha\beta]\,j}(\Phi)\right] \qquad (3.16)
\end{aligned}
$$

where the constraints on the tensors $t^{[ijk]}_{(\alpha\beta)}(\Phi)$, $t^{i}_{(\alpha\beta)\,j}(\Phi)$, $t'^{[ijk]}_{(\alpha\beta)}(\Phi)$, $t^{i}_{[\alpha\beta]\,(jk)}(\Phi)$ and $t^{i}_{[\alpha\beta]\,j}(\Phi)$ which occur in the anomalous part are easily solved in the same adapted coordinate system as above :

- the tensors $t^{[ijk]}_{(\alpha\beta)}(\Phi)$ and $t'^{[ijk]}_{(\alpha\beta)}(\Phi)$ are pure contravariant analytic skew-symmetric tensors,

- the tensor $t^{i}_{(\alpha\beta)\,j}(\Phi)$ is a mixed analytic tensor, (*i.e.* $t^{a}_{(\alpha\beta)\,b} = t^{a}_{(\alpha\beta)\,b}(\phi^c)$, $t^{\bar{a}}_{(\alpha\beta)\,\bar{b}} = t^{\bar{a}}_{(\alpha\beta)\,\bar{b}}(\bar{\phi}^{\bar{c}})$),

- the tensor $t^{\bar{c}}_{[\alpha\beta]\,(ab)} \equiv \partial_a\partial_b t^{\bar{c}}_{[\alpha\beta]}(\phi,\bar{\phi})$ and the tensor $t^{\bar{c}}_{[\alpha\beta]\,b} \equiv \partial_b t^{\bar{c}}_{[\alpha\beta]}(\phi,\bar{\phi})$ (and the complex conjugate relations), the other components vanish.

3.3.4 $N_{d_\alpha} = 3$

In that sector, there are no coboundaries, and the analysis of the cocycle condition $S^{00}_L \Delta^3_{[+1]} = 0$ with :

$$\begin{aligned}\Delta^3_{[+1]} &= d^+_\alpha d^+_\beta d^-_\gamma \int d^2x d^2\theta \{ d^- \eta_i\eta_j\eta_k t^{[ijk]}_{(\alpha\beta)\gamma} + \eta_i\eta_j t^{[ij]}_{(\alpha\beta)\gamma\,1\,k} D_+\Phi^k + \eta_i D_+\eta_j s^{(ij)}_{(\alpha\beta)\gamma\,1}\} + \\ &+ \text{ parity exchanged} \end{aligned} \tag{3.17}$$

leads to :

$$\Delta^{an.(3)}_{[+1]} = d^+_\alpha d^+_\beta d^-_\gamma d^- \int d^2x d^2\theta \eta_i\eta_j\eta_k t^{[ijk]}_{(\alpha\beta)\gamma}(\Phi) + \text{parity exchanged} \tag{3.18}$$

where the constraints on the skew-symmetric tensor $t^{[ijk]}_{(\alpha\beta)\gamma}(\Phi)$ which occurs in the anomalous part are easily solved in the same adapted coordinate system as above and again means that it is a pure contravariant analytic skew-symmetric tensor.

3.3.5 $N_{d_\alpha} = 4$

In that sector too, there are no coboundaries, and the analysis of the cocycle condition $S^{00}_L \Delta^4_{[+1]} = 0$ with :

$$\Delta^4_{[+1]} = d^+_\alpha d^+_\beta d^-_\gamma d^-_\delta \int d^2x d^2\theta \eta_i\eta_j\eta_k t^{[ijk]}_{((\alpha\beta)(\gamma\delta))} \tag{3.19}$$

leads to :

$$\Delta^{an.(4)}_{[+1]} = d^+_\alpha d^+_\beta d^-_\gamma d^-_\delta \int d^2x d^2\theta \eta_i\eta_j\eta_k t^{[ijk]}_{((\alpha\beta)(\gamma\delta))}(\Phi) \tag{3.20}$$

where the constraints on the skew-symmetric tensor $t^{[ijk]}_{((\alpha\beta)(\gamma\delta))}(\Phi)$ which occurs in the anomalous part are easily solved in the same adapted coordinate system as above and again means that it is a pure contravariant analytic skew-symmetric tensor.

This ends the analysis of the cohomology of S^{00}_L and we are now in a position to discuss the cohomology of the complete $S^0_L \equiv S^{00}_L + S^r_L$ operator.

4 The cohomology of S_L^0

It will be convenient to analyse the cohomology of S_L^0 in a coordinate system where the complex structure J_3 is constant. S_L^{00} of equ.(2.10) is unchanged (with $J_j^i(0) \to J_j^i$, field independent) and we note that

$$S_L^r = \int d^2x d^2\theta \frac{\delta A^{inv.}}{\delta \Phi^i(x,\theta)} \frac{\delta}{\delta \eta_i(x,\theta)} \tag{4.1}$$

decreases the number of η_i of one unity, whereas S_L^{00} does not change this number. Using this fact, we are able to construct the S_L^0 cohomology starting from the S_L^{00} one : indeed,

$$S_L^0 \Delta_{[\phi\pi]} \equiv (S_L^{00} + S_L^r)(\Delta_{[\phi\pi]}^{an.} + \tilde{\Delta}_{[\phi\pi]}) = 0 \;\Leftrightarrow\; S_L^{00} \tilde{\Delta}_{[\phi\pi]} = -S_L^r(\Delta_{[\phi\pi]}^{an.} + \tilde{\Delta}_{[\phi\pi]}) \; . \tag{4.2}$$

Then, when ordered by decreasing order with respect to the total number of η_i, the equation $S_L^0 \Delta_{[\phi\pi]} = 0$ is identical to the $S_L^{00} \Delta_{[\phi\pi]} = 0$ one with a right hand side given by previous order contributions.

4.1 The Faddeev-Popov negatively charged sectors

Thanks to the simplicity of $\Delta_{[-1]}$ (3.1), the cohomology of the complete S_L^0 operator in the Faddeev-Popov charge -1 sector is easily obtained : the vector $V^i[\Phi]$ should satisfy :

- $\displaystyle\int d^2x d^2\theta \frac{\delta A^{inv.}}{\delta \Phi^i(x,\theta)} V^i[\Phi(x,\theta)] = 0 \;\Leftrightarrow\; V^i[\Phi]$ is a Killing vector for the metric $g_{ij}[\Phi]$

- $J_j^i[\Phi] \nabla_i V^k = \nabla_j V^i J_i^k[\Phi] \;\Leftrightarrow\; V^i[\Phi]$ is a contravariant vector analytic with respect to $J_j^i[\Phi]$.

Let us now turn to the Faddeev-Popov neutral charge sector.

4.2 The Faddeev-Popov 0 charge sector

As explained in the appendix of (I) (see also [16],[17]), despite the non-vanishing S_L^{00} cohomology in a Faddeev-Popov positively charged sector (subsection 3.3), the cohomology of S_L^0 is a subspace of the one of S_L^{00}, *i.e.* one can always construct the cocycles for S_L^0 starting from those of S_L^{00}. It may also happen that some of the thus constructed cocycles for S_L^0 become coboundaries: this occurs when there is some cohomology for S_L^{00} in the Faddeev-Popov charge -1 sector ((I) and [25]). We have seen previously that this relies on the existence of Killing vectors for the metric $g_{ij}[\Phi]$; this is natural as such vectors signal extra isometries that constrain the invariant action or, equivalently, signal the non physically relevant character of some of the parameters of the classical action that may be reabsorbed through a conveniently chosen field and source reparametrisation [16].

As in the previous section, the analysis separates with respect to the number N_{d_α} :

4.2.1 $N_{d_\alpha} = 0$

The image of $\Delta^0_{[0]}$ (3.3) through S^r_L does not intercept $\Delta^{an.(0)}_{[+1]}$, the cohomology of S^{00}_L in the anomaly sector. As a consequence ([17] and the appendix of (I)), there will be no obstruction in the construction of the cocycles of S^0_L starting from those of S^{00}_L and there is an isomorphism between the two cohomology spaces. Consequently, the cohomology in the $N_{d_\alpha} = 0$ Faddeev-Popov neutral sector is characterized by a symmetric tensor $t_{ij}[\Phi]$ such that $g'_{ij} = g_{ij} + \hbar t_{ij}$ is a metric, hermitian with respect to the very complex structure J^i_j we started from, and such that J^i_j is covariantly constant with respect to the covariant derivative with connexion $\Gamma^k_{ij}[g'_{mn}]$. This is the necessary stability of the N=2 supersymmetric theory which ensures that, at a given perturbative order where the Slavnov identity holds (absence of anomaly up to this order), the U.V. divergences in the Green functions may be compensated for through the usual renormalization algorithm and normalisation conditions [7].

4.2.2 $N_{d_\alpha} = 1$

Here the image of $\Delta^1_{[0]}$ (3.3) through S^r_L intercepts $\Delta^{an.(1)}_{[+1]}$, the cohomology of S^{00}_L in the anomaly sector. As a consequence ([17] and the appendix of (I)), this will restrict the cohomology (3.6,3.7) in the considered sector. In fact, $S^0_L\Delta^1_{[0]} = 0$ gives (no coboundaries exist in that sector) :

$$\begin{aligned} \Delta^1_{[0]} &= \Delta^{an.(1)}_{[0]}[U^i_{\alpha j}(\Phi),\ S^{ij}_\alpha(\Phi)]_{[equ.(3.3)]} \\ \text{with } U^i_{\alpha j}(\Phi) &= -2[J_{jk}S^{ki}_\alpha(\Phi)] \ \Leftrightarrow\ S^{ij}_\alpha = -\frac{1}{2}J^{ik}U^j_{\alpha k} \end{aligned} \quad (4.3)$$

where the supplementary constraint on S^{ij}_α is such that $J^i_{\alpha j} + \hbar U^i_{\alpha j}$ - which anticommutes with J^i_j - is now also covariantly constant with respect to the covariant derivative with connexion $\Gamma^k_{ij}[g_{mn}]$.

4.2.3 $N_{d_\alpha} = 2$

Here too, the image of $\Delta^2_{[0]}$ (3.3) through S^r_L intercepts $\Delta^{an.(2)}_{[+1]}$, the cohomology of S^{00}_L in the anomaly sector, which will restrict the cohomology (3.8,3.9) in the considered sector. Thanks to the simplicity of $\Delta^2_{[0]}$ (3.3), the analysis of the cocycle condition $S^0_L\Delta^2_{[0]} = 0$ in the one η_i subsector readily shows that the cohomology space of the complete S^0_L operator is empty in this $N_{d_\alpha} = 2$, Faddeev-Popov neutral sector.

Finally, let us consider the Faddeev-Popov charge +1 sector.

4.3 The Faddeev-Popov +1 charge sector

Here too, the analysis separates with respect to the number N_{d_α}.

4.3.1 $N_{d_\alpha} = 0$

The S_L^{00} cohomology was obtained in equ.(3.11) :

$$\Delta^{an.(0)}_{[+1]} = \int d^2x d^2\theta t^{[ijk]}[\Phi](d^+)^2(d^-)^2\eta_i\eta_j\eta_k$$

and, using the algorithm described by equ.(4.2), we find the S_L^0 cohomology in the same sector to be :

$$\begin{aligned}\Delta^{an.(0)}_{[+1]} &= \int d^2x d^2\theta t^{[ijk]}[\Phi]\{(d^+)^2(d^-)^2\eta_i\eta_j\eta_k - \frac{3}{2}d^+d^-\eta_i\eta_j J_{kn}(d^+D_+\Phi^n - d^-D_-\Phi^n)\\ &- 3d^+d^-\eta_i J_{jn}J_{km}D_+\Phi^n D_-\Phi^m\\ &+ \frac{3}{4}J_{in}J_{jm}J_{kl}(d^+D_+\Phi^n D_+\Phi^m D_-\Phi^l - d^-D_-\Phi^n D_-\Phi^m D_+\Phi^l)\}\\ &+ \int d^2x d^2\theta t^1_{[nm]l}[\Phi]\left(d^+D_+\Phi^n D_+\Phi^m D_-\Phi^l - d^-D_-\Phi^n D_-\Phi^m D_+\Phi^l\right) ,\end{aligned} \tag{4.4}$$

where $t^1_{[ij]\,k}[\Phi]$ is related to the pure contravariant analytic tensor $t^{[ijk]}[\Phi]$ through (in complex coordinates) :

$$t^1_{[ab]\,\bar{c}},\quad t^1_{[\bar{a}\bar{b}]\,c} \neq 0, \quad \text{the other vanish ;}$$

$$t^1_{[ab]\,\bar{c}} = \frac{i}{4}\partial_{\bar{c}}[g_{a\bar{a}}g_{b\bar{b}}K,_{\bar{d}}\,t^{[\bar{a}\bar{b}\bar{d}]}] \ \text{ where K is the Kähler potential .}$$

4.3.2 $N_{d_\alpha} = 1$

The S_L^{00} cohomology was obtained in equ.(3.14) :

$$\Delta^{an.(1)}_{[+1]} = \int d^2x d^2\theta t^{[ijk]}_\alpha(\Phi)[d^+_\alpha d^+(d^-)^2 + d^-_\alpha d^-(d^+)^2]\eta_i\eta_j\eta_k + $$

$$+ \int d^2x d^2\theta \tilde{t}_{\alpha\,[ij]k}(\Phi)[d^+_\alpha D_+\Phi^i D_+\Phi^j D_-\Phi^k - d^-_\alpha D_-\Phi^i D_-\Phi^j D_+\Phi^k]$$

Notice first that S^r_L does not act on the second piece of $\Delta^{an.(1)}_{[+1]}$, which then is a true S_L^0 anomaly. Due to its similarity with $\Delta^{an.(0)}_{[+1]}$, the first part of $\Delta^{an.(1)}_{[+1]}$ is easily promoted to a complete S_L^0 cohomology :

$$\begin{aligned}\Delta^{an.(1)}_{[+1]} &= \int d^2x d^2\theta t^{[ijk]}_\alpha(\Phi)\{d^+_\alpha d^+(d^-)^2\eta_i\eta_j\eta_k - \frac{3}{2}d^+_\alpha d^-\eta_i\eta_j J_{kn}(d^+D_+\Phi^n - d^-D_-\Phi^n)\\ &- 3d^+_\alpha d^-\eta_i J_{jn}J_{km}D_+\Phi^n D_-\Phi^m + \frac{3}{2}J_{in}J_{jm}J_{kl}d^+_\alpha D_+\Phi^n D_+\Phi^m D_-\Phi^l + \text{parity exchange}\}\\ &+ \int d^2x d^2\theta(2t^1_{\alpha[ij]k}(\Phi) + \tilde{t}_{\alpha[ij]k}(\Phi))\left(d^+_\alpha D_+\Phi^i D_+\Phi^j D_-\Phi^k - d^-_\alpha D_-\Phi^i D_-\Phi^j D_+\Phi^k\right)\end{aligned} \tag{4.5}$$

where $t^1_{\alpha[ij]\,k}[\Phi]$ is related to the pure contravariant analytic tensor $t^{[ijk]}_\alpha[\Phi]$ as in the $N_{d_\alpha} = 0$ case, and $\tilde{t}_{\alpha[ij]k}[\Phi]$ is related to a covariant vector $t_{\alpha i}$ (see subsection 3.3.2).

4.3.3 $N_{d_\alpha} = 2$

The S_L^{00} cohomology was obtained in equ.(3.16):

$$\Delta_{[+1]}^{an.(2)} = \Delta_{[+1]}^{an.(2)}|_{(\alpha\beta)++} + \Delta_{[+1]}^{an.(2)}|_{(\alpha\beta)--} + \Delta_{[+1]}^{an.(2)}|_{(\alpha\beta)+-} + \Delta_{[+1]}^{an.(2)}|_{[\alpha\beta]}$$

where :

$$\Delta_{[+1]}^{an.(2)}|_{(\alpha\beta)++} = d_\alpha^+ d_\beta^+ \int d^2x d^2\theta \eta_i \left[(d^-)^2 \eta_j \eta_k t_{(\alpha\beta)}^{[ijk]}(\Phi) + (D_+)^2 \Phi^j t^i_{(\alpha\beta)\,j}(\Phi)\right]$$

$$\Delta_{[+1]}^{an.(2)}|_{(\alpha\beta)+-} = d_\alpha^+ d_\beta^- d^+ d^- \int d^2x d^2\theta \eta_i \eta_j \eta_k \tilde{t}_{(\alpha\beta)}^{[ijk]}(\Phi)$$

$$\Delta_{[+1]}^{an.(2)}|_{[\alpha\beta]} = d_\alpha^+ d_\beta^- \int d^2x d^2\theta \eta_i \left[D_+\Phi^j D_-\Phi^k t^i_{[\alpha\beta]\,(jk)}(\Phi) + D_+ D_- \Phi^j t^i_{[\alpha\beta]\,j}(\Phi)\right]$$

• As $S_L^{00}\Delta^2_{[+1]}$ contains at least one source η_i, and S_L^r decreases the number of η_i of one unity, the $[\alpha,\beta]$ skew-symmetric part of the looked-for cocycles of S_L^0, $\Delta^2_{[+1]}|_{[\alpha\beta]}$ should satisfy

$$\left(S_L^r \Delta^2_{[+1]}\right)|_{no\ \eta} = 0$$

which readily gives :

$$\left(\Delta^2_{[+1]}|_{[\alpha\beta]}\right)_{1\eta} = S_L^r \left(d_\alpha^+ d_\beta^- \int d^2x d^2\theta \eta_i \eta_j T^{[ij]}_{[\alpha\beta]}\right) .$$

The cocycle condition $(S_L^{00} + S_L^r)\Delta_{[+1]}^{an.(2)}|_{[\alpha\beta]} = 0$ constrains $T^{[ij]}_{[\alpha\beta]}$ to be a pure contravariant analytic skew-symmetric tensor ; then, as a consequence of (3.8,3.9), the last parenthesis is anihilated by S_L^{00} and $\Delta^2_{[+1]}|_{[\alpha\beta]}$ is in fact a S_L^0 coboundary.

• In the same way, the cocycle condition $S_L^0\Delta^2_{[+1]}|_{(\alpha\beta)+-} = 0$, when analysed by increasing number of sources η, leads to $\Delta_{[+1]}^{an.(2)}|_{(\alpha\beta)+-} = 0$.

• Finally, the cocycle condition $S_L^0\Delta^2_{[+1]}|_{(\alpha\beta)++} = 0$ is analysed along the same lines as $S_L^0\Delta^1_{[+1]} = 0$, and leads to :

$$\begin{aligned}
\Delta_{[+1]}^{an.(2)} & \\
= \quad & d_\alpha^+ d_\beta^+ \int d^2x d^2\theta t_{(\alpha\beta)}^{[ijk]}[\Phi] \left\{(d^-)^2 \eta_i \eta_j \eta_k + \frac{3}{2}[\eta_i \eta_j J_{kn} d^- D_+ \Phi^n + \eta_i J_{jn} J_{km} D_+\Phi^n D_+\Phi^m]\right\} \\
+ \quad & d_\alpha^+ d_\beta^+ \int d^2x d^2\theta \eta_i t^i_{(\alpha\beta)j}[\Phi] (D_+)^2 \Phi^j \\
+ \quad & \text{parity exchange} ,
\end{aligned} \qquad (4.6)$$

where :

- as usual, $t^{[ijk]}_{(\alpha\beta)}[\Phi]$ is a pure contravariant analytic skew-symmetric tensor, further constrained so as the corresponding $t^1_{(\alpha\beta)[ij]k}$ tensor actually vanishes, *i.e.* in complex coordinates,

$$t^{[abc]}_{(\alpha\beta)} = t^{[abc]}_{(\alpha\beta)}(\phi^d) \quad ; \quad t_{(\alpha\beta)\,[abc]} \stackrel{\text{def.}}{=} g_{a\bar{a}}g_{b\bar{b}}g_{c\bar{c}}t^{[\bar{a}\bar{b}\bar{c}]}_{(\alpha\beta)}(\bar{\phi}^{\bar{d}}) = t_{(\alpha\beta)\,[abc]}(\phi^d)$$

and the complex conjugate relations ;

- $t^i_{(\alpha\beta)j}[\Phi]$ is a mixed analytic tensor, further constrained so as the tensor $G_{(\alpha\beta)\,ij} = g_{ik}t^k_{(\alpha\beta)j}$ is a symmetric, covariantly constant tensor (hermitian with respect to the complex structure J^i_j due to previous relations (subsection 3.3.3)). This will be important in the following (subsection 5.1.6).

4.3.4 $N_{d_\alpha} = 3$

There are no coboundaries in that sector, and the S^{00}_L cohomology was obtained in equ.(3.18). Notice that $S^{00}_L\Delta^3_{[+1]}$ contains at least two sources η_i and that S^r_L decreases the number of η_i of one unity ; then, when analysed by increasing number of sources η, the cocycle condition $S^0_L\Delta^3_{[+1]} = 0$, leads to $\Delta^3_{[+1]} = 0$.

4.3.5 $N_{d_\alpha} = 4$

There are no coboundaries in that sector, and the S^{00}_L cohomology was obtained in equ.(3.20). Notice that $S^{00}_L\Delta^4_{[+1]}$ contains at least three sources η_i and S^r_L decreases the number of η_i of one unity ; here again, when analysed by increasing number of sources η, the cocycle condition $S^0_L\Delta^4_{[+1]} = 0$, leads to $\Delta^4_{[+1]} = 0$.

To sum up, the cohomology space of S^0_L in the Faddeev-Popov charge +1 sector depends on skew-symmetric contravariant analytic 3-tensors $t^{[ijk]}[\Phi]$, $t^{[ijk]}_\alpha[\Phi]$ and $t^{[ijk]}_{(\alpha\beta)}[\Phi]$, the last one endowing a further constraint, on a vector $t_{\alpha\,i}$ and on a symmetric, covariantly constant tensor $G_{(\alpha\beta)\,ij}$, hermitian with respect to the complex structure J^i_j .

We are now in a position to compute the cohomology of the complete $S_L = S^0_L + S^1_L + S^2_L$ operator. But, as an intermediate result, we comment on N = 2 supersymmetric non-linear σ models [21].

4.4 N=2 supersymmetric non-linear σ models

In the special case of N=2 supersymmetric non-linear σ models, there is no need for ghosts $d^\pm_\alpha$ and all the necessary results may be found in subsections 4.1, 4.2.1 and 4.3.1. In particular, equation (4.4) offers a candidate for an anomaly and, as a consequence, if at a given pertubative order this anomaly appears with a non zero coefficient

$$S_L^0 \Gamma|_{p^{th} order} = a(\hbar)^p \Delta^{an.}_{[+1]}, \quad a \neq 0$$

the N = 2 supersymmetry is broken as $\Delta^{an.}_{[+1]}$ cannot be reabsorbed (being a cohomology element, it is not a $S_L^0 \tilde{\Delta}_{[0]}$) and, *a priori*, we are no longer able to analyse the structure of the U.V. divergences at the next perturbative order, which is the death of the theory.

We now discuss some properties of the candidate anomaly, trying to characterize the special geometries (and manifolds $\mathcal{M}$) where such pure contravariant analytic skew-symmetric 3-tensor cannot appear.

Consider the covariant tensor

$$t_{[abc]} = g_{a\bar{a}} g_{b\bar{b}} g_{c\bar{c}} t^{[\bar{a}\bar{b}\bar{c}]}[\bar{\phi}] \, .$$

It satisfies $\nabla_d \, t_{[abc]} = 0$. The (3-0) form

$$\omega' = \frac{1}{3!} t_{[abc]} d\phi^a \wedge d\phi^b \wedge d\phi^c \tag{4.7}$$

which satisfies $d'\omega' = 0$ [10], will now be shown to be harmonic if $\mathcal{M}$ is a <u>compact</u> manifold (or if it is a <u>Ricci-flat</u> one - for example an HyperKähler manifold).

One obtains firstly from the identity :

$$\Delta t_{ijk} = g^{mn} \nabla_m \nabla_n \, t_{[ijk]} - [R_i^l \, t_{[ljk]} + \text{ perms. }] - [R_{ij}^{lm} \, t_{[lmk]} + \text{ perms. }]$$

rewritten in complex coordinates and due to $\nabla_d \, t_{[abc]} = 0$:

$$\Delta t_{[abc]} = g^{d\bar{d}} \nabla_d \nabla_{\bar{d}} \, t_{[abc]} - [R_a^d \, t_{[dbc]} + \text{ perms. }] \, .$$

On the other hand, the Ricci identity gives, still using $\nabla_d \, t_{[abc]} = 0$:

$$g^{d\bar{d}} \nabla_d \nabla_{\bar{d}} \, t_{[abc]} = g^{d\bar{d}} [R^e_{\bar{d}da} \, t_{[ebc]} + \text{ perms. }] = -[R_a^d \, t_{[dbc]} + \text{ perms. }].$$

Then $\Delta t_{[abc]} = 2 g^{d\bar{d}} \nabla_d \nabla_{\bar{d}} \, t_{[abc]} = 2[R_a^d \, t_{[dbc]} + \text{ perms. }]$. In the Ricci-flat case, this gives the claimed harmonicity of ω'. Now, when the manifold is a compact one, one may compute :

$$\begin{aligned}
(d\omega', d\omega') + (\delta\omega', \delta\omega') = (\omega', (d\delta + \delta d)\omega') = (\omega', \Delta\omega') \quad &= \quad \int_{\mathcal{M}} d\sigma 2 t^{[abc]} g^{d\bar{d}} \nabla_d \nabla_{\bar{d}} \, t_{[abc]} = \\
= \int_{\mathcal{M}} d\sigma 2 g^{d\bar{d}} \{ \nabla_d \nabla_{\bar{d}} (t^{[abc]} t_{[abc]}) - \nabla_d \, t^{[abc]} \nabla_{\bar{d}} \, t_{[abc]} \} \quad &= \quad 0 - 2(d\omega', d\omega') \\
\Rightarrow \; (\delta\omega', \delta\omega') + 3(d\omega', d\omega') = 0 \quad &\Rightarrow \quad \delta\omega' = d\omega' = \Delta\omega' = 0 \, ,
\end{aligned} \tag{4.8}$$

and, as a consequence :

$$[R_a^d \, t_{[dbc]} + \text{ perms. }] = 0 \;\Rightarrow\; 3 t^{[abc]} R_a^{a'} \, t_{a'bc} = 0 \, . \tag{4.9}$$

[10] As usual in complex geometry, see for example ([27],[28]), the differential d ≡ d' + d", the codifferential $\delta \equiv \delta' + \delta"$ and the Laplacian $\Delta \equiv d\delta + \delta d$.

Moreover, in this compact case, it results from the previous discussion that $\left[\partial_d t^1_{[ab]\,\bar{c}}\right]_{(a,b,d\ a.s.)}$ $\equiv \frac{i}{4}\partial_{\bar{c}} t_{[abd]}$ vanishes. As a consequence, $t^1_{[ab]\,\bar{c}} = \partial_b \bar{t}_{a\bar{c}} - (a \leftrightarrow b)$, which corresponds to a trivial cohomology and is then thrown away from equation (4.4).

Notice that, as *a priori* $d\omega' \neq 0$, this harmonicity is only a necessary condition which, to my knowledge, is not given in the mathematic literature. For example, in [27], page 70, only a more restrictive necessary and sufficient condition is given :

On a compact Kählerian manifold, given a (p,0) form η, the 3 following conditions are equivalent : η is closed, η is holomorphic and η is harmonic,
or in [28], theorem 9.3 :

On a compact Kählerian manifold, given a pure skew-symmetric covariant tensor, a necessary and sufficient condition for it to be analytic is that it be harmonic.
Other necessary and sufficient conditions depend on the sign of the Ricci tensor : for example [28], theorem 9.6 :

If the Ricci tensor is positive definite, there exists no contravariant analytic tensor. This results immediatly from (4.9).

It is known that the number of such harmonic (3,0) forms is given by the Hodge number $h^{(3,0)}$; then this number determines an upper bound for the dimension of the cohomology space of S^0_L in the anomaly sector.

As a first result, this proves that if the manifold $\mathcal{M}$ has a complex dimension smaller than 3, there is no anomaly candidate.

Another special case is the compact Kähler homogeneous one (N=2 supersymmetric extension of our previous works (I) for N=1 susy and [16] for the purely bosonic case) : in such a case the Ricci tensor is positive definite [29] which, due to the aforementioned theorem, forbids the existence of such analytic tensor $t^{[abc]}(\phi^d)$. As a consequence, the cohomology of S^{00}_L - and then of S^0_L - vanishes in the anomaly sector, *i.e.* the Slavnov identity is not anomalous, which means that, as expected, N=2 supersymmetry is renormalizable (at least in the absence of torsion).

Moreover, due to the "stability" [11] of the complex structure and of the classical action in the space of Kähler metrics (subsection 4.2.1), in this case of d=2, N=2 non-linear σ models, the renormalization algorithm *a priori* does not change the number of parameters with respect to the one of the classical action. Of course, as mentioned in subsections 4.1 and 4.2, in the presence of Killing vectors, *i.e.* of extra isometries, the generic symmetric, Kählerian metric tensor $t_{ij}[\Phi]$ gets some more constraints. For example, when the manifold is an homogeneous Kähler one (usual non linear σ models on coset spaces), up to infra-red analysis, our work extends to the N=2 supersymmetric case the renormalizability proof given

[11] The trivial cohomology $S^0_L \Delta_{[-1]}$ corresponds to field and source reparametrisations according to (2.7,2.8).

for the bosonic case in [16] (and for the N=1 case in Section 5 of (I)).

Of course, when $h^{(3,0)} \neq 0$ - which for example occurs for Calabi-Yau manifolds, *i.e.* compact Ricci-flat Kähler manifolds of complex dimension 3, where $h^{(3,0)} = 1$(ref.[3])[12] -, we have a true anomaly candidate [21]. Of course, as no explicit metric is at hand, one cannot compute the anomaly coefficient.

Some comments on its possible vanishing in perturbation theory will be offered in the Concluding Section.

This anomaly in global extended supersymmetry is a surprise with respect to common wisdom [18] (but see other unexpected non-trivial cohomologies in supersymmetric theories in the recent works of Brandt [19] and Dixon [20]) and the fact that if we have chosen, from the very beginning, a coordinate system adapted to the complex structure, the second supersymmetry will be linear and there will be no need for sources η_i . However, as known from chiral symmetry, even a linearly realised transformation can lead to anomalies ; moreover, here the linear supersymmetry transformations do not correspond to an ordinary group but rather to a supergroup where, contrary to ordinary compact groups [13] no general theorems exists : then there is no obvious contradiction. This emphasizes the special structure of the supersymmetry algebra.

5 The cohomology of S_L and N=4 supersymmetry

In subsection 2.1, we have split the complete linearised B.R.S. operator S_L into 3 pieces, according to their number of ghosts $d^{\pm}_{\alpha}$:

$$S_L = S_L^0 + S_L^1 + S_L^2$$

where :

$$\begin{aligned}
S_L^0 &= \int d^2x d^2\theta \{ J^i_j (d^+ D_+ \Phi^j + d^- D_- \Phi^j) \frac{\delta}{\delta \Phi^i} \\
&+ [\frac{\delta A^{inv.}}{\delta \Phi^i} + \eta_k (J^k_{j,i} - J^k_{i,j})(d^+ D_+ \Phi^j + d^- D_- \Phi^j) + J^j_i (d^+ D_+ \eta_j + d^- D_- \eta_j)] \frac{\delta}{\delta \eta_i} \} \, , \\
S_L^1 &= \int d^2x d^2\theta \{ \left[J^i_{\alpha\, j} (d^+_\alpha D_+ \Phi^j + d^-_\alpha D_- \Phi^j) + \eta_j \epsilon_{\alpha\beta 3} J^{ij}_\beta (d^+_\alpha d^- - d^-_\alpha d^+) \right] \frac{\delta}{\delta \Phi^i} \\
&+ [J^j_{\alpha\, i} (d^+_\alpha D_+ \eta_j + d^-_\alpha D_- \eta_j) + \eta_k (J^k_{\alpha\, j,i} - J^k_{\alpha\, i,j})(d^+_\alpha D_+ \Phi^j + d^- D_- \Phi^j) + \\
&+ \frac{1}{2} \eta_k \eta_l \epsilon_{\alpha\beta 3} J^{kl}_{\beta\, ,i} (d^+_\alpha d^- - d^-_\alpha d^+)] \frac{\delta}{\delta \eta_i} \} \, , \\
\text{and } S_L^2 &= -\epsilon_{\alpha\beta 3} d^+_\alpha d^-_\beta \int d^2x d^2\theta \left[\eta_j J^{ij} \frac{\delta}{\delta \Phi^i} + \frac{1}{2} \eta_k \eta_l J^{kl}_{\ \ ,i} \frac{\delta}{\delta \eta_i} \right] \, ,
\end{aligned} \tag{5.1}$$

[12] As det $\|g\| = 1$, a representative of $t^{[abc]}$ is the constant skew-symmetric tensor $\epsilon^{[abc]}$(with $\epsilon^{123} = +1$).
[13] In the appendix A of ref. [16], it is proven that any linearly realised symmetry corresponding to a compact group can be implemented to all orders of perturbation theory.

and the cohomology of S_L^0 was obtained in the previous Section. We start the analysis by the anomaly sector, as the existence of a true Slavnov anomaly would be the death of the theory.

5.1 The Faddeev-Popov +1 charge sector

Filtering the cocycle condition $S_L \Delta_{[+1]} = 0$ with respect to the number $N_{d_\alpha^\pm}$, we know that the cohomology of S_L will be a subspace of the one of S_L^0. We now analyse this restriction of the cohomology space (due to a possible Faddeev-Popov +2 charge anomaly) in successive filtration orders.

5.1.1 $N_{d_\alpha} = 0$

The cocycle condition $S_L^0 \Delta_{[+1]}^0 = 0$ has been solved in subsection 4.3.1 (equ.(4.4)) :

$$\Delta_{[+1]}^0 = \Delta_{[+1]}^{an.(0)}[t^{[ijk]}(\Phi)] + S_L^0 \Delta_{[0]}^0$$

5.1.2 $N_{d_\alpha} = 1$

At this order, and using previous result, the cocycle condition writes :

$$S_L^0(\Delta_{[+1]}^1 - S_L^1 \Delta_{[0]}^0) = -S_L^1 \Delta_{[+1]}^{an.(0)}[t^{[ijk]}(\Phi)] \tag{5.2}$$

From $S_L^0(S_L^1 \Delta_{[+1]}^{an.(0)}) = -S_L^1(S_L^0 \Delta_{[+1]}^{an.(0)}) = 0$, one sees that (5.2) *a priori* enforces some constraints on the tensor $t^{[ijk]}(\Phi)$. From the condition :

$$S_L^1 \Delta_{[+1]}^{an.(0)}[t^{[ijk]}(\Phi)] = S_L^0 \tilde{\Delta}_{[+1]}^1 \tag{5.3}$$

where the general expression for $\tilde{\Delta}_{[+1]}^1$ was given in (3.13), one obtains at decreasing orders in the number of ghosts, firstly a constraint [14] on $t^{[ijk]}$:

$$N_{d_A} = 6 \quad : \quad \nabla_i [J_\alpha^{ij} t^{[klm]}]|_{[j,k,l,m]\, a.s.} = 0 \ ,$$

then $t_\alpha^{[ijk]}(\Phi)$ in function of $t^{[ijk]}(\Phi)$. Using complex coordinates - in particular the fact that $J_{\alpha\, \bar{d}}^{d}(\phi,\bar{\phi}) = \partial_{\bar{d}} J_\alpha^d(\phi,\bar{\phi})$ - , this writes :

$$\partial_{\bar{d}} t_\alpha^{[abc]} = \frac{i}{4} J_{\alpha\, \bar{d}}^{d} \nabla_d t^{[abc]}(\phi) = \partial_{\bar{d}} \left[\frac{i}{4} J_\alpha^d(\phi,\bar{\phi}) \nabla_d t^{[abc]}(\phi) \right] \ ,$$

[14] In the compact Kähler case, we have shown in section 4.4 that $\nabla_l t^{[ijk]} = 0$ (as a consequence of $d\omega' = 0$) . Then this is not a new constraint.

$$t_\alpha^{[\bar{a}bc]} = \frac{i}{4} J^{\bar{a}}_{\alpha\, a}(\phi, \bar{\phi}) t^{[abc]}(\phi) \ ,$$

which exhibits some arbitrariness in $\tilde{\Delta}^1_{[+1]}$, defined up to a S^0_L cocycle $\Delta^{an.(1)}_{[+1]}|_{equ.(4.5)} + S^0_L \Delta^1_{[0]}$. At the next order ($N_{d_A} = 4$) we obtain the vanishing of $t^{[ijk]}(\Phi)$ *i.e.* of $\Delta^{an.(0)}_{[+1]}$.

As a consequence, equ.(5.2) writes : $S^0_L(\Delta^1_{[+1]} - S^1_L \Delta^0_{[0]}) = 0$ which, according to subsection 4.3.2 (equ.(4.5)) solves to :

$$\Delta^1_{[+1]} = \Delta^{an.(1)}_{[+1]}[t_\alpha^{[ijk]}(\Phi),\, \tilde{t}_{\alpha\, [ij]k}(\Phi)] + S^0_L \Delta^1_{[0]} + S^1_L \Delta^0_{[0]} \ . \tag{5.4}$$

5.1.3 $N_{d_\alpha} = 2$

At this order, and using previous results, the cocycle condition writes :

$$S^0_L(\Delta^2_{[+1]} - S^2_L \Delta^0_{[0]} - S^1_L \Delta^1_{[0]}) = -S^1_L \Delta^{an.(1)}_{[+1]}[t_\alpha^{[ijk]},\, \tilde{t}_{\alpha\, [ij]k}] \tag{5.5}$$

Here too, from $S^0_L(S^1_L \Delta^{an.(1)}_{[+1]}) = -S^1_L(S^0_L \Delta^{an.(1)}_{[+1]}) = 0$, one sees that (5.5) *a priori* enforces some constraints on the tensors $t_\alpha^{[ijk]}$ and $\tilde{t}_{\alpha\, [ij]k}$. From the condition :

$$S^1_L \Delta^{an.(1)}_{[+1]}[t_\alpha^{[ijk]},\, \tilde{t}_{\alpha\, [ij]k}] = S^0_L \tilde{\Delta}^2_{[+1]} \tag{5.6}$$

where the general expression for $\tilde{\Delta}^2_{[+1]}$ was given in (3.15) and is defined up to the cocycles $\Delta^{an.(2)}_{[+1]}|_{equ.(4.6)} + S^0_L \Delta^2_{[0]}$, the same analysis as in the previous subsection for $N_{d_A} = 6$, 5 and 4 again gives the vanishing of $t_\alpha^{[ijk]}(\Phi)$ and the "triviality" of the corresponding terms of $\tilde{\Delta}^2_{[+1]}$, as they reduce themselves to S^0_L cocycles. Then $\Delta^{an.(1)}_{[+1]}$ depends only on the tensor $\tilde{t}_{\alpha\, [ij]k}(\Phi)$. So $S^1_L \Delta^{an.(1)}_{[+1]}$ involves at most 3 ghosts and in $\tilde{\Delta}^2_{[+1]}$, only the terms in $t^i_{(\alpha\beta)\, 4[jk]}$, $t'^i_{(\alpha\beta)\, 4[jk]}$, $t^i_{[\alpha\beta]\, 4(jk)}$ and $t^i_{(\alpha\beta)\, 5j}$, $t^i_{[\alpha\beta]\, 5j}$ survive.

When analysed at ghost level 3 and 2, the condition (5.6) does not lead to the vanishing of these tensors, but expresses them as functions of the complex structure $J^i_{\alpha\, j}$ and the "anomaly tensor" $\tilde{t}_{\alpha\, [ij]k}$. For example, one finds (in complex coordinates) :

$$t^a_{(\alpha\beta)\, 4[bc]} = t^a_{(\alpha\beta)\, 4[b\bar{c}]} = 0 \ , \ t^a_{(\alpha\beta)\, 4[\bar{b}\bar{c}]} = \frac{1}{4}\left(J^{ad}_\alpha \tilde{t}_{\beta\, [\bar{b}\bar{c}]d} + (\alpha \leftrightarrow \beta)\right) \ ,$$

$$t^a_{(\alpha\beta)\, 5\, \bar{b}} = 0 \ , \ t^a_{(\alpha\beta)\, 5\, b, \bar{c}} = \frac{1}{2}\left(J^{ad}_\alpha \tilde{t}_{\beta\, [db]\bar{c}} + (\alpha \leftrightarrow \beta)\right) \ ,$$

and constraints such as :

$$\left(J^{\bar{a}}_{\alpha\, b} \tilde{t}_{\beta\, [cd]\bar{a}} + J^{\bar{a}}_{\alpha\, c} \tilde{t}_{\beta\, [db]\bar{a}} + J^{\bar{a}}_{\alpha\, d} \tilde{t}_{\beta\, [bc]\bar{a}} + (\alpha \leftrightarrow \beta)\right) = 0 \ .$$

Finally, from (5.5,5.6) and subsection 4.3.3, one gets :

$$\Delta^2_{[+1]} = \Delta^{an.(2)}_{[+1]}[t^{[ijk]}_{(\alpha\beta)}, t^i_{(\alpha\beta)\, 5j}] - \tilde{\Delta}^2_{[+1]} + S^0_L \Delta^2_{[0]} + S^1_L \Delta^1_{[0]} + S^2_L \Delta^0_{[0]} \ , \tag{5.7}$$

where the tensor $G_{(\alpha\beta)\, ij} = g_{ik} t^k_{(\alpha\beta)\, 5j}$ is a symmetric, covariantly constant tensor, hermitian with respect to the complex structure J^i_j (subsection 4.3.3).

5.1.4 $N_{d_\alpha} = 3$

At this order, and using previous results, the cocycle condition writes :

$$S_L^0 \ \left(\Delta^3_{[+1]} - S_L^2\Delta^1_{[0]} - S_L^1\Delta^2_{[0]}\right) = -S_L^2\left(\Delta^{an.(1)}_{[+1]}[\tilde{t}_{\alpha\,[ij]k}]\right) -$$
$$- \ S_L^1 \ \left(\tilde{\tilde{\Delta}}^2_{[+1]}[t^{an.[ijk]}_{(\alpha\beta)}, t^{an.\,i}_{(\alpha\beta)\,5j} \ ; \ t^i_{(\alpha\beta)\,4[jk]}, t'^i_{(\alpha\beta)\,4[jk]}, t^i_{[\alpha\beta]\,4(jk)}, t^i_{(\alpha\beta)\,5j}, t^i_{[\alpha\beta]\,5j}]\right) \tag{5.8}$$

where $\tilde{\tilde{\Delta}}^2_{[+1]} = \Delta^{an.(2)}_{[+1]} - \tilde{\Delta}^2_{[+1]}$. Here too, from $S_L^0(S_L^2\Delta^{an.(1)}_{[+1]} + S_L^1\tilde{\tilde{\Delta}}^2_{[+1]}) = -S_L^1 S_L^1 \Delta^{an.(1)}_{[+1]} - S_L^1 S_L^0 \tilde{\tilde{\Delta}}^2_{[+1]} \overset{(5.6)}{=} 0$, one sees that (5.8) *a priori* enforces some new constraints on the tensor $\tilde{t}_{\alpha\,[ij]k}$. From the condition :

$$S_L^2\Delta^{an.(1)}_{[+1]} + S_L^1\tilde{\tilde{\Delta}}^2_{[+1]} = S_L^0\tilde{\Delta}^3_{[+1]} \tag{5.9}$$

where the general expression for $\tilde{\Delta}^3_{[+1]}$ was given in (3.17), and thanks to the special structure of S_L^2 (5.1), one obtains through analysis of the terms in $d^+_\alpha d^+_\beta d^+_\gamma$, at the 6 and 5 order in the total number of ghosts, the vanishing of the part in $t^{an.[ijk]}_{(\alpha\beta)}(\Phi)$ of the anomaly $\Delta^{an.(2)}_{[+1]}$ involved in $\tilde{\tilde{\Delta}}^2_{[+1]}$.

Then, the analysis of the terms in $d^+_\alpha d^+_\beta d^+_\gamma d^-$ gives :

- the symmetry, in the exchange i $\leftrightarrow$ j, of : $\epsilon_{3\alpha\delta} J^{ki}_\delta \left[t^{an.\,j}_{(\beta\gamma)5\,k} + t^j_{(\beta\gamma)5\,k}\right]|_{(\alpha\beta\gamma)\ sym.}$,
- the vanishing of $t^k_{(\beta\gamma)4\,[ij]}$, and then : $\tilde{t}_{\alpha\,[ij]k} = 0$ *i.e.* $\Delta^{an.(1)}_{[+1]} = 0$.

As a consequence, equ.(5.6) gives : $S_L^0\tilde{\Delta}^2_{[+1]} = 0$, and, thanks to previous remarks, $\tilde{\Delta}^2_{[+1]}$ may be supressed, *i.e.* $t^i_{(\beta\gamma)\,5j} \equiv 0$. Then, the constraint (5.9) reduces itself to :

$$S_L^1\Delta^{an.(2)}_{[+1]}[t^{an.\,i}_{(\alpha\beta)5\,j}(\Phi)] = S_L^0\tilde{\Delta}^3_{[+1]} \tag{5.10}$$

In the analysis of the terms in $d^-_\alpha d^+_\beta d^+_\gamma$, equ.(5.10) may be shown to enforce a stronger constraint on $t^{an.\,i}_{(\alpha\beta)5\,j}(\Phi)$:

$$J^i_{\alpha\,k} t^{an.\,k}_{(\beta\gamma)5\,j} = J^k_{\alpha\,j} t^{an.\,i}_{(\beta\gamma)5\,k} \ \Leftrightarrow \ J^i_{\alpha\,k} G_{(\beta\gamma)\,ij} = -G_{(\beta\gamma)\,ik} J^i_{\alpha\,j} \quad \forall\ \alpha,\ \beta,\ \gamma\ , \tag{5.11}$$

i.e. the tensor $G_{(\alpha\beta)\,ij}$ is hermitian with respect to the complex structures $J^i_{\alpha\,k}$. Moreover, $\tilde{\Delta}^3_{[+1]} \equiv 0$. Finally, from (5.8,5.9) and subsection 4.3.4, one gets :

$$S_L^1\Delta^{an.(2)}_{[+1]} = 0 \ ; \ \Delta^3_{[+1]} = S_L^1\Delta^2_{[0]} + S_L^2\Delta^1_{[0]} \ . \tag{5.12}$$

5.1.5 $N_{d_\alpha} = 4$

At this order, and using previous results, the cocycle condition writes :

$$S_L^0(\Delta^4_{[+1]} - S_L^2\Delta^2_{[0]}) = -S_L^2\Delta^{an.(2)}_{[+1]}[t^{an.\,i}_{(\alpha\beta)\,j}] \tag{5.13}$$

Here too, from :

$$S_L^0(-S_L^2\Delta_{[+1]}^{an.(2)}) = S_L^1(S_L^1\Delta_{[+1]}^{an.(2)}) \stackrel{(5.12)}{=} 0\ ,$$

one sees that (5.13) *a priori* enforces some constraint on the tensor $t^{an.\,i}_{(\alpha\beta)5\,j}$. From the condition :

$$S_L^2\Delta_{[+1]}^{an.(2)} = S_L^0\tilde{\Delta}_{[+1]}^4 \tag{5.14}$$

where $\tilde{\Delta}_{[+1]}^4$ was given in (3.19), and the structures of S_L^2 (5.1) and $\Delta_{[+1]}^{an.(2)}$, one sees that the right hand side is proportional to $d_\alpha^+ d_\beta^- d_\gamma^+ d_\delta^-$ when no such terms may occur in the left hand side, which means that both sides should vanish. As a consequence, $\tilde{\Delta}_{[+1]}^4$ is a S_L^0 cocycle, which means that it vanishes (subsection 4.3.5).

Finally, from (5.13,5.14) and subsection 4.3.5, one gets :

$$S_L^2\Delta_{[+1]}^{an.(2)} = 0 \ ; \ \Delta_{[+1]}^4 = S_L^2\Delta_{[0]}^2\ . \tag{5.15}$$

5.1.6 Absence of N=4 supersymmetry anomaly

Putting the results of the previous subsections 5.1.1-5 together shows that the cohomology space of S_L in the Faddeev-Popov charge +1 sector depends on a single symmetric tensor $G_{(\alpha\beta)\,ij}[\Phi]$, hermitian with respect to the 3 complex structures $J^i_{A\,j}$ and covariantly constant :

$$\begin{aligned} \Delta_{[+1]} &= \Delta_{[+1]}^{an.(2)} + S_L\Delta_{[0]} \text{ where } S_L^0\Delta_{[+1]}^{an.(2)} = S_L^1\Delta_{[+1]}^{an.(2)} = S_L^2\Delta_{[+1]}^{an.(2)} = 0\,, \\ \Delta_{[+1]}^{an.(2)} &= \int d^2x d^2\theta \eta_i g^{ij} G_{(\alpha\beta)\,jk}\left(d_\alpha^+ d_\beta^+(D_+)^2\Phi^j + d_\alpha^- d_\beta^-(D_-)^2\Phi^j\right)\ . \end{aligned} \tag{5.16}$$

This is reminiscent of the right hand side of the classical Slavnov identity (2.5). As a matter of fact, it appears that $\Delta_{[+1]}^{an.(2)}$ is a S_L cocycle :

$$\begin{aligned} \Delta_{[+1]}^{an.(2)} &= S_L\left(\Delta_{[0]}[G_{(\alpha\beta)\,ij}(\Phi)]\right) \\ \Delta_{[0]} &= \int d^2x d^2\theta\{aG_{(\alpha\alpha)\,ij}D_+\Phi^i D_-\Phi^j - \frac{1}{2}\eta_i J_\alpha^{ik} G_{(\alpha\beta)\,kj}(d_\beta^+ D_+\Phi^j + d_\beta^- D_-\Phi^j) + \\ &+ \frac{1}{2}\eta_i\eta_j\epsilon_{ABC} d_A^+ d_B^-[(a+\frac{1}{2})J_C^{ik} G_{(\gamma\gamma)\,kl} g^{lj} - \frac{1}{2}J_\gamma^{ik} G_{(\gamma C)\,kl} g^{lj}]\} \end{aligned} \tag{5.17}$$

where a is an arbitrary constant. Then there is no Faddeev-Popov +1 charge cohomology, and we get the absence of supersymmetry anomaly in N=4 non-linear σ models in 2 space-time dimensions.

5.2 The Faddeev-Popov neutral charge sector

As explained before and in the appendix of (I), the existence of a non trivial cohomology for S_L^0 in the Faddeev-Popov +1 charge sector, will restrict the cocycles of S_L in the

Faddeev-Popov 0 charge sector ; this is easily understood as N=4 supersymmetry enforces new constraints on the arbitrary Kählerian metric g_{ij} corresponding to N=2 supersymmetry.

We now filter the cocycle condition $S_L\Delta_{[0]} = 0$ with respect to the number $N_{d^{\pm}_{\alpha}}$.

5.2.1 $N_{d_\alpha} = 0$

The cocycle condition $S^0_L\Delta^0_{[0]} = 0$ has been solved in subsection 4.2.1 :

$$\Delta^0_{[0]} = \Delta^{an.(0)}_{0]}[t_{(ij)}(\Phi)] + S^0_L\Delta^0_{[-1]}[V^i(\Phi)]$$

5.2.2 $N_{d_\alpha} = 1$

At this order, and using previous result, the cocycle condition writes :

$$S^0_L(\Delta^1_{[0]} - S^1_L\Delta^0_{[-1]}[V^i]) = -S^1_L\Delta^{an.(0)}_{[0]}[t_{(ij)}(\Phi)] \tag{5.18}$$

From $S^0_L(S^1_L\Delta^{an.(0)}_{[0]}) = -S^1_L(S^0_L\Delta^{an.(0)}_{[0]}) = 0$, and the existence of an S^0_L anomaly $\Delta^{an.(1)}_{[+1]}$, one sees that (5.18) enforces some constraints on the tensor $t_{(ij)}(\Phi)$:

$$S^1_L\Delta^{an.(0)}_{[0]}[t_{(ij)}] = S^0_L\tilde{\Delta}^1_{[0]}[\tilde{U}^i_{\alpha j}, \tilde{S}^{[ij]}_{\alpha}] \tag{5.19}$$

where the general expression for $\tilde{\Delta}^1_{[0]}$ was given in (3.3). As a consequence, equ.(5.18) writes : $S^0_L(\Delta^1_{[0]} - S^1_L\Delta^0_{[-1]} + \tilde{\Delta}^1_{[0]}) = 0$ which, according to subsection 4.2.2 (see in particular equ.(4.3)), solves to :

$$\begin{aligned}
\Delta^1_{[0]} &= \Delta^{an.(1)}_{[0]}[U^{an.\,i}_{\alpha j},\ S^{an.\,[ij]}_{\alpha}] - \tilde{\Delta}^1_{[0]}[\tilde{U}^i_{\alpha j}, \tilde{S}^{[ij]}_{\alpha}] + S^1_L\Delta^0_{[-1]}[V^i] \\
&\equiv \tilde{\tilde{\Delta}}^1_{[0]}[\tilde{\tilde{U}}^i_{\alpha j}, \tilde{\tilde{S}}^{[ij]}_{\alpha}] + S^1_L\Delta^0_{[-1]}[V^i] \ .
\end{aligned} \tag{5.20}$$

$\tilde{\tilde{\Delta}}^1_{[0]}$ being defined by equation (5.19) up to a S^0_L cocycle, *i.e.* up to $\Delta^{an.(1)}_{[0]}$, the constraint (5.19) especially implies the <u>hermiticity</u> of the perturbed metric $g'_{ij} = g_{ij} + \hbar t_{(ij)}$ with respect to the perturbed complex structures $J'^i_{\alpha j} = J^i_{\alpha j} + \hbar\tilde{\tilde{U}}^i_{\alpha j}$ and the <u>covariant constancy</u> of the later with respect to the covariant derivative with connexion corresponding to g'_{ij} . Moreover, the tensor $\tilde{\tilde{S}}^{[ij]}_{\alpha}$ is a pure contravariant analytic skew-symmetric tensor, with components given in complex coordinates by :

$\tilde{\tilde{S}}^{[ab]}_{\alpha} = -\frac{i}{4}g^{a\bar{c}}\left[\tilde{\tilde{U}}^{\ b}_{\alpha\,\bar{c}} + t_{\bar{c}d}J^{db}_{\alpha}\right] - (a \leftrightarrow b) \ \equiv \tilde{\tilde{S}}^{[ab]}_{\alpha}(\phi^d)$ and the complex conjugate relation,

(compare to equ.(4.3)), and one obtains the relations :

$$T_{\alpha\,[ab]} = -\frac{i}{4}\left[g_{a\bar{c}}\tilde{\tilde{U}}^{\bar{c}}_{\alpha\,b} + t_{a\bar{c}}J^{\bar{c}}_{\alpha\,b}\right] - (a \leftrightarrow b) \ \ \equiv T_{\alpha\,[ab]}(\phi^d)$$

and

$$\tilde{\tilde{U}}^{b}_{\alpha\bar{a}}\left(\text{resp. } \tilde{\tilde{S}}^{[ab]}_{\alpha}\right) = -i\epsilon_{\alpha\beta 3}\tilde{\tilde{U}}^{b}_{\beta\bar{a}}\left(\text{resp. } \tilde{\tilde{S}}^{[ab]}_{\beta}\right)$$

(compare to equ.(2.3) which implies : $J^{b}_{\alpha\bar{a}} = -i\epsilon_{\alpha\beta 3}J^{b}_{\beta\bar{a}}$) and the complex conjugate relations.

5.2.3 $N_{d_\alpha} = 2$

At this order, and using previous results, the cocycle condition writes :

$$S^0_L(\Delta^2_{[0]} - S^2_L\Delta^0_{[-1]}) = -S^2_L\Delta^{an.(0)}_{[0]} - S^1_L\tilde{\tilde{\Delta}}^1_{[0]} \tag{5.21}$$

Here too, from

$$S^0_L\left(S^2_L\Delta^{an.(0)}_{[0]} + S^1_L\tilde{\tilde{\Delta}}^1_{[0]}\right) = -S^1_L(S^1_L\Delta^{an.(0)}_{[0]} - S^0_L\tilde{\Delta}^1_{[0]}) \overset{(5.19)}{=} 0\,,$$

and the existence of an S^0_L anomaly $\Delta^{an.(2)}_{[+1]}$, one sees that (5.21) enforces some constraints on the tensors $t_{(ij)}$, $\tilde{\tilde{U}}^{i}_{\alpha j}$, $\tilde{\tilde{S}}^{[ij]}_{\alpha}$:

$$S^2_L\Delta^{an.(0)}_{[0]}[t_{(ij)}] + S^1_L\tilde{\tilde{\Delta}}^1_{[0]}[\tilde{\tilde{U}}^{i}_{\alpha j}, \tilde{\tilde{S}}^{[ij]}_{\alpha}] = S^0_L\tilde{\Delta}^2_{[0]}[\tilde{S}^{[ij]}_{[\alpha\beta]}] \tag{5.22}$$

where the general expression for $\tilde{\Delta}^2_{[0]}$ was given in (3.3). As a consequence of (5.22), equ.(5.21) writes : $S^0_L(\Delta^2_{[0]} - S^2_L\Delta^0_{[-1]} + \tilde{\Delta}^2_{[0]}) = 0$ which, according to subsection 4.2.3 solves to :

$$\Delta^2_{[0]} = -\tilde{\Delta}^2_{[0]}[\tilde{S}^{[ij]}_{[\alpha\beta]}] + S^2_L\Delta^0_{[-1]}[V^i]\ . \tag{5.23}$$

The constraint (5.22) especially implies the anticommutation of the perturbed complex structures $J'^{i}_{\alpha j}$ and $J'^{i}_{\beta j}$ and the relation :

$$\tilde{S}^{[ij]}_{[\alpha\beta]} = -\frac{1}{2}\epsilon_{\alpha\beta 3}J^{im}_{3}t_{(mn)}g^{nj}\ .$$

5.2.4 $N_{d_\alpha} = 3$

At this order, and using previous results, the cocycle condition writes :

$$\begin{aligned} S^2_L\Delta^1_{[0]} + S^1_L\Delta^2_{[0]} &= 0 \\ \Leftrightarrow\ S^1_L\tilde{\Delta}^2_{[0]}[\tilde{S}^{[ij]}_{[\alpha\beta]}] &= S^2_L\tilde{\tilde{\Delta}}^1_{[0]}[\tilde{\tilde{U}}^{i}_{\alpha j},\ \tilde{\tilde{S}}^{[ij]}_{\alpha}]\,. \end{aligned} \tag{5.24}$$

As a matter of fact, this may be shown to result from previous constraints.

5.2.5 $N_{d_\alpha} = 4$

At this order, and using previous results, the cocycle condition writes :

$$S_L^2 \Delta_{[0]}^2 = 0 \quad \Leftrightarrow \quad S_L^2 \tilde{\Delta}_{[0]}^2 [\tilde{S}_{[\alpha\beta]}^{[ij]}] = 0\,, \tag{5.25}$$

which may be shown to result from the Ricci-flatness of g'_{ij} (det $\|g'\|$ = constant), which itself results from the HyperKähler nature of the perturbed metric g'_{ij}, *i.e.* to results of the previous subsections.

5.2.6 Stability of N=4 supersymmetric non-linear σ models

Putting the results of the previous subsections 5.2.1-5 together gives :

$$\Delta_{[0]} = \Delta_{[0]}^{an.(0)}[t_{(ij)}(\Phi)] + \tilde{\tilde{\Delta}}_{[0]}^1[\tilde{\tilde{U}}^i_{\alpha j}(\Phi),\ \tilde{\tilde{S}}_\alpha^{[ij]}(\Phi)] - \tilde{\Delta}_{[0]}^2[\tilde{S}_{[\alpha\beta]}^{[ij]}(\Phi)] + S_L \Delta_{[-1]}[V^i(\Phi)] \tag{5.26}$$

Up to a trivial field and source reparametrisations (2.7,2.8), the most general Slavnov invariant integrated functional of the fields, sources and their derivatives, of Faddeev-Popov 0 charge is :

$$\begin{aligned}
\Gamma'^{class.} &= \Gamma^{class.} + \hbar\Delta_{[0]} \equiv \int d^2x d^2\theta \{ g'_{ij} D_+\Phi^i D_-\Phi^j + \\
&+ \eta_i J'^i_{A\,j}[d_A^+ D_+\Phi^j + d_A^- D_-\Phi^j] - \frac{1}{2}\epsilon_{ABC}\eta_i\eta_j J'^{ij}_C d_A^+ d_B^- \}
\end{aligned} \tag{5.27}$$

and where the results of the previous subsections consistently give :

$$\begin{aligned}
g'_{ij}[\Phi] &= g_{ij}[\Phi] + \hbar t_{(ij)}\ ,\quad J'^i_{3\,j} = J^i_{3\,j}\ ,\quad J'^i_{\alpha\,j}(\Phi) = J^i_{\alpha\,j}(\Phi) + \hbar\tilde{\tilde{U}}^i_{\alpha\,j}(\Phi)\,, \\
J'^{ij}_3(\Phi) &= J'^i_{3\,k} g'^{kj} = J^{ij}_3 - \hbar J^i_n g^{nm} t_{(mr)} g^{rj} \overset{\text{subsect. 5.2.3}}{=} J^{ij}_3(\Phi) + 2\hbar\epsilon_{\alpha\beta 3}\tilde{S}^{[ij]}_{[\alpha\beta]}(\Phi)\,, \\
J'^{ij}_\alpha(\Phi) &= J'^i_{\alpha\,k} g'^{kj} = J^{ij}_\alpha + \hbar[\tilde{\tilde{U}}^i_{\alpha\,k} g^{kj} - J^{im}_\alpha t_{(mn)} g^{nj}] \overset{\text{subsect. 5.2.2}}{=} J^{ij}_\alpha(\Phi) - 2\hbar\epsilon_{\alpha\beta 3}\tilde{\tilde{S}}_\beta^{[ij]}(\Phi)\,,
\end{aligned} \tag{5.28}$$

and the different constraints (5.19 and 5.22) may be shown to enforce N = 4 supersymmetry : *i.e.* $g'_{ij}[\Phi]$ is a symmetric metric tensor, the three J'_A 's offer a set of anticommuting, integrable and covariantly constant (with respect to the covariant derivative with connexion Γ'^i_{jk} corresponding to the metric g'_{ij}) complex structures satisfying a quaternionic multiplication law, and the metric is hermitian with respect to each of these complex structure $J'^i_{A\,j}(\Phi)$.

This proves the stability of the theory, and, thanks to the absence of anomaly, the full renormalizability of N = 4 supersymmetric non-linear σ models in two space-time dimensions.

Of course, due to a possible Faddeev-Popov -1 charge non-trivial [15] cohomology for S_L, some of the parameters of the action (5.27) may be unphysical ones . $\Delta_{[-1]}(V^i[\Phi])$ being

[15] The trivial cohomology $S_L\Delta_{[-1]}$ corresponds to field and source reparametrisations according to (2.7,2.8).

independent of $d_\alpha^{\pm}$, the cocycle condition $S_L\Delta_{[-1]}(V^i[\Phi]) = 0$ splits into the three ones $S_L^i\Delta_{[-1]}(V^i[\Phi]) = 0$, $i = 0, 1, 2$. One easily checks that $V^i[\Phi]$ should be a contravariant Killing vector for the metric g_{ij}, holomorphic with respect to the three complex structures :

$$g_{kj}\nabla_i V^k + g_{ki}\nabla_j V^k = 0 \ ; \ \ J_A^{ij}\nabla_i V^k = J_A^{ik}\nabla_i V^j \ , A = 1, 2, 3. \tag{5.29}$$

6 Concluding remarks

In the second paper of this series, we have analysed the cohomology of the B.R.S. operator associated to N = 2 and N = 4 supersymmetry in a N = 1 superfield formalism. We found an anomaly candidate for torsionless models built on compact Kähler Ricci-flat target spaces with a non vanishing Hodge number $h^{(3,0)}$. Calabi-Yau manifolds (3 complex dimensional case) where $h^{(3,0)} = 1$ (ref.[3]) are interesting examples due to their possible relevance for superstring theories. Of course, as no explicit metric is at hand, one cannot compute the anomaly coefficient.

This anomaly in global supersymmetry is a surprise with respect to common wisdom [18]. But some recent works of Brandt [19] and Dixon [20] also show the existence of new non-trivial cohomologies in supersymmetric theories and we have argued in Section 4 that the special structure of the supersymmetry algebra which does not correspond to an ordinary group but rather to a supergroup may be responsible for this peculiarity.

Our analysis then casts some doubts on the validity of the previous claims on U.V. properties of N=2 supersymmetric non linear σ models (see for example [4] or [6]) : there, the possible occurence, at 4-loops order, of (infinite) counterterms non-vanishing on-shell, even for Kähler Ricci-flat manifolds, did not " disturb" the complex structure. On the other hand, we have found a possible " instability" of the second supersymmetry, which confirms that there are some difficulties in the regularization of supersymmetry by dimensional reduction assumed in explicit perturbative calculations [5] as well as in finiteness "proofs" [4] or in higher order counterterms analysis [6]. We would like to emphasize the difference between Faddeev-Popov 0 charge cohomology which describes the stability of the classical action against radiative corrections (the usual "infinite" counterterms) and which offers no surprise, and the anomaly sector which describes the "stability" of the symmetry (the finite renormalizations which are needed, in presence of a regularization that does not respect the symmetries of the theory, to restore the Ward identities) : of course, when at a given perturbative order the Slavnov (or Ward) identities are spoiled, at the next order, the analysis of the structure of the divergences is no longer under control. In particular, the Calabi-Yau uniqueness theorem for the metric [30] supposes that one stays in the same cohomology class for the Kähler form, a fact which is not certain in the absence of a regularization that respects the N=2 supersymmetry (the possible anomaly we found expresses the impossibility of finding a regularization that respects all the symmetries of these theories).

We emphasize that the present work relies heavily on a perturbative analysis of the possible breakings of the Slavnov identity, especially through the use of the Quantum Action Principle. It may well happen that the coefficient of the anomaly candidate vanishes at any finite perturbative order [16]. However, the possibility of a non-perturbative breaking of the N=2 supersymmetry would remain open.

Of course, if one added from the very beginning extra geometrical (or physical !) constraints that would fix the classical action, we bet that our anomaly candidate would disappear : as previously mentioned, this is the case when the manifold is a compact homogeneous Kähler space.

Moreover we have been able to give the first algebraic, regularization free proof that, if one enforces N=4 supersymmetry (HyperKähler manifolds), there is no supersymmetry anomaly [17] and that the corresponding non-linear σ models are all-orders renormalizable, "à la Friedan".

The last step of our program will be the rigorous proof of the all-orders finiteness of these models. We hope to be able to report on that subject in a near future.

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