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The Connection Between the Algebraic and the Original Bethe Ansatz for the Six-Vertex Model

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Abstract. It is proved that the algebraic Bethe Ansatz method developed by Faddeev et.al. yields the same eigenfunctions for the transfer matrix of the 6-vertex model as the original coordinate Bethe Ansatz method invented by Bethe and used by Lieb in his original solution of this model.

1. The Problem.

The six-vertex model is a model of statistical mechanics defined on a two-dimensional square lattice in which the configurations are given by arrows on the bonds or links between neighbouring lattice points. The following configurations are allowed:

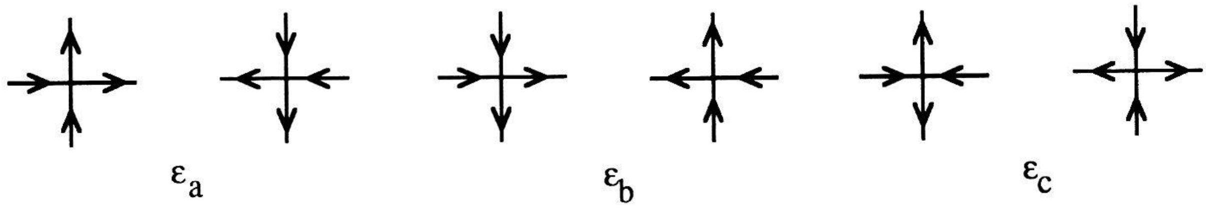

Figure 1. Allowed vertex configurations in the 6-vertex model and their energies

Assigning energies ϵ_a, ϵ_b and ϵ_c to these configurations as indicated in the figure, the total energy of a given configuration Γ becomes:

$$E_\Gamma = n_a(\Gamma)\epsilon_a + n_b(\Gamma)\epsilon_b + n_c(\Gamma)\epsilon_c, \tag{1.1}$$

where $n_a(\Gamma)$ denotes the number of vertices of type a in the configuration Γ, etc. Let M be the number of rows and N the number of columns of the lattice. The partition function of the six-vertex model is then given by

$$Z_{M,N}(a,b,c) = \sum_{\Gamma}{}' \exp\left[-\beta E_\Gamma\right], \tag{1.2}$$

where the sum runs over all allowed configurations, and β is the inverse temperature. Notice that $Z_{M,N}$ only depends on the Boltzmann factors

$$a = \exp[-\beta\epsilon_a], \quad b = \exp[-\beta\epsilon_b], \quad c = \exp[-\beta\epsilon_c]. \tag{1.3}$$

The model was solved in several special cases by Lieb [1] in 1967 and subsequently in all generality by Sutherland [2]. By 'solving' the model we mean here: obtaining an exact and explicit expression for the free energy density in the thermodynamic limit:

$$f_\beta(a,b,c) = -\frac{1}{\beta} \lim_{M,N\to\infty} \frac{1}{MN} \ln Z_{M,N}(a,b,c). \tag{1.4}$$

Assuming periodic boundary conditions, the partition function can be written in terms of the transfer matrix V, defined as the contribution of one row of vertices. Let $L_\mu^{\mu'}(\nu,\nu')$ denote the Boltzmann factor of a vertex with horizontal arrows ν and ν' and vertical arrows μ and μ'. Then the contribution of one row of vertices with lower row of vertical arrows given by $\underline{\mu} = (\mu_1,\ldots,\mu_N)$ and upper row of vertical arrows by $\underline{\mu}' = (\mu_1',\ldots,\mu_N')$ is:

$$V_{\underline{\mu},\underline{\mu}'} = \sum_{\underline{\nu}} \prod_{n=1}^{N} L_{\mu_n}^{\mu_n'}(\nu_n,\nu_{n+1}). \tag{1.5}$$

(For periodic boundary conditions, $\nu_{N+1} = \nu_1$.) The partition function becomes:

$$Z_{M,N} = \sum_{\underline{\mu}^1} \cdots \sum_{\underline{\mu}^M} \prod_{m=1}^{M} V_{\underline{\mu}^m,\underline{\mu}^{m+1}} = \operatorname{Trace} V_N^M. \tag{1.6}$$

It follows that the free energy density (1.4) is given by

$$f_\beta = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \ln \Lambda_{\max}(N), \tag{1.7}$$

where $\Lambda_{\max}(N)$ is the maximum eigenvalue of the operator V_N with matrix elements (1.5) w.r.t. the canonical basis of the N-spin Hilbert space $\mathcal{H}_N = \mathbf{C}^{2^N}$. (In fact (1.7) needs a proof because the limit in (1.4) is intended in the sense of Van Hove (see [3]), i.e. M and N must tend to infinity simultaneously. It was proved by Lieb and Wu [4] that this is in fact equivalent to taking the limits separately. Another problem is the equivalence of periodic and free boundary conditions. This is more difficult than the analogous problem for spin models (see [3]) because the set of allowed configurations imposes a nonlocal constraint. It was solved by Brascamp et.al. [5]. (A simple proof in the symmetric case was given by the author [6].) Lieb showed that V_N can be diagonalised by means of the Bethe Ansatz, originally proposed by Bethe [7] for diagonalising the Hamiltonian of the Heisenberg chain. The Bethe Ansatz eigenfunctions are given by:

$$\Psi^{(n)}(k_1, \ldots, k_n) = \sum_{1 \le r_1 < \ldots < r_n \le N} \psi_{\{k_1,\ldots,k_n\}}(r_1, \ldots, r_n) |r_1, \ldots, r_n\rangle, \tag{1.8}$$

where

$$\psi_{\{k_1,\ldots,k_n\}}(r_1, \ldots, r_n) = \sum_{\sigma \in \mathcal{S}_n} A_\sigma \prod_{j=1}^{N} \exp\left[ik_{\sigma(j)} r_j\right]. \tag{1.9}$$

Here, $|r_1, \ldots, r_n\rangle$ denotes the state of $\mathcal{H}$ with n down spins at the positions $r_1, \ldots, r_n$, and $k_1, \ldots, k_n$ are wave numbers. In order that (1.8) is an eigenstate, the latter must satisfy the Bethe Ansatz equations:

$$\exp\left[iNk_j\right] = (-1)^{n-1} \prod_{l=1}^{n} \exp\left[-i\theta(k_j, k_l)\right], \tag{1.10}$$

and the coefficients A_σ must satisfy

$$\frac{A_\sigma}{A_{\sigma'}} = -\exp\left[-i\theta(k_{\sigma(l)}, k_{\sigma(l+1)})\right] \tag{1.11}$$

if σ and σ' differ by a transposition: $\sigma'(j) = \sigma(j)$ if $j \neq l, l+1$, $\sigma'(l) = \sigma(l+1)$ and $\sigma'(l+1) = \sigma(l)$. Here the function $\theta(k, k')$ is given by

$$\exp\left[-i\theta(k, k')\right] = \frac{1 - 2\Delta e^{ik} + e^{i(k+k')}}{1 - 2\Delta e^{ik'} + e^{i(k+k')}}, \tag{1.12}$$

with

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}. \tag{1.13}$$

The inverse scattering method or algebraic Bethe Ansatz method is based on an observation by Baxter that there is a 1-parameter family of models with Boltzmann weights $(a(\lambda), b(\lambda), c(\lambda))$ such that the corresponding transfer matrices $V_N(\lambda)$ commute: $[V_N(\lambda), V_N(\mu)] = 0$. He deduced a sufficient condition for this to be the case and used it to solve the more general 8-vertex model [8]. His formalism was subsequently simplified by Faddeev and Takhtadzhan [9]. Their formulation of Baxter's star-triangle relation says that there exists a scalar 4×4 matrix $R(\lambda, \lambda')$ such that

$$R(\lambda, \lambda')\,(L_n(\lambda) \otimes L_n(\lambda')) = (\mathbf{1} \otimes L_n(\lambda'))(L_n(\lambda) \otimes \mathbf{1}) R(\lambda, \lambda'), \tag{1.14}$$

where $L_n(\lambda)$ is a 2×2-matrix with operator entries given by

$$L_n(\lambda) = \begin{pmatrix} \alpha_n(\lambda) & \beta_n \\ \gamma_n & \delta_n(\lambda) \end{pmatrix} \tag{1.15}$$

with

$$\alpha_n(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ 0 & b(\lambda) \end{pmatrix}, \qquad \beta_n = \begin{pmatrix} 0 & 0 \\ c(\lambda) & 0 \end{pmatrix}$$
$$\gamma_n = \begin{pmatrix} 0 & c(\lambda) \\ 0 & 0 \end{pmatrix}, \qquad \delta_n(\lambda) = \begin{pmatrix} b(\lambda) & 0 \\ 0 & a(\lambda) \end{pmatrix} \tag{1.16}$$

operating on the n–th spin in $\mathcal{H}_N$. Notice that the entries of $L_n(\lambda)$ are exactly the Boltzmann factors $L_{\mu_n,\mu'_n}(\nu_n, \nu_{n+1})$. The transfer matrix is given by (1.5):

$$V_N(\lambda) = \text{Trace}_2(L_1(\lambda) \dots L_N(\lambda)), \tag{1.17}$$

where the trace is taken of a 2×2–matrix, the result being an operator on $\mathcal{H}_N$. We define

$$T_N(\lambda) = L_1(\lambda) \dots L_N(\lambda). \tag{1.18}$$

As this is a 2×2–matrix with operator entries it can be written as

$$T_N(\lambda) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix}. \tag{1.19}$$

The 1-parameter family $(a(\lambda), b(\lambda), c(\lambda))$ is given by the transformation of variables

$$\begin{cases} a(\lambda) = \sin(\lambda + \eta), \\ b(\lambda) = \sin(\lambda - \eta), \\ c = \sin 2\eta. \end{cases} \tag{1.20}$$

(We have divided by an unimportant normalisation factor.) The R-matrix is given by

$$R(\lambda, \mu) = \begin{pmatrix} f(\lambda, \mu) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda, \mu) & 0 \\ 0 & g(\lambda, \mu) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda, \mu) \end{pmatrix}, \tag{1.21}$$

where the functions f and g are defined by

$$f(\lambda, \mu) = \frac{\sin(\lambda - \mu + 2\eta)}{\sin(\lambda - \mu)}$$
$$g(\lambda, \mu) = \frac{\sin 2\eta}{\sin(\lambda - \mu)}. \tag{1.22}$$

One easily checks that the relation (1.14) is satisfied. If $e_n^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes the up-spin at position n and $e_n^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the down spin at n, let $\Omega = e_1^+ \otimes e_2^+ \otimes \dots \otimes e_N^+$ be the state with all spins up (ground state). Then

$$\alpha_n(\lambda) e_n^+ = a(\lambda) e_n^+$$
$$\delta_n(\lambda) e_n^+ = b(\lambda) e_n^+ \tag{1.23}$$

and hence $A_N(\lambda)\Omega = a(\lambda)^N\Omega$ and $D_N(\lambda)\Omega = b(\lambda)^N\Omega$. Moreover,

$$\begin{aligned} \beta_n &= c\,\sigma_n^- = c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \gamma_n &= c\,\sigma_n^+ = c\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{1.24}$$

and hence $C_N(\lambda)\Omega = 0$. Using the relations (1.12) one can now derive (see [9]) that the state

$$\Phi(\lambda_1, \ldots, \lambda_n) = \prod_{j=1}^{n} B_N(\lambda_j)\,\Omega \tag{1.25}$$

is an eigenstate of the transfer operators $V_N(\lambda)$ for all values of λ provided the parameters $\lambda_1, \ldots \lambda_n$ are all different and satisfy the transcendental equations

$$\left(\frac{a(\lambda_j)}{b(\lambda_j)}\right)^N = \prod_{\substack{l=1 \\ l\neq j}}^{n} \frac{f(\lambda_j, \lambda_l)}{f(\lambda_l, \lambda_j)}. \tag{1.26}$$

In fact, one easily checks that these equations are equivalent to the Bethe Ansatz equations (1.10) if one performs the change of variables to wave numbers k_j given by

$$\exp\,[ik_j] = \frac{\sin(\lambda_j + \eta)}{\sin(\lambda_j - \eta)} = \frac{a(\lambda_j)}{b(\lambda_j)}. \tag{1.27}$$

2 Identity of Wavefunctions.

We now prove that the wavefunctions (1.25) are identical with the Bethe Ansatz wavefunctions (1.8) up to a multiplicative factor. The proof is similar but more complicated than the proof of the same fact in the case of the nonlinear Schroedinger model in [10]. In that case it has proved useful for proving the completeness of the Bethe Ansatz eigenstates. We hope that a similar thing can be done in the present case. First, we remark that, in (1.9) we can take $A_\sigma = C_\sigma(\lambda_1, \ldots, \lambda_n)$ defined by

$$C_\sigma(\lambda_1, \ldots, \lambda_n) = \prod_{1\leq i<j\leq n} f(\lambda_{\sigma(i)}, \lambda_{\sigma(j)}). \tag{2.1}$$

Indeed, if the wavenumbers k_j are given by (1.27) the relations (1.11) follow from

$$\frac{f(\lambda_j, \lambda_l)}{f(\lambda_l, \lambda_j)} = -\exp\,[-i\theta(k_j, k_l)]. \tag{2.2}$$

We now prove:

Theorem. *Let $\Phi(\lambda_1,\ldots,\lambda_n)$ be given by (1.25) where (1.26) is satisfied, and let $\Psi^{(n)}(k_1,\ldots,k_n)$ be the Bethe Ansatz wavefunction given by (1.8) and (1.9), where A_σ is replaced by $C_\sigma(\lambda_1,\ldots,\lambda_n)$. Then*

$$\Phi(\lambda_1,\ldots,\lambda_n) = c^n(a(\lambda_1)\ldots a(\lambda_n))^{-1}(b(\lambda_1)\ldots b(\lambda_n))^N\Psi^{(n)}(k_1,\ldots,k_n). \tag{2.3}$$

The case $n=1$ is simple:

$$B_N(\lambda)\Omega = (L_1(\lambda)\ldots L_N(\lambda))_{1,2}\Omega = (L_1^0(\lambda)\ldots L_N^0(\lambda))_{1,2}\Omega, \tag{2.4}$$

where we have written

$$L_m(\lambda) = L_m^0(\lambda) + \tilde{L}_m, \tag{2.5}$$

with

$$L_m^0(\lambda) = \begin{pmatrix} \alpha_m(\lambda) & \beta_m \\ 0 & \delta_m(\lambda) \end{pmatrix} \text{ and } \tilde{L}_m = \begin{pmatrix} 0 & 0 \\ \gamma_m & 0 \end{pmatrix}. \tag{2.6}$$

Now,

$$L_1^0(\lambda)\ldots L_N^0(\lambda) = \begin{pmatrix} \alpha_{1,N}(\lambda) & \beta_{1,N}(\lambda) \\ 0 & \delta_{1,N}(\lambda) \end{pmatrix} \tag{2.7}$$

with

$$\begin{cases} \alpha_{p,q}(\lambda) & = \prod_{m=p}^q \alpha_m(\lambda) \\ \delta_{p,q}(\lambda) & = \prod_{m=p}^q \delta_m(\lambda) \\ \beta_{p,q}(\lambda) & = \sum_{r=p}^q \alpha_{p,r-1}(\lambda)\beta_r\delta_{r+1,q}(\lambda). \end{cases} \tag{2.8}$$

But $\beta_r\Omega = c\,\sigma_r^-\Omega = c\,|r\rangle$ so

$$\begin{aligned} B_N(\lambda_1)\Omega &= \beta_{1,N}(\lambda_1)\Omega \\ &= \sum_{r=1}^N a(\lambda_1)^{r-1}b(\lambda_1)^{N-r}c|r\rangle \\ &= c\,a(\lambda_1)^{-1}b(\lambda_1)^N\sum_{r=1}^N e^{ik_1r}|r\rangle. \end{aligned} \tag{2.9}$$

Now consider the case $n=2$. We compute $B_N(\lambda_2)\Psi^{(1)}(k_1)$. As $\Psi^{(1)}$ contains only one down-spin, at most one $\tilde{L}_m$ can operate on it and we can write

$$\begin{aligned} B_N(\lambda_2)\Psi^{(1)}(k_1) &= (L_1^0(\lambda_2)\ldots L_N^0(\lambda_2))_{1,2}\Psi^{(1)}(k_1) \\ &+ \sum_{m=1}^N \left(L_{1,m-1}^0(\lambda_2)\tilde{L}_m L_{m+1,N}^0(\lambda_2)\right)_{1,2}\Psi^{(1)}(k_1), \end{aligned} \tag{2.10}$$

with the notation

$$L_{p,q}^0(\lambda) = \prod_{m=p}^q L_m^0(\lambda). \tag{2.11}$$

The first term in (2.10) is straightforward to compute:

$$(L^0_{1,N}(\lambda_2))_{1,2}\Psi^{(1)}(k_1) = c\,a(\lambda_2)^{-1}\,b(\lambda_2)^N \sum_{1\le r_1<r_2\le N} \{a(\lambda_2)^{-1}b(\lambda_2)e^{i(k_1r_1+k_2r_2)} + a(\lambda_2)b(\lambda_2)^{-1}e^{i(k_2r_1+k_1r_2)}\}|r_1,r_2\rangle. \tag{2.12}$$

In the second term we use

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}\begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix} = \begin{pmatrix} B\sigma A' & B\sigma B' \\ D\sigma A' & D\sigma B' \end{pmatrix}. \tag{2.13}$$

It follows that

$$\begin{aligned}
&\left(\sum_{m=1}^N L^0_{1,m-1}(\lambda_2)\tilde{L}_m L^0_{m+1,N}(\lambda_2)\right)_{1,2} \Psi^{(1)}(k_1) = \\
&= c\sum_{m=1}^N \beta_{1,m-1}(\lambda_2)\beta_{m+1,N}(\lambda_2)e^{ik_1m}\Omega \\
&= c^3 \sum_{m=1}^N\sum_{r_1=1}^{m-1} a(\lambda_2)^{r_1-1}b(\lambda_2)^{m-1-r_1} \\
&\qquad \sum_{r_2=m+1}^N a(\lambda_2)^{r_2-m-1}b(\lambda_2)^{N-r_2}e^{ik_1m}|r_1,r_2\rangle \\
&= c^3a(\lambda_2)^{-1}b(\lambda_2)^N \sum_{1\le r_1<r_2\le N} a(\lambda_2)^{-1}b(\lambda_2)^{-1}e^{ik_2(r_1+r_2)}\sum_{m=r_1+1}^{r_2-1} e^{i(k_1-k_2)m}|r_1,r_2\rangle \\
&= c^3a(\lambda_2)^{-1}b(\lambda_2)^N \sum_{1\le r_1<r_2\le N} a(\lambda_2)^{-1}b(\lambda_2)^{-1} \\
&\qquad \left\{\frac{e^{i(k_1r_1+k_2r_2)}}{e^{i(k_2-k_1)}-1} + \frac{e^{i(k_2r_1+k_1r_2)}}{e^{i(k_1-k_2)}-1}\right\}|r_1,r_2\rangle.
\end{aligned} \tag{2.14}$$

Next we use the formula

$$\frac{c^2}{e^{i(k_2-k_1)}-1} = g(\lambda_1,\lambda_2)\,a(\lambda_1)b(\lambda_2) \tag{2.15}$$

to write the sum of (2.12) and (2.14) as follows:

$$\begin{aligned}
B_N(\lambda_2)\Psi^{(1)}(k_1) &= c\,a(\lambda_2)^{-1}b(\lambda_2)^N \sum_{1\le r_1<r_2\le N} \\
&\Big\{a(\lambda_2)^{-1}\left[\alpha_{r_1}(\lambda_2)\beta_{r_1} + g(\lambda_1,\lambda_2)\beta_{r_1}\alpha_{r_1}(\lambda_1)\right]e^{i(k_1r_1+k_2r_2)} \\
&+ b(\lambda_2)^{-1}\left[\delta_{r_1}(\lambda_2)\beta_{r_1} + g(\lambda_2,\lambda_1)\beta_{r_1}\delta_{r_1}(\lambda_1)\right]e^{i(k_2r_1+k_1r_2)}\Big\}|r_2\rangle.
\end{aligned} \tag{2.16}$$

We can now use the relations (1.14) which yield, in particular,

$$\begin{cases} f(\lambda,\lambda')\beta_n\alpha_n(\lambda') = & \alpha_n(\lambda')\beta_n + g(\lambda,\lambda')\beta_n\alpha_n(\lambda), \\ f(\lambda',\lambda)\beta_n\delta_n(\lambda') = & \delta_n(\lambda')\beta_n + g(\lambda',\lambda)\beta_n\delta_n(\lambda). \end{cases} \tag{2.17}$$

Inserting this into (2.16) we obtain

$$B_N(\lambda_2)\Psi^{(1)}(k_1) = c\,a(\lambda_2)^{-1}b(\lambda_2)^N \sum_{1\le r_1<r_2\le N} \left\{ f(\lambda_1,\lambda_2)e^{i(k_1r_1+k_2r_2)} + f(\lambda_2,\lambda_1)e^{i(k_2r_1+k_1r_2)} \right\} |r_2\rangle. \tag{2.18}$$

This proves (2.3) in the case of two particles.

In the proof of the general case we proceed by induction as in [10]. Analogous to (2.10) we have

$$\begin{aligned} B_N(\lambda)\Phi^{(n)}(\lambda_1,\dots,\lambda_n) &= L^0_{1,N}(\lambda)\Phi^{(n)} \\ &+ \sum_{m_1=1}^{N} \left(L^0_{1,m_1-1}(\lambda)\tilde{L}_{m_1}L^0_{m_1+1,N}(\lambda)\right)_{1,2}\Phi^{(n)} + \dots + \\ &+ \sum_{1\le m_1<\dots<m_n\le N} \Big(L^0_{1,m_1-1}(\lambda)\tilde{L}_{m_1}L^0_{m_1+1,m_2-1}(\lambda)\dots \\ &\qquad \tilde{L}_{m_n}L^0_{m_n+1,N}(\lambda)\Big)_{1,2}\Phi^{(n)} \end{aligned} \tag{2.19}$$

Using (2.13) repeatedly we can write

$$\left(L^0_{1,m_1-1}(\lambda)\tilde{L}_{m_1}L^0_{m_1+1,m_2-1}(\lambda)\dots\tilde{L}_{m_p}L^0_{m_p+1,N}(\lambda)\right)_{1,2} = \tilde{B}_p(\lambda) \tag{2.20}$$

where $\tilde{B}_p$ is defined by

$$\tilde{B}_p(\lambda) = \beta_{1,m_1-1}(\lambda)\gamma_{m_1}\beta_{m_1+1,m_2-1}(\lambda)\dots\gamma_{m_p}\beta_{m_p+1,N}(\lambda). \tag{2.21}$$

Inserting the induction hypothesis for $\Psi^{(n)}(k_1,\dots,k_n)$ we have

$$\begin{aligned} \Phi_p^{(n+1)} &:= \tilde{B}_p(\lambda)\Phi^{(n)}(\lambda_1,\dots,\lambda_n) \\ &= \sum_{1\le m_1<\dots<m_p\le N}\ \sum_{r_1=1}^{m_1-1}\ \sum_{r_2=m_1+1}^{m_2-1}\dots\sum_{r_{p+1}=m_p+1}^{N} \\ &\quad \alpha_{1,r_1-1}(\lambda)\beta_{r_1}\delta_{r_1+1,m_1-1}(\lambda)\gamma_{m_1}\alpha_{m_1+1,r_2-1}(\lambda)\beta_{r_2}\dots \\ &\quad \dots\gamma_{m_p}\alpha_{m_p+1,r_{p+1}-1}(\lambda)\beta_{r_{p+1}}\delta_{r_{p+1}+1,N}(\lambda) \\ &\quad \sum_{1\le s_1<\dots<s_n\le N} \phi^{(n)}_{\{\lambda_1,\dots,\lambda_n\}}(s_1,\dots,s_n)|s_1\dots s_n\rangle, \end{aligned} \tag{2.22}$$

where $\phi^{(n)}_{\{\lambda_1,\ldots,\lambda_n\}}(s_1,\ldots,s_n) = c^n \prod_{i=1}^n a(\lambda_i)^{-1} b(\lambda_i)^N \psi^{(n)}_{\{k_1,\ldots,k_n\}}(s_1,\ldots,s_n)$. For each $a = 1,\ldots,p$ there must exist j_a such that $m_a = s_{j_a}$. Moreover, $s_j \neq r_i$ because $\beta^2_{r_i} = 0$. We can therefore write $\{r_1,\ldots,r_{p+1}\} \cup \{s_1,\ldots,s_n\} \setminus \{m_1,\ldots,m_p\} = \{r'_1,\ldots,r'_{n+1}\}$ and $r_a = r'_{i_a}$ with $1 \le i_1 < \ldots < i_{p+1} \le n+1$ and $r_a < m_a < r_{a+1} \iff r'_{i_a} < m_a < r'_{i_{a+1}}$ for $a = 1,\ldots,p$. Inserting these arguments into $\psi^{(n)}_{\{k_1,\ldots,k_n\}}(s_1,\ldots,s_n)$ we obtain

$$
\begin{aligned}
&\psi^{(n)}_{\{k_1,\ldots,k_n\}}(r'_1,\ldots,r'_{i_1-1},r'_{i_1+1},\ldots,r'_{j_1},m_1,r'_{j_1+1},\ldots,r'_{i_2-1},\ldots,m_p,\\
&\qquad r'_{j_p+1},\ldots,r'_{i_{p+1}-1},r'_{i_{p+1}+1},\ldots,r'_{n+1}) =\\
&\sum_{\sigma\in\mathcal{S}_n} C_\sigma(\lambda_1,\ldots,\lambda_n) \prod_{a=1}^{p+1} \left\{ \prod_{i=j_{a-1}+1}^{i_a-1} e^{ik_{\sigma(i)}r'_i} \prod_{i=i_a+1}^{j_a} e^{ik_{\sigma(i-1)}r'_i} \right\} \prod_{a=1}^{p} e^{ik_{\sigma(j_a)}m_a}
\end{aligned}
\tag{2.23}
$$

where $j_0 = 0$ and $j_{p+1} = n+1$. The factor $\alpha_{m_a+1,r'_{i_{a+1}}-1}(\lambda)$ in (2.22) operates on $i_{a+1}-1-j_a$ down spins and $(r'_{i_{a+1}}-1-m_a)-(i_{a+1}-1-j_a)$ up spins and therefore contributes a factor $a(\lambda)^{r'_{i_{a+1}}-m_a-i_{a+1}+j_a} b(\lambda)^{i_{a+1}-1-j_a}$ $(a = 0,\ldots,p)$. Here $m_0 = 0$. Similarly, the operator $\delta_{r'_{i_a}+1,m_a-1}(\lambda)$ contributes a factor $a(\lambda)^{j_a-i_a} b(\lambda)^{m_a-1-r'_{i_a}-j_a+i_a}$ $(a = 1,\ldots,p+1)$. Here $m_{p+1} = N+1$. In total, we obtain a factor

$$
\begin{aligned}
&a(\lambda)^{(r'_{i_1}+\ldots+r'_{i_{p+1}})-(m_1+\ldots+m_p)-2(i_1+\ldots+i_{p+1})+2(j_1+\ldots+j_p)+n+1}\\
&b(\lambda)^{(m_1+\ldots+m_p)-(r'_{i_1}+\ldots+r'_{i_{p+1}})-2(j_1+\ldots+j_p)-2(p+1)+2(i_1+\ldots+i_{p+1})+N-n} =\\
&= a(\lambda)^{-1-2p} b(\lambda)^N \exp\left[ik(r'_{i_1}+\ldots+r'_{i_{p+1}}-(m_1+\ldots+m_p))\right]\\
&\quad\times \exp\left[ik(-2(i_1+\ldots+i_{p+1})+2(j_1+\ldots+j_p)+2p+2+n)\right].
\end{aligned}
\tag{2.24}
$$

Next we can perform the summation over m_a for $a = 1,\ldots,p$:

$$
\sum_{m_a=r'_{j_a}+1}^{r'_{j_a+1}-1} e^{-ikm_a+ik_{\sigma(j_a)}m_a} = \frac{e^{i(k_{\sigma(j_a)}-k)r'_{j_a+1}} - e^{i(k_{\sigma(j_a)}-k)(r'_{j_a}+1)}}{e^{i(k_{\sigma(j_a)}-k)}-1}.
\tag{2.25}
$$

Thus we find, with $\lambda = \lambda_{n+1}$,

$$
\begin{aligned}
\Psi^{(n+1)}_p &= c^{-1} a(\lambda) b(\lambda)^{-N} \tilde{B}_p(\lambda) \Psi^{(n)}(\lambda_1,\ldots,\lambda_n)\\
&= c^{2p} \sum_{1\le r'_1<\ldots<r'_{n+1}\le N} \sum_{1\le i_1<\ldots<i_{p+1}\le n+1}\\
&\sum_{j_1=i_1}^{i_2-1} \cdots \sum_{j_p=i_p}^{i_{p+1}-1} \sum_{\sigma\in\mathcal{S}_n} C_\sigma(\lambda_1,\ldots,\lambda_n) a(\lambda)^{-2p} e^{ik(r'_{i_1}+\ldots+r'_{i_{p+1}})}\\
&\exp\left[ik(-2(i_1+\ldots+i_{p+1})+n+2(j_1+\ldots+j_p+p+1))\right]\\
&\prod_{a=1}^{p+1} \left\{ \prod_{i=j_{a-1}+1}^{i_a-1} e^{ik_{\sigma(i)}r'_i} \prod_{i=i_a}^{j_a-1} e^{ik_{\sigma(i)}r'_{i+1}} \right\}\\
&\prod_{a=1}^{p} \left\{ \frac{e^{i(k_{\sigma(j_a)}-k)r'_{j_a+1}} - e^{i(k_{\sigma(j_a)}-k)(r'_{j_a}+1)}}{e^{i(k_{\sigma(j_a)}-k)}-1} \right\} |r'_1,\ldots,r'_{n+1}\rangle.
\end{aligned}
\tag{2.26}
$$

Next we distinguish various terms in this expression depending on the choice of j_a $(a = 1, \ldots, p)$ and the $\pm$-term in the corresponding factor in the last product.

$\Psi_{p,0}^{(n+1)}$ will contain the terms with $(j_a, \pm) = (i_a, -)$ for $a = 1, \ldots, s$ and $(j_a, \pm) = (i_{a+1} - 1, +)$ for $a = s + 1, \ldots, p$, for some $s \in \{0, 1, \ldots, p\}$.

$\Psi_{p,-}^{(n+1)}$ consists of the terms for which there is $s \in \{1, \ldots, p-1\}$ and $r \in \{0, \ldots, s-1\}$ such that $(j_a, \pm) = (i_a, -)$ for $a = 1, \ldots, r$ and $(j_a, \pm) = (i_{a+1}-1, +)$ for $a = r+1, \ldots, s-1$, $(j_s, \pm) = (i_{s+1} - 1, +)$, $(j_{s+1}, \pm) = (i_{s+1}, -)$ and $(j_a, \pm)$ arbitrary for $a \geq s + 2$.

$\Psi_{p,+}^{(n+1)}$ contains the terms for which there are $s \in \{1, \ldots, p\}$ and $r \in \{0, \ldots, s - 1\}$ such that $(j_a, \pm) = (i_a, -)$ for $a = 1, \ldots, r$, $(j_a, \pm) = (i_{a+1} - 1, +)$ for $a = r + 1, \ldots, s - 1$, $(j_s, \pm) \notin \{(i_s, -), (i_{s+1} - 1, +)\}$ and $(j_a, \pm)$ arbitrary for $a = s + 1, \ldots, p$.

We will denote $\Psi_{p,\pm}^{(n+1)}(r, s)$ the sum of terms with given r and s. Presently, we show that

$$\Psi_{p,-}^{(n+1)}(r, s) + \Psi_{p-1,+}^{(n+1)}(r, s) = 0 \text{ for } p = 1, \ldots, n \tag{2.27}$$

and all values of r and s, and also, $\Psi_{0,-}^{(n+1)} = \Psi_{n,+}^{(n+1)} = 0$.

The terms $\Psi_{p,0}^{(n+1)}$ add up to the desired result:

$$\sum_{p=0}^{n} \Psi_{p,0}^{(n+1)} = \sum_{\tau \in S_{n+1}} C_\tau(\lambda_1, \ldots, \lambda_{n+1}) \sum_{1 \leq r_1 < \ldots < r_{n+1} \leq N} \prod_{j=1}^{n+1} e^{ik_{\tau(j)} r_j} |r_1, \ldots r_{n+1}\rangle. \tag{2.28}$$

To see this we write out $\Psi_{p,0}^{(n+1)}$:

$$\begin{aligned} \Psi_{p,0}^{(n+1)} = c^{2p} & \sum_{1 \leq r_1 < \ldots < r_{n+1} \leq N} \sum_{1 \leq i_1 < \ldots < i_{p+1} \leq n+1} \\ & \sum_{\sigma \in S_n} C_\sigma(\lambda_1, \ldots, \lambda_n) \sum_{s=0}^{p} \exp\left[ik(r_{i_{s+1}} - 2i_{s+1} + n + 2s + 2)\right] a(\lambda)^{-2p} \\ & \prod_{a=1}^{s} \frac{1}{e^{i(k - k_{\sigma(i_a)})} - 1} \prod_{a=s+1}^{p} \frac{1}{e^{i(k_{\sigma(i_{a+1}-1)} - k)} - 1} \\ & \prod_{i=1}^{i_{s+1}-1} e^{ik_{\sigma(i)} r_i} \prod_{i=i_{s+1}+1}^{n+1} e^{ik_{\sigma(i-1)} r_i} |r_1, \ldots, r_{n+1}\rangle \end{aligned} \tag{2.29}$$

Next we insert (2.15):

$$\Psi_{p,0}^{(n+1)} = c^{-n-1} \sum_{1\le r_1<\ldots<r_{n+1}\le N} \sum_{1\le i_1<\ldots<i_{p+1}\le n+1}$$
$$\sum_{\sigma\in\mathcal{S}_n} C_\sigma(\lambda_1,\ldots,\lambda_n) \sum_{s=0}^{p} \prod_{\substack{i=1\\ i\neq i_a}}^{i_{s+1}-1} a(\lambda)^{-1}\alpha_{r_i}(\lambda)\beta_{r_i}$$
$$\prod_{a=1}^{s} a(\lambda)^{-1} g(\lambda_{\sigma(i_a)},\lambda)\beta_{r_{i_a}}\alpha_{r_{i_a}}(\lambda_{\sigma(i_a)}) \prod_{\substack{i=i_{s+1}+1\\ i\neq i_a}}^{n+1} b(\lambda)^{-1}\delta_{r_i}(\lambda)\beta_{r_i} \tag{2.30}$$
$$\prod_{a=s+1}^{p} b(\lambda)^{-1} g(\lambda,\lambda_{\sigma(i_{a+1}-1)})\beta_{r_{i_{a+1}}}\delta_{r_{i_{a+1}}}(\lambda_{\sigma(i_{a+1}-1)})$$
$$\prod_{i=1}^{i_{s+1}-1} e^{ik_{\sigma(i)}r_i} e^{ikr_{i_{s+1}}} \prod_{i=i_{s+1}+1}^{n+1} e^{ik_{\sigma(i-1)}r_i}\Omega$$

We now write $C_\tau(\lambda_1,\ldots,\lambda_{n+1})$ in terms of $C_{\tau'}(\lambda_1,\ldots,\lambda_n)$ where τ' is defined by

$$\begin{cases} \tau'(i) = \tau(i) & \text{if } i < \bar\tau(n+1) \\ \tau'(i) = \tau(i+1) & \text{if } i \ge \bar\tau(n+1) \end{cases} \quad (1\le i\le n) \tag{2.31}$$

(We have written $\bar\tau$ for the inverse permutation.) We have:

$$C_\tau(\lambda_1,\ldots,\lambda_{n+1}) = C_{\tau'}(\lambda_1,\ldots,\lambda_n) \prod_{i<\bar\tau(n+1)} f(\lambda_{\tau(i)},\lambda_{n+1}) \prod_{i>\bar\tau(n+1)} f(\lambda_{n+1},\lambda_{\tau(i)}). \tag{2.32}$$

Inserting this into the right-hand side of (2.28) and using the identities (2.17) we find that it equals the left-hand side of (2.28) with $\Psi_{p,0}^{(n+1)}$ given by (2.30), where the products in (2.32) are expanded with s factors $g(\lambda_{\sigma(i_a)},\lambda)\beta_{r_{i_a}}\alpha_{r_{i_a}}(\lambda_{\sigma(i_a)})$ from the first product and $p-s$ factors $g(\lambda,\lambda_{\sigma(i_{a+1}-1)})\beta_{r_{i_{a+1}}}\delta_{r_{i_{a+1}}}(\lambda_{\sigma(i_{a+1}-1)})$ from the second product.

It remains to prove the identity (2.27). Now, in computing $\Psi_{p,-}^{(n+1)}(r,s)$ we must take $j_a = i_a$ and choose the second term in the last product of (2.26) for $a=1,\ldots,r$ and also for $a=s+1$; for $a=r+1,\ldots,s$ we must take $j_a = i_{a+1}-1$ and choose the first term in (2.26). Then inserting (2.15) for all values $a=1,\ldots,s$, we obtain

$$
\begin{aligned}
\Psi_{p,-}^{(n+1)}(r,s) = c^{2(p-s-1)} & \sum_{1\le r_1<\ldots<r_{n+1}\le N} \sum_{1\le i_1<\ldots<i_{p+1}\le n+1} \\
& \sum_{j_{s+2}=i_{s+2}}^{i_{s+3}-1} \cdots \sum_{j_p=i_p}^{i_{p+1}-1} \sum_{\sigma\in S_n} C_\sigma(\lambda_1,\ldots,\lambda_n) \\
& a(\lambda)^{-2p} \exp\left[ik(r_{i_{r+1}} + r_{i_{s+2}} + \cdots + r_{i_{p+1}} - r_{i_{s+1}})\right] \\
& \exp\left[ik\big(-2(i_{r+1} + i_{s+2} + \cdots + i_{p+1}) + n \right. \\
& \qquad \left. + 2(i_{s+1} + j_{s+2} + \cdots + j_p + p + 1 - s + r)\big)\right] \\
& \prod_{i=1}^{i_{r+1}-1} e^{ik_{\sigma(i)}r_i} \prod_{i=i_{r+1}+1}^{i_{s+1}} e^{ik_{\sigma(i-1)}r_i} e^{ik_{\sigma(i_{s+1})}r_{i_{s+1}}} \\
& \prod_{a=s+2}^{p+1} \left\{ \prod_{i=j_{a-1}+1}^{i_a-1} e^{ik_{\sigma(i)}r_i} \prod_{i=i_a+1}^{j_a} e^{ik_{\sigma(i-1)}r_i} \right\} \\
& \prod_{a=1}^{r} g(\lambda_{\sigma(i_a)},\lambda) a(\lambda_{\sigma(i_a)}) b(\lambda) \\
& \prod_{a=r+1}^{s} g(\lambda,\lambda_{\sigma(i_{a+1}-1)}) a(\lambda) b(\lambda_{\sigma(i_{a+1}-1)}) \\
& g(\lambda_{\sigma(i_{s+1})},\lambda) a(\lambda_{\sigma(i_{s+1})}) b(\lambda) \\
& \prod_{a=s+2}^{p} \left\{ \frac{e^{i(k_{\sigma(j_a)}-k)r_{j_a+1}} - e^{i(k_{\sigma(j_a)}-k)(r_{j_a}+1)}}{e^{i(k_{\sigma(j_a)}-k)} - 1} \right\} |r_1,\ldots,r_{n+1}\rangle.
\end{aligned} \tag{2.33}
$$

In $\Psi_{p-1,+}^{(n+1)}(r,s)$ the variable j_s ranges from i_s to $i_{s+1}-1$, but when the minus sign is chosen then $j_s \neq i_s$ and when the $+$-sign is chosen, $j_s \neq i_{s+1}-1$. To make this range uniform for both signs we redefine j_s in case of the $+$-sign: $j_s' = j_s + 1$ in case of the $+$-sign. Thus, j_s' ranges from $i_s + 1$ to $i_{s+1} - 1$ for both choices of the sign. Inserting (2.15) again in all terms $a = 1, \ldots, s$ we obtain

$$\Psi^{(n+1)}_{p-1,+}(r,s) = c^{2(p-s-1)} \sum_{1\le r_1<\ldots<r_{n+1}\le N} \sum_{1\le i_1<\ldots<i_p\le n+1} \sum_{j'_s=i_s+1}^{i_{s+1}-1}$$
$$\sum_{j_{s+1}=i_{s+1}}^{i_{s+2}-1} \cdots \sum_{j_{p-1}=i_{p-1}}^{i_p-1} \sum_{\sigma\in\mathcal{S}_n} C_\sigma(\lambda_1,\ldots,\lambda_n)$$
$$a(\lambda)^{-2(p-1)} \exp\left[ik(r_{i_{r+1}} + r_{i_{s+1}} + \cdots + r_{i_p} - r_{j'_s})\right]$$
$$\exp\big[ik\big(-2(i_{r+1} + i_{s+1} + \cdots + i_p) + n$$
$$+ 2(j'_s + j_{s+1} + \cdots + j_{p-1} + p + 1 - s + r\big)\big]$$
$$\prod_{i=1}^{i_{r+1}-1} e^{ik_{\sigma(i)}r_i} \prod_{i=i_{r+1}}^{j'_s-1} e^{ik_{\sigma(i)}r_{i+1}} e^{ik_{\sigma(j'_s)}r_{j'_s}}$$
$$\prod_{a=s+1}^{p} \left\{ \prod_{i=j_{a-1}+1}^{i_a-1} e^{ik_{\sigma(i)}r_i} \prod_{i=i_a}^{j_a-1} e^{ik_{\sigma(i)}r_{i+1}} \right\}$$
$$\prod_{a=1}^{r} g(\lambda_{\sigma(i_a)},\lambda)a(\lambda_{\sigma(i_a)})b(\lambda)$$
$$\prod_{a=r+1}^{s-1} g(\lambda,\lambda_{\sigma(i_{a+1}-1)})a(\lambda)b(\lambda_{\sigma(i_{a+1}-1)})$$
$$a(\lambda)^{-1}b(\lambda)\left\{g(\lambda,\lambda_{\sigma(j'_s-1)})b(\lambda)b(\lambda_{\sigma(j'_s-1)}) + g(\lambda_{\sigma(j'_s)},\lambda)a(\lambda)a(\lambda_{\sigma(j'_s)})\right\}$$
$$\prod_{a=s+1}^{p-1} \left\{ \frac{e^{i(k_{\sigma(j_a)}-k)r_{j_a+1}} - e^{i(k_{\sigma(j_a)}-k)(r_{j_a}+1)}}{e^{i(k_{\sigma(j_a)}-k)} - 1} \right\} |r_1,\ldots,r_{n+1}\rangle. \tag{2.34}$$

Comparing the two expressions, we see that, if we change the indices in (2.34) according to $j'_s \to i_{s+1}$, $i_a \to i_{a+1}$ $(a = s+1,\ldots,p)$ and $j_a \to j_{a+1}$ for $a = s+1,\ldots,p-1$, they cancel provided the following identity holds:

$$\begin{aligned}&[g(\lambda,\lambda_1)b(\lambda)b(\lambda_1) + g(\lambda_2,\lambda)a(\lambda)a(\lambda_2)]C_\sigma \\ &+ [g(\lambda_1,\lambda)a(\lambda)a(\lambda_1) + g(\lambda,\lambda_2)b(\lambda)b(\lambda_2)]C_{\tilde\sigma} \\ &= g(\lambda_1,\lambda)g(\lambda_2,\lambda)\left[a(\lambda_2)b(\lambda_1)C_\sigma + a(\lambda_1)b(\lambda_2)C_{\tilde\sigma}\right].\end{aligned} \tag{2.35}$$

Here we have written λ_1 instead of $\lambda_{\sigma(i_{s+1}-1)}$ and λ_2 instead of $\lambda_{\sigma(i_{s+1})}$ and we have defined $\tilde\sigma$ as the permutation that differs from σ only by a transposition of $i_{s+1}-1$ and i_{s+1}. It is easily seen that the only difference between these two permutations in (2.33) and (2.34) is given by the left- and right-hand sides of (2.35).

To prove (2.35), notice first that, by the definition of C_σ,

$$C_{\tilde\sigma} = \frac{f(\lambda_2,\lambda_1)}{f(\lambda_1,\lambda_2)} C_\sigma. \tag{2.36}$$

Using (2.17) with $\lambda = \lambda_1$ and $\lambda' = \lambda_2$ this implies

$$\frac{a(\lambda_2)}{b(\lambda_2)}C_\sigma - \frac{b(\lambda_2)}{a(\lambda_2)}C_{\tilde\sigma} = \left(\frac{b(\lambda_1)}{b(\lambda_2)}C_\sigma + \frac{a(\lambda_1)}{a(\lambda_2)}C_{\tilde\sigma}\right) g(\lambda_1,\lambda_2). \tag{2.37}$$

Combining the two identities (2.17) with $\lambda' = \lambda_2$ we have also

$$g(\lambda_2,\lambda)a(\lambda_2)b(\lambda_2) = a(\lambda)a(\lambda_2) - b(\lambda)b(\lambda_2) + g(\lambda_2,\lambda)a(\lambda)b(\lambda). \tag{2.38}$$

Inserting this into the right-hand side of (2.35) we have

$$\begin{aligned} g(\lambda_1,\lambda)g(\lambda_2,\lambda)[a(\lambda_2)b(\lambda_1)C_\sigma + a(\lambda_1)b(\lambda_2)C_{\tilde\sigma}] = \\ - g(\lambda_1,\lambda)b(\lambda)b(\lambda_1)C_\sigma + g(\lambda_1,\lambda)a(\lambda)a(\lambda_1)C_{\tilde\sigma} \\ + g(\lambda_1,\lambda)a(\lambda)b(\lambda_1)\left(a(\lambda_2)/b(\lambda_2)\right)C_\sigma \\ - g(\lambda_1,\lambda)a(\lambda_1)b(\lambda)\left(b(\lambda_2)/a(\lambda_2)\right)C_{\tilde\sigma} \\ + a(\lambda)b(\lambda)g(\lambda_1,\lambda)g(\lambda_2,\lambda)\left[\frac{b(\lambda_1)}{b(\lambda_2)}C_\sigma + \frac{a(\lambda_1)}{a(\lambda_2)}C_{\tilde\sigma}\right]. \end{aligned} \tag{2.39}$$

The relation (2.15) can be written in terms of $a(\lambda)$'s and $b(\lambda)$'s:

$$[a(\lambda)b(\lambda') - a(\lambda')b(\lambda)]\,g(\lambda,\lambda') = -c^2. \tag{2.40}$$

This implies easily

$$b(\lambda)g(\lambda_1,\lambda)g(\lambda_2,\lambda) = -b(\lambda_1)g(\lambda_1,\lambda)g(\lambda_1,\lambda_2) + b(\lambda_2)g(\lambda_2,\lambda)g(\lambda_1,\lambda_2). \tag{2.41}$$

With this and (2.37), equation (2.39) becomes

$$\begin{aligned} & g(\lambda_1,\lambda)g(\lambda_2,\lambda)\left[a(\lambda_2)b(\lambda_1)C_\sigma + a(\lambda_1)b(\lambda_2)C_{\tilde\sigma}\right] \\ &= -g(\lambda_1,\lambda)b(\lambda)b(\lambda_1)C_\sigma + g(\lambda_1,\lambda)a(\lambda)a(\lambda_1)C_{\tilde\sigma} \\ &\quad + g(\lambda_1,\lambda)a(\lambda)b(\lambda_1)\left(a(\lambda_2)/b(\lambda_2)\right)C_\sigma - g(\lambda_1,\lambda)a(\lambda_1)b(\lambda)\left(b(\lambda_2)/a(\lambda_2)\right)C_{\tilde\sigma} \\ &\quad + a(\lambda)\left[-b(\lambda_1)g(\lambda_1,\lambda) + b(\lambda_2)g(\lambda_2,\lambda)\right]\left[\frac{a(\lambda_2)}{b(\lambda_2)}C_\sigma - \frac{b(\lambda_2)}{a(\lambda_2)}C_{\tilde\sigma}\right] \\ &= -g(\lambda_1,\lambda)b(\lambda)b(\lambda_1)C_\sigma + g(\lambda_2,\lambda)a(\lambda)a(\lambda_2)C_\sigma + g(\lambda_1,\lambda)a(\lambda)a(\lambda_1)C_{\tilde\sigma} \\ &\quad - g(\lambda_1,\lambda)\left[b(\lambda)a(\lambda_1) - a(\lambda)b(\lambda_1)\right]\frac{b(\lambda_2)}{a(\lambda_2)}C_{\tilde\sigma} - g(\lambda_2,\lambda)a(\lambda)b(\lambda_2)\frac{b(\lambda_2)}{a(\lambda_2)}C_{\tilde\sigma} \\ &= \left[-g(\lambda_1,\lambda)b(\lambda)b(\lambda_1) + g(\lambda_2,\lambda)a(\lambda)a(\lambda_2)\right]C_\sigma \\ &\quad + g(\lambda_1,\lambda)a(\lambda)a(\lambda_1)C_{\tilde\sigma} - \left[a(\lambda)b(\lambda_2)g(\lambda_2,\lambda) - c^2\right]\frac{b(\lambda_2)}{a(\lambda_2)}C_{\tilde\sigma}, \end{aligned} \tag{2.42}$$

where we have used (2.40). Using (2.40) again, this equals the left-hand side of (2.35). QED

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