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Superdiffeomorphisms of the Three-dimensional BF System and Finiteness

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Abstract. We discuss the ultraviolet behaviour of the three-dimensional BF system in curved space-time by making use of a local supersymmetry Ward identity. The model is shown to be finite at all orders of perturbation theory.

1 Introduction

In the recent years, much attention has been devoted to the study of topological field theories of the Schwarz type [1] because of their connection to mathematics. A prominent example of such a class of models is the Chern-Simons theory in three space-time dimensions [2, 3] which, when regarded as a pure gauge field model, can provide an interpretation of topological invariants of three-manifolds in terms of vaccum expectation values of Wilson lines [4].

It turns out however that the correct evaluation of these topological quantities in a quantum field theoretical approach necessarily entails the discussion of the ultraviolet behaviour of the Chern-Simons theory in a well-defined perturbative lagrangian framework. Indeed,

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from the standpoint of the renormalization theory, the topological models are expected to be an example of finite field theories.

Actually, in a series of papers [5, 6], the BRS-algebraic renormalization technique [7] was employed in the specific case of the Chern-Simons model in both flat and curved space-times. As a result, the model was proven to be finite to all orders of perturbation theory without any reference to a particular regularization scheme. It should be emphasized moreover that the mentioned finiteness proof makes use of a supersymmetric structure [8, 9] which is manifest when a Landau type gauge is chosen. The topological vector supersymmetry builds up, together with the BRS-symmetry and translations, a supersymmetry algebra of the Wess-Zumino type and is a common feature of a large class of topological theories.

Furthermore, the same finiteness properties were also shown to persist in the more general situation of the topological BF system [10, 11] in three [12] or more space-time dimensions [13]. This last fact motivates us here to extend the study carried out in [12] to a curved background and to exploit the local counterpart of the aforementioned vector supersymmetry. This is the aim of the present paper: to follow the strategy of ref.[6] and to show that the three-dimensional BF system defined on a curved space-time is finite to all orders.

The curved space-time is assumed here to be a three-manifold $\mathcal{M}$ in which the locality properties of the renormalized perturbation theory are valid. Thus, a consistent ultra-violet subtraction scheme is supposed to exist in momentum space as a necessary element of our renormalization study. This would not be possible in the case of an arbitrary manifold in which momentum conservation does not hold in general: only flat space is translational invariant. Nevertheless, one can consider a manifold $\mathcal{M}$ resulting from the continuous deformation of a flat manifold, the metric being regarded as an external field. Hence, $\mathcal{M}$ has to be topologically equivalent to a flat space (i.e. $\mathbb{R}^3$) and it clearly possesses a flat limit since any external field vanishes at the infinity.

The work is organized as follows: in Section 2 we describe the theory in the classical approximation pointing out some details of its construction on the three-manifold, in Section 3 we proceed to the stability analysis of the model and prove its perturbative finiteness. Section 4 contains some concluding remarks.

2 The Classical Approximation

We begin our investigation by defining the BF system on the Riemannian three-manifold $\mathcal{M}$. The complete action is understood here to be invariant under diffeomorphisms and is composed of three parts [12]:

$$\Sigma = \Sigma_{BF} + \Sigma_{gf} + \Sigma_s. \tag{2.1}$$

To describe the pure BF part Σ_{BF} one introduces a pair of vector fields A^a_μ (the gauge connection) and B^a_μ which transform covariantly under diffeomorphisms and take values in the adjoint representation of a gauge group $\mathbf{G}$ with structure constants f^{abc}. The action

Σ_{BF} writes according to:

$$\Sigma_{BF} = \frac{1}{2} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\rho} \left\{ F^a_{\mu\nu} B^a_\rho + \frac{1}{3} \alpha f^{abc} B^a_\mu B^b_\nu B^c_\rho \right\}, \tag{2.2}$$

where a cosmological constant α is also included and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{2.3}$$

We remark that the outstanding topological properties of Σ_{BF} are a direct consequence of its metric independence, a fact that can be noticed by a simple inspection of (2.2).

To proceed to quantization we incorporate to the model a couple of scalar ghost fields (c^a, φ^a), two scalar Lagrange multiplier fields (b^a, d^a), as well as the antighosts $(\bar{c}^a, \bar{\varphi}^a)$, also scalars under diffeomorphism transformations. An off-shell nilpotent s-operation is also present and is a symmetry of Σ_{BF}:

$$\begin{array}{ll} sA^a_\mu = -(D_\mu c)^a - \alpha f^{abc} B^b_\mu \varphi^c, & \\ sB^a_\mu = -(D_\mu \varphi)^a - f^{abc} B^b_\mu c^c, & \\ sc^a = \frac{1}{2} f^{abc} \left(c^b c^c + \alpha \varphi^b \varphi^c \right), & \\ s\varphi^a = f^{abc} \varphi^b c^c, & \\ s\bar{c}^a = b^a, & sb^a = 0, \\ s\bar{\varphi}^a = d^a, & sd^a = 0. \end{array} \tag{2.4}$$

In the present work, one adopts a Landau type gauge-fixing which has the following structure:

$$\Sigma_{gf} = -s \int d^3x \left\{ \sqrt{g} g^{\mu\nu} \left[(\partial_\mu \bar{c}^a) A^a_\nu + (\partial_\mu \bar{\varphi}^a) B^a_\nu \right] \right\} \tag{2.5}$$

where $g^{\mu\nu}$ stands for the inverse of the metric $g_{\mu\nu}$ and g for its determinant. We realize here that (2.5) depends explicitly on the background graviational configuration, representing an unavoidable nontopological contribution to Σ. Let us observe however that the metric has a purely nonphysical character here: appearing only in the BRS-trivial gauge-fixing part, $g_{\mu\nu}$ has to be interpreted as a gauge parameter. In practice, it is useful to enlarge the BRS operation [6], allowing $g_{\mu\nu}$ to transform under s as below (see also [14]):

$$sg_{\mu\nu} = \hat{g}_{\mu\nu}, \qquad s\hat{g}_{\mu\nu} = 0. \tag{2.6}$$

The dimensions and Faddeev-Popov ($\Phi\Pi$) ghost charges of the fields introduced thus far are given in Table 1.

The third term of the complete action (2.1) is related to the renormalization of the non-linear parts of the transformation laws in (2.4). These latters are coupled to external sources [15]: a pair of contravariant vector densities of weight one $(\Omega^{\mu a}, \tau^{\mu a})$ and two scalar densities (L^a, D^a), also of weight one. Their dimensions and ghost charges are given in Table 2. The action of external sources reads as follows:

$$\begin{aligned} \Sigma_s &= \int d^3x \left\{ -\Omega^{a\mu} \left[(D_\mu c)^a + \alpha f^{abc} B^b_\mu \varphi^c \right] + \frac{1}{2} L^a f^{abc} (c^b c^c + \alpha \varphi^b \varphi^c) + \right. \\ &- \left. \tau^{a\mu} \left[(D_\mu \varphi)^a + f^{abc} B^b_\mu c^c \right] + D^a f^{abc} \varphi^b c^c \right\}. \end{aligned} \tag{2.7}$$

	A_μ	B_μ	c	φ	b	d	$\bar{c}$	$\bar{\varphi}$	$g_{\mu\nu}$	$\hat{g}_{\mu\nu}$
dim	1	1	0	0	1	1	1	1	0	0
ΦΠ	0	0	1	1	0	0	-1	-1	0	1

Table 1: Dimensions and Faddeev-Popov charges of the fields.

	Ω^μ	τ^μ	L	D
dim	2	2	3	3
ΦΠ	-1	-1	-2	-2

Table 2: Dimensions and ΦΠ charges of the sources.

At this stage, one can translate the BRS invariance into a Slavnov-Taylor identity:

$$\begin{aligned}\mathcal{S}(\Sigma) &= \int d^3x \left(\frac{\delta\Sigma}{\delta\Omega^{a\mu}}\frac{\delta\Sigma}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta\tau^{a\mu}}\frac{\delta\Sigma}{\delta B^a_\mu} + \frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta c^a} + \right. \\ &+ \left. \frac{\delta\Sigma}{\delta D^a}\frac{\delta\Sigma}{\delta\varphi^a} + b^a\frac{\delta\Sigma}{\delta\bar{c}^a} + d^a\frac{\delta\Sigma}{\delta\bar{\varphi}^a} + \hat{g}_{\mu\nu}\frac{\delta\Sigma}{\delta g_{\mu\nu}} \right) = 0 \end{aligned} \tag{2.8}$$

where the non-topological effects in (2.1) are seen to be duly controlled by the last term, which corresponds to a doublet containing the BRS-transforming metric.

After establishing the invariance of the BF system under the diffeomorphisms and the s-operation, one may eventually ask whether it is possible to observe a local vector supersymmetry in the specific case of the three-dimensional BF system in a Landau type gauge. Indeed, in much the same way as it occurs for the Chern-Simons model defined on a manifold [6], the superdiffeomorphisms – as they have been named – can be shown to exist here and to form, beside the BRS symmetry and the diffeomorphisms, a supersymmetry algebra [8, 9]. We propose the following set of superdiffeomorphic transformations on the fields:

$$\begin{array}{ll} \delta^S_{(\xi)}A^a_\mu = -\epsilon_{\mu\nu\rho}\xi^\nu\left(\tau^{a\rho} + \sqrt{g}g^{\rho\sigma}\partial_\sigma\bar{\varphi}^a\right), & \delta^S_{(\xi)}c^a = -\xi^\mu A^a_\mu, \\ \delta^S_{(\xi)}b^a = \xi^\mu\partial_\mu\bar{c}^a, & \delta^S_{(\xi)}\bar{c}^a = 0, \\ \delta^S_{(\xi)}B^a_\mu = -\epsilon_{\mu\nu\rho}\xi^\nu\left(\Omega^{a\rho} + \sqrt{g}g^{\rho\sigma}\partial_\sigma\bar{c}^a\right), & \delta^S_{(\xi)}\varphi^a = -\xi^\mu B^a_\mu, \\ \delta^S_{(\xi)}d^a = \xi^\mu\partial_\mu\bar{\varphi}^a, & \delta^S_{(\xi)}\bar{\varphi}^a = 0, \\ \delta^S_{(\xi)}g_{\mu\nu} = 0, & \delta^S_{(\xi)}\hat{g}_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \\ \delta^S_{(\xi)}\Omega^{a\mu} = -\xi^\mu L^a, & \delta^S_{(\xi)}L^a = 0, \\ \delta^S_{(\xi)}\tau^{a\mu} = -\xi^\mu D^a, & \delta^S_{(\xi)}D^a = 0, \end{array} \tag{2.9}$$

here ξ^μ is the local parameter of superdiffeomorphisms, a contravariant vector field of dimension minus one; $\mathcal{L}_\xi$ is the Lie derivative in the direction ξ^μ. We define a Ward operator for superdiffeomorphisms:

$$\mathcal{W}^S_{(\xi)} = \int d^3x \sum_\phi \delta^S_{(\xi)}\phi \, \frac{\delta}{\delta\phi} \tag{2.10}$$

	K_μ	M_μ
dim	-1	-1
$\Phi\Pi$	2	1

Table 3: Dimensions and $\Phi\Pi$ charges of the external fields.

the summation being performed over all the fields transforming in (2.9).

One notices however that the complete action (2.1) is not invariant under the action of $\mathcal{W}^S_{(\xi)}$, the associated Ward identity being broken by two different pieces:

$$\mathcal{W}^S_{(\xi)}\Sigma = \Delta^{cl}_{(\xi)} + \int d^3x \; s\left(g_{\mu\rho}\xi^\rho \Xi^\mu\right), \tag{2.11}$$

where

$$\begin{aligned}\Delta^{cl}_{(\xi)} &= \int d^3x \left[-\Omega^{a\mu}(\mathcal{L}_\xi A^a_\mu) + L^a(\mathcal{L}_\xi c^a) - \tau^{a\mu}(\mathcal{L}_\xi B^a_\mu) + D^a(\mathcal{L}_\xi \varphi^a)\right] \\ &+ \int d^3x \left\{\xi^\mu \epsilon_{\mu\nu\rho}\left[\Omega^{a\nu} s(\sqrt{g} g^{\rho\sigma}\partial_\sigma\bar{\varphi}^a) + \tau^{a\nu} s(\sqrt{g} g^{\rho\sigma}\partial_\sigma \bar{c}^a)\right]\right\}\end{aligned} \tag{2.12}$$

and with

$$\Xi^\mu = \epsilon^{\mu\nu\rho}\partial_\nu \bar{c}^a \partial_\rho \bar{\varphi}^a. \tag{2.13}$$

The first piece in the r.h.s. of (2.11) is a classical breaking term (linear in the quantum fields) and is already present in the algebraic renormalization study of the BF system in three-dimensional flat space [12]. The second piece, being non-linear in the quantum fields, is a hard breaking term and its presence in (2.11) requires a more careful attention. In the present situation, one is able to control the hard breaking piece by means of a standard procedure [16], which consists of absorbing its effects into the complete action (2.1). To do this, we couple the field polynomial Ξ^μ and its s-variation to two external covariant vector fields (K_μ, M_μ) as shown below:

$$\Sigma_{K,M} = \int d^3x \left[K_\mu \Xi^\mu - M_\mu (s\Xi^\mu)\right] \tag{2.14}$$

moreover, we impose the transformation laws:

$$\begin{aligned} sM_\mu &= K_\mu, \quad \delta^S_{(\xi)} M_\mu = g_{\mu\rho}\xi^\rho, \\ sK_\mu &= 0, \qquad \delta^S_{(\xi)} K_\mu = \mathcal{L}_\xi M_\mu - \hat{g}_{\mu\rho}\xi^\rho. \end{aligned} \tag{2.15}$$

Dimensions and $\Phi\Pi$ ghost charges of the external fields are displayed in Table 3. We redefine then (2.1), adding a fourth term to it, namely $\Sigma_{K,M}$, and we observe that with this redefinition the Ward identity of superdiffeomorphisms is now rendered unbroken (apart of the harmless linear breaking piece):

$$\mathcal{W}^S_{(\xi)}\Sigma = \Delta^{cl}_{(\xi)}, \tag{2.16}$$

with

$$\Sigma = \Sigma_{BF} + \Sigma_{gf} + \Sigma_s + \Sigma_{K,M}. \tag{2.17}$$

The Slavnov-Taylor identity eq.(2.8) is modified in accordance with the above redefinition:

$$\begin{aligned} \mathcal{S}(\Sigma) &= \int d^3x \left(\frac{\delta\Sigma}{\delta\Omega^{a\mu}} \frac{\delta\Sigma}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta\tau^{a\mu}} \frac{\delta\Sigma}{\delta B^a_\mu} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta c^a} + \frac{\delta\Sigma}{\delta D^a} \frac{\delta\Sigma}{\delta\varphi^a} + \right. \\ &+ \left. b^a \frac{\delta\Sigma}{\delta\bar{c}^a} + d^a \frac{\delta\Sigma}{\delta\bar{\varphi}^a} + K_\mu \frac{\delta\Sigma}{\delta M_\mu} + \hat{g}_{\mu\nu} \frac{\delta\Sigma}{\delta g_{\mu\nu}} \right) = 0. \end{aligned} \tag{2.18}$$

From the new Slavnov-Taylor identity one reads off the linearized BRS operator:

$$\begin{aligned} \mathcal{S}_\Sigma &= \int d^3x \left(\frac{\delta\Sigma}{\delta\Omega^{a\mu}} \frac{\delta}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta A^a_\mu} \frac{\delta}{\delta\Omega^{a\mu}} + \frac{\delta\Sigma}{\delta\tau^{a\mu}} \frac{\delta}{\delta B^a_\mu} + \frac{\delta\Sigma}{\delta B^a_\mu} \frac{\delta}{\delta\tau^{a\mu}} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta\Sigma}{\delta c^a} \frac{\delta}{\delta L^a} + \right. \\ &+ \left. \frac{\delta\Sigma}{\delta D^a} \frac{\delta}{\delta\varphi^a} + \frac{\delta\Sigma}{\delta\varphi^a} \frac{\delta}{\delta D^a} + b^a \frac{\delta}{\delta\bar{c}^a} + d^a \frac{\delta}{\delta\bar{\varphi}^a} + K_\mu \frac{\delta}{\delta M_\mu} + \hat{g}_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \right). \end{aligned} \tag{2.19}$$

and we remark that it is a nilpotent operator.

The two Landau gauge-fixing conditions are:

$$\frac{\delta\Sigma}{\delta b^a} = \partial_\mu \left(\sqrt{g} g^{\mu\nu} A^a_\nu\right) - \partial_\mu \left(\epsilon^{\mu\nu\rho} M_\nu \partial_\rho \bar{\varphi}^a\right) \tag{2.20}$$

and

$$\frac{\delta\Sigma}{\delta d^a} = \partial_\mu \left(\sqrt{g} g^{\mu\nu} B^a_\nu\right) - \partial_\mu \left(\epsilon^{\mu\nu\rho} M_\nu \partial_\rho \bar{c}^a\right). \tag{2.21}$$

The commutation of the gauge conditions (2.20) and (2.21) with (2.18) yields two ghost equations [17]:

$$\frac{\delta\Sigma}{\delta\bar{c}^a} + \partial_\mu \left(\sqrt{g} g^{\mu\nu} \frac{\delta\Sigma}{\delta\Omega^{a\nu}} \right) = -\partial_\mu \left[s(\sqrt{g} g^{\mu\nu}) A^a_\nu \right] + \epsilon^{\mu\nu\rho} \partial_\mu \left(K_\nu \partial_\rho \bar{\varphi}^a - M_\nu \partial_\rho d^a \right) \tag{2.22}$$

and

$$\frac{\delta\Sigma}{\delta\bar{\varphi}^a} + \partial_\mu \left(\sqrt{g} g^{\mu\nu} \frac{\delta\Sigma}{\delta\tau^{a\nu}} \right) = -\partial_\mu \left[s(\sqrt{g} g^{\mu\nu}) B^a_\nu \right] + \epsilon^{\mu\nu\rho} \partial_\mu \left(K_\nu \partial_\rho \bar{c}^a - M_\nu \partial_\rho b^a \right). \tag{2.23}$$

The algebraic structure of the Landau gauge-fixing allows us to obtain two antighost equations [18]. The first one controls the coupling of the ghost c:

$$\mathcal{G}^a \Sigma = \Delta^a_{\mathcal{G}}, \tag{2.24}$$

where

$$\mathcal{G}^a = \int d^3x \left(\frac{\delta}{\delta c^a} + f^{abc} \bar{c}^b \frac{\delta}{\delta b^c} + f^{abc} \bar{\varphi}^b \frac{\delta}{\delta d^c} \right) \tag{2.25}$$

and

$$\Delta^a_{\mathcal{G}} = \int d^3x f^{abc} \left(\Omega^{b\mu} A^c_\mu - L^b c^c + \tau^{b\mu} B^c_\mu - D^b \varphi^c \right). \tag{2.26}$$

	ξ^μ	ϵ^μ
dim	-1	-1
$\Phi\Pi$	2	1

Table 4: Dimensions and $\Phi\Pi$ charges of the gauge parameters.

The other antighost equation controls the coupling of the ghost φ:

$$\mathcal{F}^a\Sigma = \Delta^a_{\mathcal{F}}, \tag{2.27}$$

where

$$\mathcal{F}^a = \int d^3x \left(\frac{\delta}{\delta\varphi^a} + \alpha f^{abc}\bar{c}^b \frac{\delta}{\delta d^c} + f^{abc}\bar{\varphi}^b \frac{\delta}{\delta b^c} \right) \tag{2.28}$$

and

$$\Delta^a_{\mathcal{F}} = \int d^3x f^{abc} \left(\tau^{b\mu} A^c_\mu - D^b c^c + \alpha\Omega^{b\mu} B^c_\mu - \alpha L^b \varphi^c \right). \tag{2.29}$$

One can also write down a Ward identity for diffeomorphisms:

$$\mathcal{W}^D_{(\epsilon)}\Sigma = 0, \tag{2.30}$$

where $\mathcal{W}^D_{(\epsilon)}$ denotes their respective Ward operator,

$$\mathcal{W}^D_{(\epsilon)} = \int d^3x \sum_{fields\ f} \left(\mathcal{L}_{(\epsilon)} f\right) \frac{\delta}{\delta f} \tag{2.31}$$

the summation running on $f = A_\mu$, B_μ, c, φ, b, d, $\bar{c}$, $\bar{\varphi}$, $g_{\mu\nu}$, $\hat{g}_{\mu\nu}$, Ω^μ, τ^μ, L, D, K_μ and M_μ. At this point, we anticipate the ghosty interpretation of the vector parameters of superdiffeomorphisms and diffeomorphisms by attributing a definite statistics to them. Dimensions and $\Phi\Pi$ charges read as in Table 4.

To summarize our results, we collect all the constraints obeyed by the complete action:
(i) the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0, \tag{2.32}$$

(ii) the two gauge-fixing conditions, eqs.(2.20) and (2.21),
(iii) the two ghost equations, eqs.(2.22) and (2.23),
(iv) the $\bar{c}$-antighost equation

$$\mathcal{G}^a\Sigma = \Delta^a_{\mathcal{G}}, \tag{2.33}$$

(v) the $\bar{\varphi}$-antighost equation

$$\mathcal{F}^a\Sigma = \Delta^a_{\mathcal{F}}, \tag{2.34}$$

(vi) the local supersymmetry Ward identity

$$\mathcal{W}^S_{(\xi)}\Sigma = \Delta^{cl}_{(\xi)} \tag{2.35}$$

and
(vii) the diffeomorphisms Ward identity

$$\mathcal{W}^D_{(\epsilon)}\Sigma = 0. \tag{2.36}$$

The graded linear algebra obeyed by the functional operators $\mathcal{S}_\Sigma$, $\mathcal{W}^S_{(\xi)}$ and $\mathcal{W}^D_{(\epsilon)}$ is:

$$\begin{array}{rcl} \{\mathcal{S}_\Sigma, \mathcal{S}_\Sigma\} & = & 0, \\ \left\{\mathcal{S}_\Sigma, \mathcal{W}^D_{(\epsilon)}\right\} & = & 0, \\ \left\{\mathcal{W}^D_{(\epsilon)}, \mathcal{W}^D_{(\epsilon')}\right\} & = & -\mathcal{W}^D_{([\epsilon,\epsilon'])}, \\ \left\{\mathcal{W}^S_{(\xi)}, \mathcal{S}_\Sigma\right\} & = & \mathcal{W}^D_{(\xi)}, \\ \left\{\mathcal{W}^S_{(\xi)}, \mathcal{W}^D_{(\epsilon)}\right\} & = & \mathcal{W}^S_{([\xi,\epsilon])}, \\ \left\{\mathcal{W}^S_{(\xi)}, \mathcal{W}^S_{(\xi')}\right\} & = & 0, \end{array} \tag{2.37}$$

where we introduced the notation for the graded Lie bracket of the vector fields ξ, ϵ and ϵ',

$$[\epsilon, \epsilon']^\mu = \mathcal{L}_\epsilon \epsilon'^\mu = \epsilon^\lambda \partial_\lambda \epsilon'^\mu + \epsilon'^\lambda \partial_\lambda \epsilon^\mu \tag{2.38}$$

and

$$[\xi, \epsilon]^\mu = \mathcal{L}_\xi \epsilon^\mu = \xi^\lambda \partial_\lambda \epsilon^\mu - \epsilon^\lambda \partial_\lambda \xi^\mu. \tag{2.39}$$

3 Stability Analysis

In this section one is concerned with the evaluation of the most general local counterterm to the complete action Σ in (2.17). To do this we make use of the Slavnov-Taylor identity and the whole set of Ward identities derived previously. We stress however that these identities can only be promoted to the quantum level if they can be shown to be anomaly free, an issue that is also considered here.

A possible general counterterm $\tilde{\Sigma}$ – if it exists – is an integrated local polynomial in the fields and sources with dimension three and neutral $\Phi\Pi$ charge. The perturbed action is then given by:

$$\Sigma' = \Sigma + \tilde{\Sigma} \tag{3.1}$$

the local deformation $\tilde{\Sigma}$ being constrained by the following set of stability requirements:

$$\mathcal{G}^a \tilde{\Sigma} = 0, \quad \mathcal{F}^a \tilde{\Sigma} = 0, \tag{3.2}$$

$$\mathcal{S}_\Sigma \tilde{\Sigma} = 0, \quad \mathcal{W}^S_{(\xi)} \tilde{\Sigma} = 0, \quad \mathcal{W}^D_{(\epsilon)} \tilde{\Sigma} = 0, \tag{3.3}$$

$$\frac{\delta\tilde{\Sigma}}{\delta b^a} = 0, \quad \frac{\delta\tilde{\Sigma}}{\delta d^a} = 0, \quad \frac{\delta\tilde{\Sigma}}{\delta \bar{c}^a} + \partial_\mu \left(\sqrt{g} g^{\mu\nu} \frac{\delta\tilde{\Sigma}}{\delta \Omega^{a\nu}}\right) = 0, \quad \frac{\delta\tilde{\Sigma}}{\delta \bar{\varphi}^a} + \partial_\mu \left(\sqrt{g} g^{\mu\nu} \frac{\delta\tilde{\Sigma}}{\delta \tau^{a\nu}}\right) = 0. \tag{3.4}$$

The first two conditions in (3.4) rule out in $\tilde{\Sigma}$ any possible dependence on the two Lagrange multiplier fields b^a and d^a. On the other hand, the two last local constraints in (3.4) imply that the general counterterm $\tilde{\Sigma}$ depends on $\bar{c}^a$, $\bar{\varphi}^a$, $\Omega^{a\mu}$ and $\tau^{a\mu}$ only through the combinations:

$$\hat{\Omega}^{a\mu} = \Omega^{a\mu} + \sqrt{g} g^{\mu\nu} \partial_\nu \bar{c}^a \tag{3.5}$$

and

$$\hat{\tau}^{a\mu} = \tau^{a\mu} + \sqrt{g} g^{\mu\nu} \partial_\nu \bar{\varphi}^a . \tag{3.6}$$

The antighost equations in (3.2), can then be rewritten in a simpler form:

$$\int d^3x \frac{\delta \tilde{\Sigma}}{\delta c^a} = \int d^3x \frac{\delta \tilde{\Sigma}}{\delta \varphi^a} = 0. \tag{3.7}$$

Both the gauge-fixing and the ghost equations (3.4) are known to be anomaly free and therefore to hold at all orders of the perturbative expansion [17]. The same is also true in the context of the antighost equations given in (3.2) (see ref.[18]).

At this stage, the deformation is seen to depend on the fields and sources as:

$$\tilde{\Sigma} = \tilde{\Sigma} \left(A_\mu, B_\mu, c, \varphi, g_{\mu\nu}, \hat{g}_{\mu\nu}, \hat{\Omega}^\mu, \hat{\tau}^\mu, L, D, K_\mu, M_\mu \right) . \tag{3.8}$$

Further information on $\tilde{\Sigma}$ can be obtained by considering the remaining constraints in (3.3). We combine the three conditions (3.3) into one single cohomology problem [19]:

$$\delta \tilde{\Sigma} = 0, \tag{3.9}$$

where

$$\delta = \mathcal{S}_\Sigma + \mathcal{W}^S_{(\xi)} + \mathcal{W}^D_{(\epsilon)} + \mathcal{U}_{(\xi)} + \mathcal{V}_{(\epsilon)}, \tag{3.10}$$

with

$$\mathcal{U}_{(\xi)} = \int d^3x [\epsilon, \xi]^\mu \frac{\delta}{\delta \xi^\mu} \tag{3.11}$$

and

$$\mathcal{V}_{(\epsilon)} = \int d^3x \left\{ \tfrac{1}{2} [\epsilon, \epsilon]^\mu - \xi^\mu \right\} \frac{\delta}{\delta \epsilon^\mu} . \tag{3.12}$$

One easily verifies that the extended operator δ is nilpotent:

$$\delta\delta = 0. \tag{3.13}$$

To determine the general counterterm we study the BRS-cohomology $\mathcal{H}^*(\delta)$ of δ in the neutral ghost charge sector. Likewise, to characterize the eventual anomalies in the theory one investigates the sector with one unit of $\Phi\Pi$ charge.

In order to simplify the notation we rewrite the basic fields and sources in terms of differential p-forms (scalar fields are 0-forms):

$$A = A^a_\mu T^a dx^\mu, \qquad B = B^a_\mu T^a dx^\mu, \qquad K = K_\mu dx^\mu, \qquad M = M_\mu dx^\mu,$$

$$\hat{\Omega} = \tfrac{1}{2}\epsilon_{\mu\nu\rho}\left(\Omega^{a\mu} + \sqrt{g}g^{\mu\sigma}\partial_\sigma \bar{c}^a\right)T^a dx^\nu \wedge dx^\rho, \qquad \hat{\tau} = \tfrac{1}{2}\epsilon_{\mu\nu\rho}\left(\tau^{a\mu} + \sqrt{g}g^{\mu\sigma}\partial_\sigma \bar{\varphi}^a\right)T^a dx^\nu \wedge dx^\rho,$$

$$L = \tfrac{1}{6}\epsilon_{\mu\nu\rho}L^a T^a dx^\mu \wedge dx^\nu \wedge dx^\rho, \qquad D = \tfrac{1}{6}\epsilon_{\mu\nu\rho}D^a T^a dx^\mu \wedge dx^\nu \wedge dx^\rho, \tag{3.14}$$

where the T^a's are the generators of $\mathbf{G}$. One also introduces the exterior derivative operator, $d = dx^\mu \partial_\mu$. To characterize the cohomology $\mathcal{H}^*(\delta)$ we define a filtration operator $\mathcal{N}$:

$$\mathcal{N} = \int_{\mathcal{M}} \sum_{fields\ f} f\frac{\delta}{\delta f} + \int d^3x \left(\epsilon^\mu \frac{\delta}{\delta \epsilon^\mu} + \xi^\mu \frac{\delta}{\delta \xi^\mu}\right). \tag{3.15}$$

The functional operator $\mathcal{N}$ induces a separation on δ,

$$\delta = \delta_0 + \delta_1, \tag{3.16}$$

with

$$[\mathcal{N}, \delta_0] = 0, \qquad [\mathcal{N}, \delta_1] = \delta_1. \tag{3.17}$$

It is useful to display the explicit expression of δ_0:

$$\begin{aligned} \delta_0 &= Tr\int_{\mathcal{M}}\left[(dc)\frac{\delta}{\delta A} + (dB)\frac{\delta}{\delta\hat{\Omega}} + (d\varphi)\frac{\delta}{\delta B} + (dA)\frac{\delta}{\delta\hat{\tau}} + (d\hat{\Omega})\frac{\delta}{\delta L} + (d\hat{\tau})\frac{\delta}{\delta D}\right] \\ &+ \int_{\mathcal{M}} K\frac{\delta}{\delta M} + \int d^3x\left(\hat{g}_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} - \xi^\mu\frac{\delta}{\delta\epsilon^\mu}\right) \end{aligned} \tag{3.18}$$

which, as a consequence of (3.13), is also a coboundary operator. We notice then that the three pairs of variables (M, K), $(g_{\mu\nu}, \hat{g}_{\mu\nu})$ and $(\epsilon^\mu, -\xi^\mu)$ appear in (3.18) in BRS-doublets, implying that the δ_0-cohomology $\mathcal{H}^*(\delta_0)$ is independent of those fields [20]. Hence, the δ-cohomology is also independent of the fields K, M, $g_{\mu\nu}$, $\hat{g}_{\mu\nu}$, ϵ^μ and ξ^μ. Now, since the metric tensor is not present in the δ-cohomology our counterterm analysis reduces essentially to that of ref.[13]: it is reasonable to expect here the same finiteness properties of the flat space case.

Let us first focus our attention on the characterization of $\mathcal{H}^*(\delta_0)$. If this is the case, our task reduces to the determination of an integrated polynomial $\tilde{\Delta}(A, B, c, \varphi, \hat{\Omega}, \hat{\tau}, L, D)$ which solves:

$$\delta_0\tilde{\Delta} = \delta_0 \int_{\mathcal{M}} \omega^0_3 = 0 \tag{3.19}$$

where ω^q_p is a local polynomial in the fields and sources of form degree p and $\Phi\Pi$ charge q. The general form of ω^q_p is determined by considering the following set of descent equations:

$$\delta_0\omega^0_3 + d\omega^1_2 = 0, \quad \delta_0\omega^1_2 + d\omega^2_1 = 0, \quad \delta_0\omega^2_1 + d\omega^3_0 = 0, \quad \delta_0\omega^3_0 = 0. \tag{3.20}$$

The simplest way for evaluating ω_3^0 consists of solving a local cohomology problem: the last equation in (3.20). The solution ω_0^3 is:

$$\omega_0^3 = xTr(c^3) + yTr(c^2\varphi) + wTr(c\varphi^2) + zTr(\varphi^3) \tag{3.21}$$

where x, y, w and z are constants. The expression of the associated non-trivial δ_0-cocycle with zero ghost number is obtained by means of well-known techniques (see for instance [21]):

$$\begin{aligned} \omega_3^0 = & - xTr\left(\tfrac{1}{3}A^3 + \hat{\tau}\{A,\varphi\} + Dc^2\right) + \\ & - yTr\left(A^2B + \hat{\Omega}\{A,c\} + \hat{\tau}\{A,\varphi\} + \hat{\tau}\{B,c\} + D\{\varphi,c\} + Lc^2\right) + \\ & - wTr\left(AB^2 + \hat{\Omega}\{B,c\} + \hat{\Omega}\{A,\varphi\} + \hat{\tau}\{B,\varphi\} + L\{c,\varphi\} + D\varphi^2\right) + \\ & - zTr\left(\tfrac{1}{3}B^3 + \hat{\Omega}\{B,\varphi\} + L\varphi^2\right) + \delta_0\sigma_3^{-1} + d\sigma_2^0 \end{aligned} \tag{3.22}$$

where the two last pieces stand for trivial δ_0- and d-cocycles. The integrated polynomial $\tilde{\Delta}$ reads as:

$$\tilde{\Delta} = \Delta + \delta_0\hat{\Delta} = \int_{\mathcal{M}} \omega_3^0 + \delta_0 \int_{\mathcal{M}} \sigma_3^{-1}. \tag{3.23}$$

As in ref.[6], we observe that Δ is invariant under the complete operator δ and since it does not depend on the vector parameter ξ^μ, it has to belong to $\mathcal{H}^*(\delta)$. This means that the general solution of the cohomology problem in (3.9) – i.e. the general counterterm $\tilde{\Sigma}$ – is given by:

$$\tilde{\Sigma} = \Delta + \delta\hat{\Delta}, \tag{3.24}$$

with Δ and $\hat{\Delta}$ as in (3.23).

To obtain $\hat{\Delta}$ one has to select all possible diffeomorphism invariant integrated monomials of dimension zero and $\Phi\Pi$ charge minus one. We list here the eight independent contributions:

$$\begin{array}{llll} \hat{\Delta}_1 = Tr\int_{\mathcal{M}} Lc\ , & \hat{\Delta}_2 = Tr\int_{\mathcal{M}} L\varphi\ , & \hat{\Delta}_3 = Tr\int_{\mathcal{M}} Dc\ , & \hat{\Delta}_4 = Tr\int_{\mathcal{M}} D\varphi\ , \\ \hat{\Delta}_5 = Tr\int_{\mathcal{M}} \hat{\Omega}A\ , & \hat{\Delta}_6 = Tr\int_{\mathcal{M}} \hat{\Omega}B\ , & \hat{\Delta}_7 = Tr\int_{\mathcal{M}} \hat{\tau}A\ , & \hat{\Delta}_8 = Tr\int_{\mathcal{M}} \hat{\tau}B\ . \end{array} \tag{3.25}$$

We observe however that ξ^μ, being an infinitesimal vector parameter, is not present in the original expression of the complete action (2.17) and thus, we are allowed to discard from our analysis any deformation depending on it [6]. This is indeed the case of the counterterms $\delta\hat{\Delta}_i$'s: they cannot be linearly combined in such a way that the resulting polynomial is ξ^μ-independent.

Finally, one has to consider the following expression for the general counterterm $\tilde{\Sigma}$:

$$\tilde{\Sigma} = \Delta. \tag{3.26}$$

Here, the conclusion is that the r.h.s. of (3.26) is also unacceptable as a deformation of Σ since it clearly represents a violation of the two antighost equations in (3.7). This fact expresses that no deformations are allowed.

Still, in order to extend this last result to all orders of perturbation theory one has to discuss the absence of δ-anomalies. At the non-integrated level, the possible anomaly in the condition (3.9) is related to a cocycle ω_3^1 standing in the following set of descent equations:

$$\delta_0\omega_3^1 + d\omega_2^2 = 0, \quad \delta_0\omega_2^2 + d\omega_1^3 = 0, \quad \delta_0\omega_1^3 + d\omega_0^4 = 0, \quad \delta_0\omega_0^4 = 0. \tag{3.27}$$

The 0-form with $\Phi\Pi$ charge four which solves the last equation in (3.27) is given by the linear combination of two objects: $Tr(c^2\varphi^2)$ and $Tr(c\varphi c\varphi)$. Starting from the first candidate one would arrive at the following non-trivial cocycle:

$$\begin{aligned} \omega_3^1 &= Tr\Big(c^2\varphi L + c^2 L\varphi + c^2 B\hat{\Omega} + c^2\hat{\Omega}B + cA\varphi\hat{\Omega} + \\ &+ cA\hat{\Omega}\varphi + cAB^2 + c\hat{\tau}\varphi B + c\hat{\tau}B\varphi + cD\varphi^2 + \\ &+ Ac\varphi\hat{\Omega} + Ac\hat{\Omega}\varphi + AcB^2 + A^2\varphi B + A^2 B\varphi + \\ &+ A\hat{\tau}\varphi^2 + \hat{\tau}c\varphi B + \hat{\tau}cB\varphi + \hat{\tau}A\varphi^2 + Dc\varphi^2\Big). \end{aligned} \tag{3.28}$$

However, one observes that (3.28) is forbidden by the antighost equations (3.7). The same reasoning is also true for the anomaly candidate steming from $Tr(c\varphi c\varphi)$. Hence, the condition (3.9) is free of anomalies: our arguments are seen to hold beyond the classical approximation.

We arrive at the main conclusion of the present work: the three-dimensional BF system defined on a manifold $\mathcal{M}$ is finite to all orders.

4 Concluding Remarks

The problem of the renormalization of the three-dimensional BF system on a curved space-time was discussed by using the BRS-algebraic technique.

It was shown that the model is finite to all orders due to the superdiffeomorphisms Ward identity and the others stability constraints. However, we stress once more that the finiteness proof presented here should only be taken as a rigorous result in the restricted case of a topologically flat space-time manifold which possesses an asymptotically flat metric. Under these assumptions, we are allowed to make use in this paper of the Quantum Action Principles and to arrive to the aforesaid proof to all orders.

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References

[1] A.S. Schwarz, Baku International Topological Conference, Abstracts, vol. 2, p.345, 1987.

[2] S.S. Chern and J. Simons, Ann. Math. **99**(1974)48.

[3] E. Witten, Comm. Math. Phys. **121**(1989)351.

[4] E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. **B330**(1990)575; **B336** (1990)581.

[5] F. Delduc, C. Lucchesi, O. Piguet and S.P. Sorella, Nucl. Phys. **B346**(1990)313.

[6] C. Lucchesi and O. Piguet, Phys. Lett. **B271** (1991)350; Nucl. Phys. **B381**(1992)281.

[7] C.M. Becchi, A. Rouet and R. Stora, Comm. Math. Phys. **42**(1975)127; Ann. Phys. (N.Y.) **98**(1976)287.

[8] F. Delduc, F. Gieres and S.P. Sorella, Phys. Lett. **B225**(1989)367.

[9] D. Birmingham, M. Rakowski and G. Thompsom, Nucl. Phys. **B329**(1990)83.

[10] M. Blau and G. Thompson, Phys. Lett. **B228**(1989)64; Ann. Phys. (N.Y.) **205** (1991)130.

[11] G.T. Horowitz, Comm. Math. Phys. **125**(1989)417; G.T. Horowitz and M. Srednicki, Comm. Math. Phys. **130**(1990)83.

[12] N. Maggiore and S.P. Sorella, Nucl. Phys. **B377** (1992)236.

[13] C. Lucchesi, O. Piguet and S.P. Sorella, Nucl. Phys. **B395**(1993)661.

[14] O. Piguet and K. Sibold, Nucl. Phys. **B253**(1985)517.

[15] C.M. Becchi,"The Renormalization of Gauge Theories", in the Proceedings of Les Houches Summer School 1983, eds. B.S. De Witt and R. Stora, North Holland, Amsterdam, 1984.

[16] K. Symanzik,"Renormalization of Theories with Broken Symmetries", in the Cargése Lectures in Physics 1970, vol.5, 179, ed. D. Bessis, Gordon and Breach, 1972.

[17] O. Piguet and A. Rouet, Phys. Rep. **76 C**(1981)1.

[18] A. Blasi, O. Piguet and S.P. Sorella, Nucl. Phys. **B356**(1991)154.

[19] C.M. Becchi, A. Blasi, G. Bonneau, R. Collina and F. Delduc, Comm. Math. Phys. **120**(1988)121.

[20] J.A. Dixon,"Cohomology and Renormalization of Gauge Theories", preprints Imperial College, 1977; Comm. Math. Phys. **139**(1991)495.

[21] S.P. Sorella, Comm. Math. Phys. **189**(1993)456.