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Algebraic Solution for Certain Noncentral Potentials

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Abstract. The Green's function relative to certain noncentral potentials is constructed via the so(2,1) algebraic approach in cylindrical parabolic coordinates. The energy spectrum and the correctly normalized wave functions are deduced. A particular case of Smorodinsky-Winternitz potentials is also analyzed.

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1 Introduction

The application of the algebraic approach of Kleinert [1] to quantum mechanical problems has seen rapid expansion in recent years. A variant of this method was proposed by Milshtein and Strakhovenko [2] (hereafter, MS) to build the Green's function for a Dirac electron in a static Coulomb field. It has received renewed attention following the development of

path integration techniques and it has been demonstrated that a certain number of systems displaying dynamical symmetry can be treated with this variant of the algebraic technique [3, 4, 5, 6].
We shall want to present here a rigorous algebraic treatment of the nonrelativistic quantum mechanical system corresponding to the three dimensional noncentral potential

$$V_1(\mathbf{r}) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\beta}{y^2} + \frac{\gamma x}{y^2\sqrt{x^2+y^2}} + \alpha' z^2 + \frac{\beta'}{z^2}, \tag{1}$$

where $\alpha, \beta, \gamma, \alpha'$, and β' are positive constants. It looks intractable in cartesian coordinates as well as in polar coordinates. If $\alpha = \alpha' = 0$, we have a separable potential in different coordinates systems which belongs to the large class of Smorodinsky-Winternitz potentials with dynamical symmetries [7]. This class of potentials has been discussed by path integration very recently [8, 9, 10].
The algebraic solution for the potential (1) via the MS technique is facilitated by using the cylindrical parabolic variables (u_1, u_2, z) defined by

$$x = u_1^2 - u_2^2, \qquad y = 2u_1u_2, \qquad z = z, \tag{2}$$

with $-\infty < u_1, u_2, z < \infty$. Note that due to the strong singularities at $y = 0$ and $z = 0$, creating impenetrable barriers, the potential (1) can be analyzed into four completely separated regions such that is sufficient to consider only the domain $0 < y, z < \infty$ and $x \in \Re$, respectively $0 < u_1, u_2, z < \infty$.

2 Green's function via so(2,1) algebra

By using the Schwinger's integral representation [11], the Green's function associated with the potential (1) is given by

$$G(\mathbf{r}, \mathbf{r}'; E) = \int_0^\infty ds \exp\left\{-\frac{is}{\hbar}\left[-\frac{\hbar^2}{2M}\nabla^2 + V_1(\mathbf{r}) - E - i0\right]\right\}\delta(\mathbf{r} - \mathbf{r}'). \tag{3}$$

Since the z-variable is separable from others, we can express the Green's function as a Fourier transform of a product of two kernels,

$$G(\mathbf{r}, \mathbf{r}'; E) = \int_0^\infty ds \exp\left[\frac{is}{\hbar}(E + i0)\right] K(\vec{\rho}, \vec{\rho}'; s) K(z, z'; s), \tag{4}$$

where

$$K(\vec{\rho}, \vec{\rho}'; s) = \exp\left\{-\frac{is}{\hbar}\left[-\frac{\hbar^2}{2M}\nabla_\rho^2 + V_1(\vec{\rho})\right]\right\}\delta(\vec{\rho} - \vec{\rho}') =$$

$$\frac{1}{4\rho}\exp\left\{-\frac{i}{\hbar}\frac{s}{4\rho}\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right) + \frac{\beta - \gamma}{u_1^2} + \frac{\beta + \gamma}{u_2^2} - 4\alpha\right]\right\}\prod_{j=1}^{2}\delta(u_j - u_j');$$

$$\vec{\rho} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{5}$$

and

$$K(z,z';s) = \exp\left\{-\frac{is}{\hbar}\left[-\frac{\hbar^2}{2M}\nabla_z^2 + V_1(z)\right]\right\}\delta(z-z')$$
$$= \exp\left\{-\frac{is}{\hbar}\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2} + \alpha' z^2 + \frac{\beta'}{z^2}\right]\right\}\delta(z-z'). \tag{6}$$

Both of expressions (5) and (6) can be treated algebraically using the non compact Lie algebra so(2,1) characterized by the commutation relations

$$[T_1, T_2] = -iT_1, \qquad [T_2, T_3] = -iT_3, \text{and} \qquad [T_1, T_3] = -iT_2. \tag{7}$$

The form of the Hamiltonian in (5) and (6) makes it convenient to use the differential realization of the operators:

$$T_1(\xi) = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial \xi^2} - \frac{\mu(\mu-1)}{\xi^2}\right],$$
$$T_2(\xi) = -\frac{i}{2}\left[\xi\frac{\partial}{\partial \xi} + \frac{1}{2}\right],$$
$$T_3(\xi) = \frac{M}{4\hbar^2}\xi^2, \tag{8}$$

for $0 < \xi < \infty$.

A. Axial propagator

First, we see that the kernel (6) is the propagator of an harmonic oscillator with a constant frequency, constrained to a centrifugal repulsion. It can easily be expressed in terms of these operators as

$$K(z,z';s) = \exp\left\{-\frac{is}{\hbar}\left[T_1(z) + 2\hbar^2 w^2 T_3(z)\right]\right\}\delta(z-z'), \tag{9}$$

where $w = \left(\frac{2\alpha'}{M}\right)^{\frac{1}{2}}$.
By using the same procedure as the one described in our previous paper [6] based upon the Baker-Campbell-Hausdorff formulas [12], we get

$$K(z,z';s) = \frac{Mw}{i\hbar\sin(ws)}(zz')^{\frac{1}{2}} I_{2\nu}\left(\frac{Mwzz'}{i\hbar\sin(ws)}\right)\exp\left\{\frac{iMw}{2\hbar}(z^2+z'^2)\mathrm{cotg}(ws)\right\}, \tag{10}$$

where $\nu = \frac{1}{4}\left(1 + \frac{8M\beta'}{\hbar^2}\right)^{\frac{1}{2}}$.
The energy spectrum and the wave functions can be obtained from the Green's function which is the Fourier transform of the propagator (10)

$$G(z,z';\tilde{E}) = \int_0^\infty ds \exp\left[\frac{i\tilde{E}}{\hbar}s\right] K(z,z';s). \tag{11}$$

To evaluate this, we make use of a standard integral involving Bessel functions [13]

$$\int_0^\infty dq \frac{e^{2pq}}{\sinh q} \exp\left[-\frac{1}{2}(x+y)\coth q\right] I_{2\nu}\left(\frac{\sqrt{xy}}{\sinh q}\right) = \frac{\Gamma(p+\nu+1/2)}{\sqrt{xy}\Gamma(2\nu+1)} M_{-p,\nu}(x) W_{-p,\nu}(y);$$

$$\left(Re(1/2+\nu+p)>0, Re(\nu)>0, y>x\right), \tag{12}$$

where $M_{-p,\nu}(x)$ and $W_{-p,\nu}(y)$ are the standard Whittaker functions. This yields

$$G(z,z';\tilde{E}) = \frac{\Gamma(p+\nu+1/2)}{iw\sqrt{zz'}\Gamma(2\nu+1)} M_{-p,\nu}\left(\frac{Mw}{\hbar}z'^2\right) W_{-p,\nu}\left(\frac{Mw}{\hbar}z^2\right), \tag{13}$$

where $p = -\frac{\tilde{E}}{2\hbar w}$ and $z > z'$.
The poles of the Green's function, coming from the Γ-function in the numerator, are

$$-\frac{\tilde{E}}{2\hbar w} + \nu + \frac{1}{2} = -m, \qquad m = 0, 1, 2 \ldots, \infty, \tag{14}$$

or

$$\tilde{E} = 2\hbar w(m + \nu + \frac{1}{2}). \tag{15}$$

From the residues of the Green's function (13), corresponding to the poles (14), we extract the wave functions

$$Z_m(z) = A_m z^{-\frac{1}{2}} M_{m+\nu+1/2,\nu}\left(\frac{Mw}{\hbar}z^2\right), \tag{16}$$

with the normalized factor

$$A_m = \left[\frac{2\Gamma(m+2\nu+1)}{m!\Gamma^2(2\nu+1)}\right]^{\frac{1}{2}}. \tag{17}$$

Consequently, the propagator (6) describing the motion along the z axis has the spectral representation

$$K(z,z';s) = \sum_{m=0}^{\infty} Z_m(z) Z_m^*(z') \exp\left(-\frac{i}{\hbar}\tilde{E}_m s\right). \tag{18}$$

B. Polar Green's function

In order to evaluate the $\vec{\rho}$ part of the problem, let us go back to the expression (4) and let us insert (18). This leads to

$$G(\mathbf{r},\mathbf{r}';E) = \sum_{m=0}^{\infty} Z_m(z) Z_m^*(z') G_m(\vec{\rho},\vec{\rho}\,';E-\tilde{E}_m), \tag{19}$$

where

$$G_m(\vec{\rho},\vec{\rho}\,';E-\tilde{E}_m) = \int_0^\infty ds \exp\left[\frac{i}{\hbar}s(E-\tilde{E}_m+i0)\right] K(\vec{\rho},\vec{\rho}\,';s). \tag{20}$$

Now we perform the time transformation defined by

$$\tau = \frac{s}{4\rho} \quad , \tag{21}$$

the Green's function (20) is then written as

$$G_m(\vec{\rho}, \vec{\rho}'; E - \tilde{E}_m) = \int_0^\infty d\tau \exp\left[\frac{i}{\hbar}(4\alpha + i0)\tau\right]^2 \prod_{j=1}^{2} K(u_j, u_j'; \tau), \tag{22}$$

with

$$K(u_j, u_j'; \tau) = \exp\left\{-\frac{i\tau}{\hbar}\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial u_j^2} - \frac{\lambda_j(\lambda_j - 1)}{u_j^2}\right) - 4(E - \tilde{E}_m)u_j^2\right]\right\}\delta(u_j - u_j')$$

$$= \exp\left\{-\frac{i\tau}{\hbar}\left[T_1(u_j) + 2\hbar^2\Omega^2 T_3(u_j)\right]\right\}\delta(u_j - u_j'), \tag{23}$$

where

$$\lambda_j = \frac{1}{2} + \left[\frac{1}{4} + \frac{2M}{\hbar^2}\left(\beta + (-1)^j\gamma\right)\right]^{\frac{1}{2}}, \qquad (j = 1, 2) \tag{24}$$

and

$$\Omega = \left[-\frac{8}{M}(E - \tilde{E}_m\right]^{\frac{1}{2}}. \tag{25}$$

Thus the kernel (23) is identical in form with the radial propagator of an harmonic oscillator placed in an inverse square potential. As the solution is quite similar to that of (6), we obtain

$$G_m(\vec{\rho}, \vec{\rho}'; E - \tilde{E}_m) = \int_0^\infty d\tau \exp\left[\frac{i}{\hbar}(4\alpha + i0)\tau\right]\left(\frac{M\Omega}{i\hbar \sin\Omega\tau}\right)^2 \prod_{j=1}^{2}(u_j u_j')^{\frac{1}{2}}$$

$$\times \exp\left\{\frac{iM\Omega}{2\hbar}(u_j^2 + u'^2_j)\text{cotg}\Omega\tau\right\} I_{2\mu_j}\left(\frac{M\Omega u_j u'_j}{i\hbar \sin\Omega\tau}\right), \tag{26}$$

with

$$\mu_j = \frac{1}{4}\left[1 + \frac{8M}{\hbar^2}\left(\beta + (-1)^j\gamma\right)\right]^{\frac{1}{2}}, \qquad (j = 1, 2). \tag{27}$$

Using the polar coordinates (ρ, ϕ) in two dimensions,

$$u_1 = \sqrt{\rho}\cos\frac{\phi}{2}, \qquad u_2 = \sqrt{\rho}\sin\frac{\phi}{2}, \tag{28}$$

and the Bateman's expansion formula [14]

$$\frac{1}{2} z I_\nu(z \sin\alpha \sin\beta) I_\mu(z \cos\alpha \cos\beta) = (\sin\alpha \sin\beta)^\nu (\cos\alpha \cos\beta)^\mu$$

$$\times \sum_{n=0}^{\infty}(\nu + \mu + 2n + 1)\frac{n!\Gamma(\nu + \mu + n + 1)}{\Gamma(\nu + n + 1)\Gamma(\mu + n + 1)} I_{\mu+\nu+2n+1}(z) P_n^{(\nu,\mu)}(\cos 2\alpha) P_n^{(\nu,\mu)}(\cos 2\beta), \tag{29}$$

the Green's function associated with the $\vec{\rho}$ part of the problem can then be given as follows

$$G_m(\vec{\rho},\vec{\rho}';E-\tilde{E}_m)=\sum_{n=0}^{\infty}G_n(\rho,\rho';E-\tilde{E}_m)\Phi_n^*(\phi')\Phi_n(\phi), \tag{30}$$

where

$$\Phi_n(\phi)=\left[\frac{2(n+q)n!\Gamma(n+2q)}{\Gamma(2\mu_1+n+1)\Gamma(2\mu_2+n+1)}\right]^{\frac{1}{2}}\left(\sin\frac{\phi}{2}\right)^{2\mu_2+\frac{1}{2}}\left(\cos\frac{\phi}{2}\right)^{2\mu_1+\frac{1}{2}}P_n^{(2\mu_2,2\mu_1)}(\cos\phi), \tag{31}$$

with

$$q=\mu_1+\mu_2+\frac{1}{2} \tag{32}$$

The $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomials. Note that the angular wave functions (31) are correctly normalized.

The radial Green's function $G_n(\rho,\rho';E-\tilde{E}_m)$ appearing in (30) has the form

$$G_n(\rho,\rho';E-\tilde{E}_m)=\frac{2M\Omega}{i\hbar}\int_0^{\infty}\frac{d\tau}{\sin\Omega\tau}\exp[\frac{i}{\hbar}(4\alpha+i0)\tau+\frac{iM\Omega}{2\hbar}(\rho+\rho')\text{cotg}\Omega\tau]$$
$$\times I_{2n+2q}\left(\frac{M\Omega(\rho\rho')^{\frac{1}{2}}}{i\hbar\sin\Omega\tau}\right). \tag{33}$$

Eventually, with the help of the formula (12), we get

$$G_n(\rho,\rho';E-\tilde{E}_m)=\frac{2}{i\Omega}(\rho\rho')^{\frac{-1}{2}}\frac{\Gamma(p+n+q+\frac{1}{2})}{\Gamma(2n+q+1)}M_{-p,n+q}\left(\frac{M\Omega}{\hbar}\rho'\right)W_{-p,n+q}\left(\frac{M\Omega}{\hbar}\rho\right), \tag{34}$$

where

$$p=-\left(\frac{2\alpha}{\hbar\Omega}\right). \tag{35}$$

The discrete energy spectrum can be obtained from the poles of the gamma function $\Gamma(p+n+q+\frac{1}{2})$, i.e., when $p+n+q+\frac{1}{2}=-n', (n'=0,1,2,..)$.

The energy eigenvalues are then given by

$$E_{Nm}-\tilde{E}_m=-\frac{M\alpha^2}{2\hbar^2(\mu_1+\mu_2+N)^2} \tag{36}$$

where $N=n'+n+1$.

The radial wave functions, properly normalized, are obtained at the poles of (34). This result in

$$R_{Nm}(\rho)=\frac{1}{(N+q-1/2)\Gamma(1+2n+2q)}\left[\frac{M\alpha}{\hbar^2}\frac{\Gamma(2q+N+n)}{\Gamma(N-n)}\right]^{\frac{1}{2}}\rho^{-\frac{1}{2}}M_{N+q-1/2,n+q}\left(\frac{M\Omega}{\hbar}\rho\right). \tag{37}$$

Adding (15) and (36) gives the energy levels of the system

$$E_{Nm}=2\hbar w(m+\nu+\frac{1}{2})-\frac{M\alpha^2}{2\hbar^2(\mu_2+\mu_2+N)^2}. \tag{38}$$

The normalized wave functions of the bound states are

$$\Psi_{Nmn}(\mathbf{r})=R_{Nm}(\rho)\Phi_n(\phi)Z_m(z), \tag{39}$$

where $R_{Nm}(\rho)$, $\Phi_n(\phi)$ and $Z_m(z)$ are given by (37), (31) and (16) respectively.

3 Special case

By setting $\alpha = \alpha' = 0$ in the expression (1), we obtain the potential

$$V_2(\mathbf{r}) = \frac{\beta}{y^2} + \frac{\gamma x}{y^2\sqrt{x^2+y^2}} + \frac{\beta'}{z^2}, \tag{40}$$

which belongs to the class of the three-dimensional maximally super-integrable Smorodinsky-Winternitz potentials. This potential has already been studied by the path integral approach in ref [8, 9, 10]. It is obvious that the physical system moving in this potential is characterized by the absence of bound states. In order to find the corresponding Green's function, we shall first let w approach zero in (10); thus we can readily obtain

$$K(z,z';s) = \frac{M}{i\hbar s}\sqrt{zz'}I_{2\nu}\left(\frac{Mzz'}{i\hbar s}\right)\exp\left\{\frac{iM}{2\hbar s}(z^2+z'^2)\right\}. \tag{41}$$

With the help of the formula [13] which allows the separation of the variables z, z' and s,

$$\int_0^\infty dx \exp(-\alpha x)J_\nu(2\beta\sqrt{x})J_\nu(2\gamma\sqrt{x}) = \frac{1}{\alpha}I_\nu\left(\frac{2\beta\gamma}{\alpha}\right)\exp\left(-\frac{\beta^2+\gamma^2}{\alpha}\right), \tag{42}$$

the propagator (41) takes the following form:

$$K(z,z';s) = \sqrt{zz'}\int_0^\infty k_z dk_z J_{2\nu}(k_z z)J_{2\nu}(k_z z')\exp\left(-\frac{i}{\hbar}\frac{\hbar^2 k_z^2}{2M}s\right). \tag{43}$$

In this case the Green's function (19) becomes

$$\begin{aligned} G(\mathbf{r},\mathbf{r}';E) &= \int_0^\infty dk_z Z_{k_z}(z)Z^*_{k_z}(z')G_{k_z}(\vec{\rho},\vec{\rho}';\hat{E}) \\ &= \sum_{n=0}^\infty \Phi^*_n(\phi')\Phi_n(\phi)\int_0^\infty dk_z Z_{k_z}(z)Z^*_{k_z}(z')G_{k_z}(\rho,\rho';\hat{E}) \end{aligned} \tag{44}$$

where $G_n(\rho,\rho';\hat{E})$ is the radial part of the Green's function, with

$$\hat{E} = E - \left(\frac{\hbar^2 k_z^2}{2M}\right). \tag{45}$$

By letting α approach zero, thus from (34) it follows that

$$\begin{aligned} G_n(\rho,\rho';\hat{E}) &= \frac{2}{i\Omega}\frac{1}{\sqrt{\rho\rho'}}\frac{\Gamma(n+q+\frac{1}{2})}{\Gamma(2n+2q+1)}M_{0,n+q}\left(\frac{M\Omega}{\hbar}\rho'\right)W_{0,n+q}\left(\frac{M\Omega}{\hbar}\rho\right) \\ &= \frac{i\hbar}{2\pi}\frac{1}{\sqrt{\rho\rho'}}\frac{\Gamma(n+q+\frac{1}{2})}{\Gamma(2n+2q+1)}\int_C \frac{d\sigma}{\hat{E}+i0-\frac{\hbar^2\sigma^2}{2M}}M_{0,n+q}\left(\frac{M\Omega}{\hbar}\rho'\right)W_{0,n+q}\left(\frac{M\Omega}{\hbar}\rho\right). \end{aligned} \tag{46}$$

We take for C the closed contour

$$c : \left\{ \begin{array}{l} \sigma = k, k \in [-R, R] \\ \sigma = Re^{i\theta}, \theta \in (\pi, 2\pi) \end{array} \right. \tag{47}$$

and consider the limit $R \longrightarrow \infty$. Taking into account the asymptotic behaviour [13] of the Whittaker functions, it is easy to show that the integral over the semi-circle vanishes. So we get

$$G_n(\rho, \rho'; \hat{E}) = \frac{i\hbar}{2\pi} \frac{1}{\sqrt{\rho\rho'}} \frac{\Gamma(n+q+\frac{1}{2})}{\Gamma(2n+2q+1)} \int_{-\infty}^{+\infty} \frac{dk}{\hat{E} + i0 - \frac{\hbar^2 k^2}{2M}} \times M_{0,n+q}\left(-2ik\rho'\right) W_{0,n+q}\left(2ik\rho\right). \tag{48}$$

Now we replace

$$M_{\lambda,\mu}(z) = e^{-i\pi(\mu+\frac{1}{2})} M_{-\lambda,\mu}(-z) \tag{49}$$

and use the relation [13]

$$M_{\lambda,\mu}(z) = \Gamma(2\mu+1) e^{i\pi\lambda} \left[\frac{W_{-\lambda,\mu}(-z)}{\Gamma(\mu-\lambda+\frac{1}{2})} + e^{-i\pi(\mu+\frac{1}{2})} \frac{W_{\lambda,\mu}(z)}{\Gamma(\mu+\lambda+\frac{1}{2})} \right], \tag{50}$$

valid for $\arg z \in (\frac{\pi}{2}, \frac{3\pi}{2}), 2\mu \neq -1, -2, -3, \ldots$, to find

$$G_n(\rho, \rho'; E) = \frac{i\hbar}{2\pi} \frac{M}{\hbar^2} \frac{1}{\sqrt{\rho\rho'}} \frac{\Gamma^2(n+q+\frac{1}{2})}{\Gamma^2(2n+2q+1)} \int_{\frac{\hbar^2 k_z^2}{2M}}^{+\infty} \frac{dE_k}{E + i0 - \frac{\hbar^2(k_z^2+k^2)}{2M}} \times \frac{1}{k} M_{0,n+q}\left(-2ik\rho'\right) W_{0,n+q}\left(2ik\rho\right). \tag{51}$$

Thanks to the relations between the $M_{0,\mu}(z)$ Whittaker functions, the first and second-kind Bessel functions [13]

$$M_{0,\mu}(z) = 2^{2\mu}\Gamma(\mu+1)\sqrt{z} I_\mu(\frac{z}{2}) \tag{52}$$

$$I_\mu(z) = e^{-\frac{i\mu\pi}{2}} J_\mu(iz), \tag{53}$$

as well as the doubling formula of the gamma function

$$\sqrt{\pi}\Gamma(2\mu+1) = 2^{2\mu}\Gamma(\mu+1)\Gamma(\mu+\frac{1}{2}), \tag{54}$$

the Green's function (51) can be written as follows:

$$G_n(\rho, \rho'; E) = \frac{iM}{\hbar} \int_{\frac{\hbar^2 k_z^2}{2M}}^{+\infty} \frac{dE_k}{E + i0 - \frac{\hbar^2}{2M}(k_z^2 + k^2)} J_{n+q}(k\rho') J_{n+q}(k\rho). \tag{55}$$

The complete wave functions and the energy spectrum can then be easily deduced

$$\Psi_{E_k,n,k_z}(\rho, \phi, z) = \left(\frac{M}{\hbar^2} z k_z\right)^{\frac{1}{2}} J_{n+q}(k\rho)\Phi_n(\phi) J_{2\nu}(k_z z), \tag{56}$$

$$E_{k,k_z} = \frac{\hbar^2}{2M}(k^2 + k_z^2). \tag{57}$$

4 Conclusion

By using the cylindrical coordinates, it becomes apparent that the potential $V_1(\mathbf{r})$ possesses $SO(2,1) \otimes SO(2,1) \wedge SO(2,1)$ as a dynamical group of symmetry. We have thus calculated the Green's function in the Schwinger's integral representation via the $so(2,1)$ algebraic approach. The purely scattering potential $V_2(\mathbf{r})$ which belongs to Smorodinsky-Winternitz class of potentials may also be considered as a particular case of the potential $V_1(\mathbf{r})$.

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