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Localisation in a Single-Band Approximation to Random Schroedinger Operators in a Magnetic Field

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Abstract We study random Schrödinger operators in an external magnetic field restricted to the first Landau band. It is shown that for a general class of possibly unbounded potential distributions, the Hamiltonian is well-defined as a self-adjoint integral operator and has a single-band spectrum almost surely. A theorem is proved giving conditions under which the eigenstates corresponding to eigenvalues near the edge of the spectrum are localised. This theorem is analogous to the one formulated by von Dreifus and Klein in the case of lattice models. The conditions are checked in the case of unbounded distributions of the random potential.

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1. Introduction

When electrons move in a random potential, subtle interference effects may localise their wavepacket in an essentially finite region of space. Since Anderson's first prediction [An] of the phenomenon,both the physics and mathematics of the subject have been much developped [LiPa], [CaLa]. It is well established that, typically, the states corresponding to energies near the edge of spectral bands are exponentially localised. Mathematical proofs of this fact exist for a wide class of one-dimensional lattice or continuum models, where in fact all states are localised. We refer the reader to [GoMoPa], [KuSo] for early works and the book [CaLa] for more recent material. In two or higher dimensions exponential localisation near a band edge is proven mostly for lattice models (see [FrSp] and the review [MaSc] for the first proofs, and [Sp], [DrKl], [CaLa], [AiMo] for more recent progress) and for a special class of continuous models where the random potential takes constant uncorrelated values on cells or blocks of $\mathbb{R}^d$ [HoMa], [KoSi]. In [BeGrMaSc] it was noticed that the original techniques and results of [FrSp] and [FrMaScSp] remain unchanged if the hopping matrix of the lattice model has a complex phase factor representing the effect of a magnetic field. For a continuum setting with magnetic fields (let us say in $d = 2$), while the existence of localised states is accepted, it has not yet been established in a rigorous way. This problem is particularly important in connection with the Quantum Hall Effect where the following picture is an essential ingredient for the explanation of the effect [GiPr], [Ma]. The random potential broadens the discrete, highly degenerate, Landau levels which become "Landau bands", with a density of states taking large values near the original levels and being small for energies between the levels. The latter energies are at the "edges" of the Landau bands and correspond to localised states. Since the conductivity is non-vanishing this implies that there must exist at least one energy, presumably near the center of the band, where the localisation length diverges [Ha]. See [Ku] for a rigorous discussion based on the Kubo formula. Whether there exists in fact a whole non-zero range of energies corresponding to "extended states", and what is the nature of the spectrum (e.g. absolutely or singular continuous) is still not at all clear (see for example [Th], [AoAn], [Tr]). In this paper we are concerned only with the first, easier aspect, of localisation at the edges of the Landau bands. At first sight it would seem that a magnetic field would have a localising effect. However, this cannot be so since in two dimensions without magnetic fields, scaling theories [AALR], [Wel] predict all states to be localised while as we have just discussed the magnetic field must "delocalise" at least one energy. The classical picture does not take into account the Aharonov-Bohm type phases introduced by the field, which may drastically modify the underlying interference effects.

In the present work we consider an electron, allowed to move in an infinite two-dimensional plane $\mathbb{R}^2 = \mathbb{C}$, submitted to a uniform, perpendicular magnetic field B. The random potential $V(z)$, $z \in \mathbb{R}^2$ (or $\mathbb{C}$) takes constant values $v_x \in \mathbb{R}$ for $z \in B(x)$, where $B(x)$ is a unit square centered at $x \in \mathbb{Z}^2$. The v_x, $x \in \mathbb{Z}^2$, are independent identically distributed random variables. Our precise hypotheses on the probability distribution are stated in Sect. 2. In axial gauge $A(z) = \frac{B}{2}(-\mathrm{Im}z, \mathrm{Re}z)$ the usual Hamiltonian is

$$\tilde{H}(V) = (-i\nabla - A(z))^2 + V(z). \tag{1.1}$$

In our model, however, we only consider electron states belonging to the first Landau level. This means that we consider states in the infinite-dimensional subspace of $L^2(\mathbb{R}^2)$ corresponding to the projection P_0 onto the first Landau level. The Hamiltonian is

$$H(V) = P_0 \tilde{H}(V) P_0 = \frac{B}{2} P_0 + P_0 V P_0 \tag{1.2}$$

where the first term comes from the projection of the kinetic part. Since this term only changes the energy by a constant we will drop it. Our Hamiltonian thus reduces to $H(V) = P_0 V P_0$. The projection P_0 has an integral kernel so $H(V)$ is a random integral operator. The main result of this paper is Theorem 4.1. In this theorem we obtain sufficient conditions for an eigenstate to be exponentially localised. This theorem is then applied to show that, if the distribution of the potential is unbounded then the eigenfunctions corresponding to high enough energies are exponentially localised (Theorem 5). Our result is in particular applicable in the physical range of energies corresponding to the first Landau band (centered at the origin for convenience), i.e. $E << -1$ and $1 << E << B$, for B large. In a companion paper [DoMaPu] the same theorem is applied to prove localisation in the case when the distribution is bounded.

We now discuss briefly the approximation involved in (1.2). For large enough magnetic field B this approximation is considered to be good because the overlap of the eigenfunctions corresponding to different Landau bands is small. This is particularly so in the case of bounded potential distributions but should also hold in the unbounded case if the distribution decays rapidly enough at infinity. The density of states associated with (1.1) has been studied by several authors. Wegner succeeded to compute it analytically [We3] in the case when the potential has a white noise distribution. Soon after, this was extended to other distributions using a different technique [BrGrIt], [KlPe]. Finally, in [MaPu], it was shown that the true density of states converges to the one of the projected Hamiltonian in the limit $B \to \infty$. Further results in the same direction have been obtained recently [Wa1].

In the model studied here the kernel of the Hamiltonian has Gaussian decay. Apart from its continuum nature, the model is therefore very similar to a lattice model with an infinite range hopping matrix. For this reason it turns out that the techniques of [Sp] and [DrKl] can be adapted to our situation. However, the modifications required are highly non-trivial due to the continuum setting of the model. In particular, the relevant Green identities are considerably more complicated.

A full understanding of the phenomenon of localisation involves the study of Lifshitz tails [LiPa], i.e. the rate of decay of the density of states near the band edges. This is a problem in its own right but it only arises in the case of bounded potentials. We therefore leave the discussion of this problem to [DoMaPu].

The paper is organised in the following way. In Section 2 we define the Hamiltonian, discuss its self-adjointness and gather some general material specific to our setting. Then the restriction of the Hamiltonian and the relevant Green identities are treated in Section 3. Section 4 contains the main theorem which gives the existence of localised states under two conditions (as in [DrKl]). These conditions are verified in Section 5 for the case of unbounded potentials and sufficiently high energies. The main theorem (Theorem 4.1) is proved in Sections 6 and 7.

While this paper was being written we received a preprint by Combes and Hislop [CoH2] on the same subject. At the I.A.M.P satellite conference on 'Disordered Systems' (Paris, July 1994) W.M Wang informed us that she had also been working on this problem, see [Wa2].

2. Definition and Self-Adjointness of the Hamiltonian

As explained in the introduction, the Hamiltonian of the model we consider is given by

$$H(V) = P_0\, V\, P_0, \tag{2.1}$$

where P_0 is the projection operator onto the first Landau level of a free particle in a magnetic field and V is a random block potential. In axial gauge, P_0 is given by its kernel

$$P_0(z,z') = \frac{B}{2\pi}\exp\left[-\frac{B}{4}|z-z'|^2 + \frac{i}{2}Bz\wedge z'\right] \tag{2.2}$$

for $z, z' \in \mathbb{R}^2 = \mathbb{C}$. Here the exterior product is given by $z\wedge z' = \mathrm{Re}(z)\mathrm{Im}(z') - \mathrm{Im}(z)\mathrm{Re}(z')$. Notice that P_0 is invariant under *magnetic translations* defined by

$$T(\vec{a}) = e^{i(\vec{p}-\vec{A}(z))\cdot\vec{a}}. \tag{2.3}$$

Indeed, one easily derives that

$$\left(T(\vec{a})\psi\right)(z) = e^{i\frac{B}{2}a\wedge z}\psi(z+a) \tag{2.4}$$

and hence that

$$T(\vec{a})P_0T(\vec{a})^{-1} = P_0 \tag{2.5}$$

and also

$$T(\vec{a})H(V)T(\vec{a})^{-1} = H(V(\cdot + a)). \tag{2.6}$$

The random potential V is given by

$$V = \sum_{x\in\mathbb{Z}^2} v_x\, 1_{B(x)}, \tag{2.7}$$

where the v_x are real-valued, i.i.d. random variables with distribution given by a probability measure μ, and the $B(x)$ are unit blocks centred at x:

$$B(x) = \{z\in\mathbb{R}^2 \mid x_i - \frac{1}{2} \le z_i < x_i + \frac{1}{2}\ (i=1,2)\}. \tag{2.8}$$

We shall write $\mathbb{P}$ for the product measure $\mathbb{P} = \prod_{x\in\mathbb{Z}^2}\mu$ describing the distribution of the potential V, and $\mathbb{E}$ for the expectation w.r.t. this measure. We shall need several assumptions on the measure μ in the following:

1. μ is absolutely continuous with respect to the Lebesgue measure with density $\rho(v)$.
2. ρ is continuous on its support, which is either a closed interval $\text{supp}(\rho) = [a, b]$ or a half-line $\text{supp}(\rho) = [0, \infty)$ or the whole real line: $\text{supp}(\rho) = \mathbb{R}$.
3. In case $\text{supp}(\rho) = [0, \infty)$ or $\mathbb{R}$,

$$\mu\{v \in \mathbb{R} |\, |v| > L\} \leq L^{-\eta} \tag{2.9}$$

   for some $\eta > 1/2$.(In fact any $\eta > 0$ will do).
4. Also in case $\text{supp}(\rho) = [0, \infty)$ or $\mathbb{R}$, we shall assume that all moments exist and satisfy

$$\int v^k \, \mu(dv) \leq M^k \, k! \tag{2.10}$$

   for some constant $M > 0$.

The last assumption is needed only in the proof of self-adjointness of the Hamiltonian. We have not been able to prove the self-adjointness without this assumption. Notice that all the above assumptions are satisfied for the normal distribution and also for the exponential distribution.

To prove the self-adjointness of the Hamiltonian we first need a suitable domain. Obviously, $H(V)\psi = 0$ for $\psi \in \mathcal{H}_0^\perp$, the orthogonal complement of the range of P_0. Moreover, $H(V)u_m$ is well-defined for all $m \geq 0$, where u_m is the unit vector in the range of P_0 given by

$$u_m(z) = (2^{m+1}\pi m!)^{-1/2} B^{(m+1)/2} z^m \exp{[-\frac{1}{4}B|z|^2]}. \tag{2.11}$$

This follows immediately from the following useful lemma:

Lemma 2.1 *For almost every V, there exists $C_V > 0$ such that $|V(z)| \leq C_V(1+|z|^2)$ for all $z \in \mathbb{R}^2$.*

Proof. By assumption 3 about the measure μ,

$$\begin{aligned} \sum\nolimits_{x\in\mathbb{Z}^2} \mathbb{P}\{V \,|\, |v_x| > |x|^2\} &= \sum_{x\in\mathbb{Z}^2} \mu\{v \,|\, |v| > |x|^2\} \\ &\leq 1 + \sum_{x\in\mathbb{Z}^2\setminus\{0\}} |x|^{-2\eta} = 1 + 8\sum_{L=1}^{\infty} L^{1-2\eta} < \infty. \end{aligned} \tag{2.12}$$

By the Borel-Cantelli lemma, therefore, with probability 1, $|v_x| \leq |x|^2$ except for a finite number of x's. QED

Theorem 2.1 *The Hamiltonian $H(V)$ defined by (2.1) is almost surely essentially self-adjoint on the span of $\{u_m\}_{m=0}^\infty$ and $\mathcal{H}_0^\perp$.*

Proof. By Nelson's analytic vector theorem ([ReSi], Theorem X.39) it suffices to prove that, for almost all V,

$$\sum_{k=0}^{\infty} \frac{||(P_0 V P_0)^k u_m||}{k!} t^k < \infty \tag{2.13}$$

for some $t > 0$. To prove (2.13) we use the fact that if $F(V)$ is a positive function of V and if the expectation $\mathbb{E}(F(V)) < \infty$ then $F(V) < \infty$ for almost every V. Noticing also that by the Cauchy-Schwarz inequality, $\mathbb{E}(||A^k u||) \leq \left(\mathbb{E}(||A^k u||^2)\right)^{1/2}$ we conclude that it suffices to show that

$$\sum_{k=0}^{\infty} \frac{\left(\mathbb{E}(||(P_0 V P_0)^k u_m||^2)\right)^{1/2}}{k!} t^k < \infty \tag{2.14}$$

for some $t > 0$. Now,

$$\begin{aligned}\mathbb{E}(||(P_0 V P_0)^k u_m||^2) = \int dz \int dz' \, \overline{u_m(z)} u_m(z') \int dz_1 \dots dz_{2k} P_0(z, z_1) \\ P_0(z_1, z_2) \cdots P_0(z_{2k}, z') \mathbb{E}(V(z_1) V(z_2) \cdots V(z_{2k})).\end{aligned} \tag{2.15}$$

The $z_1, \dots, z_{2k}$ are in a certain number of distinct blocks $B(x_1), \dots, B(x_l)$. If p_i ($i = 1, \dots, l$) is the number of z's in block $B(x_i)$ then

$$\mathbb{E}(V(z_1) \cdots V(z_{2k})) = \int v^{p_1} \mu(dv) \cdots \int v^{p_l} \mu(dv) \leq \prod_{i=1}^{l} M^{p_i} p_i! \leq M^{2k} (2k)! \tag{2.16}$$

by Assumption 4, equation (2.10). If $K_B(z, z')$ denotes the absolute value of the kernel (2.2),

$$K_B(z, z') = \frac{B}{2\pi} e^{-B|z-z'|^2/4}, \tag{2.17}$$

then we can write

$$\begin{aligned}&\mathbb{E}(||(P_0 V P_0)^k u_m||^2) \\ &\quad = \int dz \overline{u_m(z)} \int dz' \, u_m(z') \int dz_2 \cdots \int dz_{2k-1} P_0(z, z_2) \\ &\quad\quad P_0(z_2, z_3) \cdots P_0(z_{2k-1}, z') \mathbb{E}(V(z) V(z_2) \cdots V(z_{2k-1}) V(z')) \\ &\quad \leq M^{2k} (2k)! \int dz |u_m(z)| \int dz' |u_m(z')| \\ &\quad\quad \int dz_2 \cdots \int dz_{2k-1} K_B(z, z_2) \cdots K_B(z_{2k-1}, z').\end{aligned} \tag{2.18}$$

Next we can iterate the identity

$$\int dz' K_{\alpha B}(z, z') K_B(z', z'') = 2 \, K_{\alpha B/(\alpha+1)}(z, z'') \tag{2.19}$$

to write

$$\begin{aligned}&\mathbb{E}(||(P_0 V P_0)^k u_m||^2) \\ &\leq \frac{1}{4} (2M)^{2k} (2k)! \int dz |u_m(z)| \int dz' |u_m(z')| K_{B/(2k-1)}(z, z') \\ &\leq (2M)^{2k} (2k)! \frac{B}{8(2k-1)\pi} \left(\int dz |u_m(z)| \right)^2 .\end{aligned} \tag{2.20}$$

Inserting (2.11) we see that $||u_m||_1 < \infty$. Therefore, if we insert (2.20) into (2.14) we find

$$\sum_{k=0}^{\infty} \frac{\left(\mathbb{E}(||(P_0 V P_0)^k u_m||^2)\right)^{1/2}}{k!} t^k$$
$$\leq 1 + \left(\frac{B}{8\pi}\right)^{1/2} ||u_m||_1 \sum_{k=1}^{\infty} (2k-1)^{-1/2} \frac{\sqrt{(2k)!}}{k!} (2M)^k t^k < \infty \tag{2.21}$$

provided $t < 1/4M$. QED

REMARK. If μ has bounded support then V is almost surely bounded and $H(V)$ is obviously self-adjoint.

We next prove as in [KuSo] that the spectrum of $H(V)$ is the support of μ. This uses the following simple lemma:

Lemma 2.2 *Let $E \in \mathbb{R}$ and suppose that for all $\delta > 0$ there exist Ω with $\mathbb{P}(\Omega) > 0$ and $\psi \in P_0(L^2(\mathbb{R}^2)) = \mathcal{H}_0$ with $||\psi|| = 1$ such that $||(H(V) - E)\psi|| < \delta$ for all $V \in \Omega$. Then $E \in \sigma(H(V))$.*

Theorem 2.2 *With probability 1, the spectrum of $H(V)$ is given by $\sigma(H(V)) = supp(\mu) \cup \{0\}$.*

Proof. We follow [KuSo]. Notice first that it follows from the ergodicity with respect to magnetic translations (2.3) that the spectrum $\sigma(H(V))$ is almost surely independent of V. To see that it equals the support of μ, suppose first that $E \neq 0$, $E \notin \text{supp}(\mu)$. Obviously, if A is a self-adjoint operator satisfying $A \geq \alpha \mathbf{1}$ for some $\alpha > 0$ then $||A\psi|| \geq \alpha ||\psi||$ for all $\psi \in D(A)$. Since $\text{supp}(\mu)$ is an interval, if $d = d(E, \text{supp}(\mu))$ then either $V - E \geq d\mathbf{1}$ or $E - V \geq d\mathbf{1}$. In either case we have

$$\begin{aligned} ||(H(V) - E)\phi||^2 &= ||P_0(V - E)P_0\phi||^2 + |E|\,||(\mathbf{1} - P_0)\phi||^2 \\ &\leq d||P_0\phi||^2 + |E|\,||(\mathbf{1} - P_0)\phi||^2 \leq (d \wedge |E|)||\phi||^2. \end{aligned}$$

Moreover, one easily sees that $(H(V) - E)(L^2)$ is dense in L^2. It follows that $H(V) - E$ is invertible, that is E is not in the spectrum of $H(V)$.

Next suppose that $E \in \text{supp}(\mu)$ and let $\psi \in \mathcal{H}_0$ with $||\psi|| = 1$. For $R > 0$, write $\Lambda(R)$ for the disk with radius R and centre 0, and put $\psi_R = P_0 1_{\Lambda(R)}\psi$. Given $\delta > 0$, choose R large enough so that $||\psi_R|| > 1/2$. Define $\Omega' = \{V \,|\, |V(z) - E| < \delta/2 \forall z \in \Lambda(2R)\}$. Since $E \in \text{supp}(\mu)$ and $\Lambda(2R)$ intersects only a finite number of unit blocks, $\mathbb{P}(\Omega') > 0$. Taking $C > |E| + \delta/2$ large enough it follows from Lemma 2.1 that $\Omega = \Omega' \cap \{V \,|\, |V(z)| \leq C(1 + |z|^2 \forall z \in \mathbb{R}^2\}$ also satisfies $\mathbb{P}(\Omega) > 0$. Note that $||\psi - \psi_R|| \leq ||1_{\Lambda(R)^c}\psi||$. Now,

$$\begin{aligned} ||(H(V) - E)\psi_R||^2 &= ||P_0(V - E)P_0 1_{\Lambda(R)}\psi||^2 \\ &\leq ||(V - E)P_0 1_{\Lambda(R)}\psi||^2 \\ &= \int_{\Lambda(2R)} d^2z\, (V(z) - E)^2 |\psi_R(z)|^2 \\ &\quad + \int_{\Lambda(2R)^c} d^2z\, (V(z) - E)^2 |\psi_R(z)|^2. \end{aligned} \tag{2.22}$$

The first integral is bounded by

$$\int_{\Lambda(2R)} d^2z\,(V(z)-E)^2|\psi_R(z)|^2 \leq \frac{1}{4}\delta^2||\psi_R||^2 \leq \delta^2/4. \tag{2.23}$$

In the second integral we have

$$\begin{aligned} |\psi_R(z)|^2 &= \frac{B^2}{4\pi^2}\left(\int_{\Lambda(R)} d^2z'\, e^{-B|z-z'|^2/4}|\psi(z')|\right)^2 \\ &\leq \frac{B^2}{4\pi^2}\left(\int_{\Lambda(R)} d^2z'\, e^{-B|z-z'|^2/2}\right)^2 \end{aligned} \tag{2.24}$$

by the Cauchy-Schwarz inequality and the fact that $||\psi|| = 1$. Since $z \in \Lambda(2R)^c$ and $z' \in \Lambda(R)$ we can bound the exponential in the integrand by $\exp[-BR^2/4] \times \exp[-B|z-z'|^2/4]$. Thus we have

$$\begin{aligned} &\int_{\Lambda(2R)^c} d^2z\,(V(z)-E)^2|\psi_R(z)|^2 \\ &\leq \frac{B^2}{4\pi^2}e^{-BR^2/4}\int_{\Lambda(R)} d^2z' \int_{\Lambda(2R)^c} d^2z\,\left(C(1+|z|^2)+|E|\right)^2 e^{-B|z-z'|^2/4} \\ &\leq \frac{B^2R}{4\pi}e^{-BR^2/4}\int_{\mathbb{R}^2} d^2\zeta[2C+|E|+2R^2+2|\zeta|^2]^2 e^{-B|\zeta|^2/4} \\ &\leq 2\pi BRe^{-BR^2/4}[(2C+2R^2+|E|^2)^2+16B^{-1}] < \delta^2/2 \end{aligned} \tag{2.25}$$

if R is large enough. QED

We want to prove that the Hamiltonian $H(V)$ has almost surely pure-point spectrum in the neighbourhood of the edges of its spectrum given by Theorem 2.2. This will be done by proving that the corresponding generalised eigenfunctions decay exponentially. The pure-point spectrum is then a consequence of the following general result:

Theorem 2.3 *Let $\mathcal{H} \subset E$ be a Hilbert subspace of a conuclear space E. Suppose that $\tau : E \longrightarrow E$ is a continuous linear map such that its restriction T to a dense subset D of $\mathcal{H}$ defines a self-adjoint operator on $\mathcal{H}$. Then, with respect to the spectral measure of T, almost every $\lambda \in \mathbb{R}$ is a generalised eigenvalue, that is, there exists a non-zero $\xi \in E$ such that $\tau(\xi) = \lambda\xi$.*

Corollary. *If, for a given $\lambda \in \sigma(T)$, $\tau(\xi) = \lambda\xi$ implies that $\xi \in D$ for any $\xi \in E$ then $\lambda \in \sigma_{pp}(T)$.*

The standard references for this theorem are [Ber] and [Mau]. However, we prefer the approach developed by Thomas in [Tho1] and [Tho2]. We apply the theorem to the case $E = \mathcal{S}'(\mathbb{R}^2)$ and $\mathcal{H} = L^2(\mathbb{R}^2)$. To prove that the theorem is indeed applicable we must show:

Lemma 2.3. *For almost all V, $H(V)$ maps $\mathcal{S}'(\mathbb{R}^2)$ continuously into itself.*

Proof. We need to define $H(V)\xi$ for $\xi \in \mathcal{S}'(\mathbb{R}^2)$. This is, however, straightforward if we first prove that $H(V)$ maps $\mathcal{S}(\mathbb{R}^2)$ into itself. We can then define $H(V)$ on $\mathcal{S}'(\mathbb{R}^2)$ by duality as the adjoint map. This is consistent because of the fact that $H(V)$ is Hermitian on $L^2(\mathbb{R}^2)$. Continuity in the strong topology of $\mathcal{S}'(\mathbb{R}^2)$ then follows from the corollary of Prop. 6 of Chap. 4, §4.2 in Bourbaki [Bo].

To prove that $H(V)$ is continuous on $\mathcal{S}(\mathbb{R}^2)$ consider the general seminorm $p_{m,\alpha}$, where m is a non-negative integer and $\alpha = (\alpha_1, \alpha_2)$ is a double index ($\alpha_i \in \mathbb{Z}_+$), is defined by

$$p_{r,m}(\psi) = \sup_{|\alpha|\le r} \sup_{z\in\mathbb{C}} (1+|z|^2)^m |\partial^\alpha \psi(z)|. \tag{2.26}$$

It suffices to show that for all m and r, there exists a constant K and integers $m', r' \geq 0$ such that

$$p_{r,m}(H(V)\psi) \leq K p_{r',m'}(\psi). \tag{2.27}$$

Now,

$$(H(V)\psi)(z) = \int d^2z' \int d^2z'' P_0(z,z')V(z')P_0(z',z'')\psi(z'') \tag{2.28}$$

so differentiating underneath the integral sign,

$$\begin{aligned} p_{r,m}(H(V)\psi) = &\sup_{|\alpha|\le r} \sup_{z\in\mathbb{C}} (1+|z|^2)^m \\ &\left| \int d^2z' \int d^2z'' \partial_z^\alpha P_0(z,z')V(z')P_0(z',z'')\psi(z'') \right|. \end{aligned} \tag{2.29}$$

It is not difficult to prove the following relation for the derivative of P_0 analogous to formula (I.18b) in Simon [Si2]:

$$\partial_z^\alpha P_0(z,z') = F_{\alpha_1}(z_1 - z')F_{\alpha_2}(z_2 + iz')P_0(z,z') \tag{2.30}$$

where $z_1 = \mathrm{Re}(z)$, $z_2 = \mathrm{Im}(z)$ and

$$F_n(z) = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} \left(-\frac{B}{4}\right)^k \left(-\frac{B}{2}z\right)^{n-2k}. \tag{2.31}$$

We use the following rough estimate to bound $F_n(z)$:

$$\frac{n!}{k!(n-2k)!} \leq \binom{[n/2]}{k} 4^k \left(\left[\frac{n+1}{2}\right]\right)! \tag{2.32}$$

This yields

$$|F_n(z)| \leq ([(n+1)/2])! \left(B + \left(\frac{B}{2}|z|\right)^2\right)^{n/2} \leq a_n(1+|z|^2)^{n/2} \tag{2.33}$$

for some constant a_n. Using Lemma 2.1 and the simple estimate $1+|z-z'|^2 \leq 2(1+|z|^2)(1+|z'|^2)$ we obtain

$$p_{r,m}(H(V)\psi) \leq \sup_{|\alpha|\leq r} 2^{|\alpha|/2} C_V a_\alpha p_{m+|\alpha|/2}\left(K_B(1+|z|^2)^{1+|\alpha|/2} K_B|\psi|\right). \tag{2.34}$$

(Here we have written p_m for $p_{0,m}$ and $|\alpha| = \alpha_1 + \alpha_2$ and $a_\alpha = a_{\alpha_1} a_{\alpha_2}$.) It now follows from Lemma 2.4 below that there is a constant K such that

$$p_{r,m}(H(V)\psi) \leq K p_{m+r+1}(\psi). \tag{2.35}$$

QED

Lemma 2.4 *Given $m \in \mathbb{Z}$, $m \geq 0$, there exists a constant $C_{B,m}$ such that*

$$\int K_B(z,z')(1+|z'|^2)^{-m} d^2z' \leq C_{B,m}(1+|z|^2)^{-m}. \tag{2.36}$$

Proof. It is easy to prove that $\ln(1+t) \leq \frac{1}{4\epsilon} + \epsilon t$ for $t \geq 0$ and $\epsilon > 0$. Therefore, if we take

$$R^2 = c_1 + |z|^2/4 \text{ with } c_1 = \left(\frac{4m}{B}\right)^2 + \frac{4}{B}\ln 2 \tag{2.36}$$

then

$$\int_{|z-z'|\geq R} K_B(z,z')d^2z' = 2e^{-BR^2/4} \leq (1+|z|^2)^{-m}. \tag{2.38}$$

On the other hand, if $|z'-z| \leq R$ then $|z|^2 \leq (|z'|+R)^2 \leq 2(|z'|^2+R^2) \leq 2(c_1+|z'|^2)+|z|^2/2$ and hence

$$(1+|z|^2)^{-m} \geq 4^{-m}(1+c_1+|z'|^2)^{-m} \geq (4(1+c_1))^{-m}(1+|z'|^2)^{-m}. \tag{2.39}$$

Inserting this into the integral we have

$$\begin{aligned}\int_{|z'-z|\leq R} & K_B(z,z')(1+|z'|^2)^{-m} d^2z' \\ &\leq (4(1+c_1))^m (1+|z|^2)^{-m} \int K_B(z,z')d^2z' \\ &= 2(4(1+c_1))^m(1+|z|^2)^{-m}.\end{aligned} \tag{2.40}$$

It follows that we can take $C_{B,m} = 1 + 2(4(1+c_1))^m$. QED

The distributions $H(V)\xi$ are in fact regular:

Lemma 2.5 *For almost all V, the distributions $H(V)\xi$ are in fact C^∞ functions for all $\xi \in \mathcal{S}'(\mathbb{R}^2)$.*

Proof. It is clear that $H(V)\xi$ is the function given by

$$(H(V)\xi)(z') = \langle H(z,z')|\xi(z)\rangle, \tag{2.41}$$

where

$$H(z,z') = \int d^2z''\, P_0(z,z'')V(z'')P_0(z'',z') \tag{2.42}$$

and the angled brackets denote the (anti-)duality between $\mathcal{S}$ and $\mathcal{S}'$. (We take it to be antilinear in the first argument.) As $\xi \in \mathcal{S}'(\mathbb{R}^2)$, there exist integers m and r and a constant $M > 0$ such that

$$|\langle\phi|\xi\rangle| \le M\, p_{r,m}(\phi) \tag{2.43}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^2)$. It therefore suffices to show that, for any double index β and any compact set $K \subset \mathbb{R}^2$,

$$\sup_{z'\in K} p_{r,m}\left(\partial^\beta_{z'} H(\cdot,z')\right) < \infty. \tag{2.44}$$

This is proved in the same way as (2.35) in the proof of Lemma 2.3. QED

This lemma has the following important consequence:

Lemma 2.6 *Suppose that $\xi \in \mathcal{S}'(\mathbb{R}^2)$ is a generalised eigenfunction of $H(V)$ with eigenvalue $E \neq 0$. Assume that ξ is exponentially decaying* on blocks, *that is,*

$$\langle 1_{B(x)}|\,|\xi|\rangle \le Ae^{-m|x|} \tag{2.45}$$

for some constants A and $m > 0$. Then ξ is a C^∞- function which decays exponentially with rate m: There exists a constant A' such that

$$|\xi(z)| \le A'\, e^{-m|z|}. \tag{2.46}$$

Proof. As ξ is an eigenfunction of $H(V) = P_0VP_0$ and $P_0^2\xi = P_0\xi$ even for $\xi \in \mathcal{S}'$ it follows that $P_0\xi = \xi$, that is

$$\xi(z) = \int P_0(z,z')\xi(z')d^2z'. \tag{2.47}$$

Taking absolute values we find

$$\begin{aligned} |\xi(z)| &\le \int K_B(z,z')|\xi(z')|d^2z' \\ &= \sum_{y\in\mathbb{Z}^2}\int_{B(y)} K_B(z,z')|\xi(z')|d^2z'. \end{aligned} \tag{2.48}$$

Now suppose $z \in B(x)$. We split the above sum into two pieces: the first, denoted S_1, will contain the terms for which $|x-y|_\infty \le 1$, the other, denoted S_2, the remaining terms. Here $|\cdot|_\infty$ is the sup-norm which is convenient for the square lattice: $|x|_\infty = \sup_{i=1,2}|x_i|$. Obviously, S_1 contains only 9 terms. In these we can replace $K_B(z,z')$ by its maximum value. The remaining integral is just (2.45). Thus

$$S_1 \le A\frac{B}{2\pi}\sum_{|y-x|_\infty\le 1} e^{-m|y|} \le A_1' e^{-m|z|} \tag{2.49}$$

where

$$A_1' = 9A\frac{B}{2\pi}e^{\frac{3m}{\sqrt{2}}}. \tag{2.50}$$

(We use the fact that $|y| \geq |z| - |y-z| \geq |z| - 3/\sqrt{2}$.) The second part of the sum can be bounded as follows. In the kernel $K_B(z,z')$ we use $|z-z'| \geq |x-y| - |x-z| - |y-z'| \geq |x-y| - \sqrt{2}$ and in the bound (2.45), $|y| \geq |z| - |x-y| - 1/\sqrt{2}$. Inserting this we have

$$S_2 \leq A\frac{B}{2\pi}\sum_{|y-x|_\infty \geq 2} e^{-B(|x-y|-\sqrt{2})^2/4}e^{-m|y|} \leq A_2' e^{-(m-\epsilon)|z|}, \tag{2.51}$$

where

$$A_2' = A\frac{B}{2\pi}e^{m/\sqrt{2}}\sum_{y'\in\mathbb{Z}^2} e^{-B|y'|-\sqrt{2})^2/4}e^{m|y'|} < \infty. \tag{2.52}$$

We can thus take $A' = A_1' + A_2'$. QED

3. Truncation and Green's Functions

For regions $\Lambda \subset \mathbb{R}^2$ we define truncated Hamiltonians $H_\Lambda = H_\Lambda(V)$ by

$$H_\Lambda(V) = P_\Lambda V_\Lambda P_\Lambda^*, \tag{3.1}$$

where

$$P_\Lambda = 1_\Lambda P_0 \tag{3.2}$$

and $V_\Lambda = 1_\Lambda V$. $H_\Lambda(V)$ acts on $L^2(\Lambda)$. If $E \notin \sigma(H_\Lambda(V))$ then the Green function

$$G_\Lambda(V,E) = (H_\Lambda(V) - E)^{-1} \tag{3.3}$$

is well-defined. In particular, we shall consider the regions

$$\Lambda_L(x) = \{z \in \mathbb{R}^2 | \, |z-x|_\infty < L/2\} \tag{3.4}$$

for $x \in \mathbb{Z}^2$ and $L > 0$, $\Lambda_L(x)$ is a square of size L^2 and, if L is an integer, it is a union of unit squares $B(y)$. We derive some important relations for the Green function. In the following we write G_Λ or $G_\Lambda(V)$ or $G_\Lambda(E)$ for $G_\Lambda(V,E)$ when there is no ambiguity. Using the resolvent identity

$$B^{-1} - A^{-1} = A^{-1}(A-B)B^{-1} \tag{3.5}$$

with $A = H_{\Lambda_1} \oplus H_{\Lambda_2} - E$ and $B = H_{\Lambda_1\cup\Lambda_2} - E$ this gives, when $\Lambda_1 \cap \Lambda_2 = \emptyset$,

$$\begin{aligned} G_{\Lambda_1\cup\Lambda_2} &= G_{\Lambda_1} \oplus G_{\Lambda_2} + (G_{\Lambda_1} \oplus G_{\Lambda_2})(H_{\Lambda_1} \oplus H_{\Lambda_2} - H_{\Lambda_1\cup\Lambda_2})G_{\Lambda_1\cup\Lambda_2} \\ &= G_{\Lambda_1} \oplus G_{\Lambda_2} - (G_{\Lambda_1} \oplus G_{\Lambda_2})\Gamma_{\Lambda_1,\Lambda_2}G_{\Lambda_1\cup\Lambda_2}, \end{aligned} \tag{3.6}$$

where

$$\Gamma_{\Lambda_1,\Lambda_2} = P_{\Lambda_1} V_{\Lambda_1 \cup \Lambda_2} P_{\Lambda_2}^* + P_{\Lambda_2} V_{\Lambda_1 \cup \Lambda_2} P_{\Lambda_1}^* + P_{\Lambda_1} V_{\Lambda_2} P_{\Lambda_1}^* + P_{\Lambda_2} V_{\Lambda_1} P_{\Lambda_2}^*. \tag{3.7}$$

In particular, if $\Lambda_1 \cup \Lambda_2 = \mathbb{R}^2$, we can write $\Lambda_1 = \Lambda$ and $\Lambda_2 = \Lambda^c$ and (3.6) becomes

$$G = G_\Lambda \oplus G_{\Lambda^c} - (G_\Lambda \oplus G_{\Lambda^c})\Gamma_{\Lambda,\Lambda^c} G \tag{3.8}$$

with

$$\Gamma_{\Lambda,\Lambda^c} = 1_\Lambda H 1_{\Lambda^c} + 1_{\Lambda^c} H 1_\Lambda + P_\Lambda V_{\Lambda^c} P_\Lambda^* + P_{\Lambda^c} V_\Lambda P_{\Lambda^c}^*. \tag{3.9}$$

Now suppose that $\xi \in \mathcal{S}'(\mathbb{R}^2)$ is a generalised eigenfunction of H with eigenvalue $E \notin \sigma(H_\Lambda)$. Then it follows that for $\epsilon > 0$,

$$\begin{aligned} \xi &= i\epsilon G(E - i\epsilon)\xi \\ &= i\epsilon \left(G_\Lambda(E - i\epsilon) \oplus G_{\Lambda^c}(E - i\epsilon)\right)\xi - \left(G_\Lambda(E - i\epsilon) \oplus G_{\Lambda^c}(E - i\epsilon)\right)\Gamma_{\Lambda,\Lambda^c}\xi. \end{aligned} \tag{3.10}$$

For $z \in \Lambda$ this yields

$$\xi(z) = i\epsilon(G_\Lambda(E - i\epsilon)\xi)(z) - (G_\Lambda(E - i\epsilon)\Gamma_{\Lambda,\Lambda^c}\xi)(z). \tag{3.11}$$

Taking $\epsilon \longrightarrow 0$ this yields

$$\begin{aligned} \xi(z) &= -(G_\Lambda(E)\Gamma_{\Lambda,\Lambda^c}\xi)(z) \\ &= -(G_\Lambda(E)(H 1_{\Lambda^c} + P_0 V_{\Lambda^c} P_\Lambda^*)\xi)(z), \end{aligned} \tag{3.12}$$

where we have used (3.9).

We finish this short section with an important definition.

Definition. *Fix constants $\beta \in (0,1)$ and $s \in (\frac{1}{2}, 1)$. Given a potential V, a square $\Lambda_L(x)$ is called (m, E)-regular for some $m > 0$ and $E \in \mathbb{R}$ if the following two conditions are satisfied:*

(RA)
$$d(E, \sigma(H_{\Lambda_L(x)}(V))) > \frac{1}{2} e^{-L^\beta}$$

(RB)
$$\langle 1_{\Lambda_1(x)} \,|\, |G_{\Lambda_L(x)} 1_{\tilde{\Lambda}_L(x)} \phi| \rangle \le e^{-mL} ||1_{\tilde{\Lambda}_L(x)} \phi||_2$$

for all $\phi \in L^2$, where $\tilde{\Lambda}_L(x) = \Lambda_L(x) \setminus \Lambda_{\tilde{L}}(x)$ with $\tilde{L} = L - L^s$. $\Lambda_L(x)$ is called singular if it is not regular.

4. The Main Theorem

In this section we state the main theorem which is an analogue of the main theorem proved in [DrKl]. It states conditions under which the spectrum of our random Schroedinger

Hamiltonian is pure-point near the edges. We shall check these conditions in the next section in the case of unbounded support: $\text{supp}(\mu) = \mathbb{R}$. The case of bounded support is more complicated and will be analysed in a separate paper [DoMaPu].

Theorem 4.1 *Fix constants $\beta \in (0,1)$, $s \in (\frac{1}{2}, 1)$, $p > 2$, $q > 4p + 12$, $\gamma \in (\beta, 1)$ and $\epsilon_0 > 0$. There exists $Q_0 > 0$ depending on all these constants such that the following holds:*

If, for some $E_0 \in \mathbb{R} \setminus [-\epsilon_0, \epsilon_0]$ the conditions (P1) *and* (P2) *are satisfied, where*

(P1) *There exist $L_0 > Q_0$ and $m_0 \geq L_0^{-1+\gamma}$ such that*

$$\mathbb{P}\left\{\Lambda_{L_0}(0) \textit{ is } (m_0, E_0)\textit{-regular}\right\} \geq 1 - L_0^{-p} \tag{4.1}$$

(P2) *There exists $\eta \in (0, |E_0|/2)$ such that, for all $E \in (E_0 - \eta, E_0 + \eta)$ and for all $L > L_0$,*

$$\mathbb{P}\left\{d\left(E, \sigma\left(H_{\Lambda_L(0)}\right)\right) < e^{-L^\beta}\right\} < L^{-q}, \tag{4.2}$$

then, for all $m \in (0, m_0)$, there exists $\delta > 0$ depending on m, m_0, L_0, β and η such that, almost surely, $\sigma(H) \cap (E_0 - \delta, E_0 + \delta)$ is pure-point and the corresponding eigenfunctions decay exponentially with rate $\geq m$.

As in [DrKl], the proof of this theorem can be split into two parts: one in which condition (P1) is iterated to pairs of larger and larger blocks, and one in which the iterated condition is shown to imply exponential decay. The latter part is formulated in the following analogue of Theorem 2.3 of [DrKl]:

Theorem 4.2 *Let $I \subset \mathbb{R}$, $L_0 > 1$, $\beta \in (0,1)$, $s \in (\frac{1}{2}, 1)$, $\alpha \in (1,2)$, $p > \alpha$ and $m > 0$. Define $L_{k+1} = L_k^\alpha$ for $k = 0, 1, 2, \ldots$. Suppose that, for any $k = 0, 1, 2, \ldots$ and any $x, y \in \mathbb{Z}^2$ with $|x - y|_\infty \geq L_k + 1$,*

$$\mathbb{P}\left\{\exists E \in I : \Lambda_{L_k}(x) \textit{ and } \Lambda_{L_k}(y) \textit{ are } (m, E)\textit{-singular}\right\} < L^{-2p}. \tag{4.3}$$

Then, with probability 1, the generalised eigenfunctions of H corresponding to generalised eigenvalues in I decay exponentially with rate $\geq m'$ for any $m' < m$.

The proof of this theorem is similar to the one of [DrKl], but more complicated. We shall indicate the main differences in Section 7. The deterministic part of Theorem 4.1 is the analogue of Theorem 2.2 in [DrKl]. However, it is considerably more complicated due to the infinite range of the Hamiltonian and the more difficult form of the resolvent identity (3.12). Before we formulate the theorem it is convenient to define the conditions

(K1) There exists $m_0 > L^{-1+\gamma}$ such that $R(L_0, m_0)$ holds, where

R(L,m) For all $x, y \in \mathbb{Z}^2$ with $|x - y|_\infty \geq L + 1$,

$$\mathbb{P}\left\{\exists E \in I : \Lambda_L(x) \text{ and } \Lambda_L(y) \text{ are } (m, E)\text{-singular}\right\} < L^{-2p} \tag{4.4}$$

and

(K2) For all $L > L_0$ and all E with $d(E, I) \leq \frac{1}{2}e^{-L^\beta}$,

$$\mathbb{P}\left\{d\left(E, \sigma\left(H_{\Lambda_L(0)}\right)\right) < e^{-L^\beta}\right\} < L^{-q}. \tag{4.5}$$

Theorem 4.3 *Let $I \subset \mathbb{R} \setminus [-\epsilon_0, \epsilon_0]$. Fix $\beta \in (0,1)$, $s \in (1/2, 1)$, $p > 2$, and $q > 4p + 12$. There exists $Q(\beta, s, p, q, \epsilon_0) > 0$ such that the following holds: If, for some $L_0 \geq Q$ the conditions* (K1) *and* (K2) *hold then there exists $\alpha \in (1, 2s)$ such that, with $L_{k+1} = L_k^\alpha$ for $k = 0, 1, 2, \ldots$,* R(L_k, m) *holds for all $m < m_0$ and $k = 1, 2, \ldots$.*

This theorem will be proved in Section 7. We now prove Theorem 4.1 assuming that Theorems 4.2 and 4.3 are valid.

Proof of Theorem 4.1
Suppose that (P1) and (P2) are satisfied for some L_0. By the resolvent identity (3.5),

$$G_{\Lambda_{L_0}(0)}(E) = G_{\Lambda_{L_0}(0)}(E_0) + (E - E_0)G_{\Lambda_{L_0}(0)}(E)G_{\Lambda_{L_0}(0)}(E_0). \tag{4.6}$$

If $||G_{\Lambda_{L_0}(0)}(E_0)|| \leq \exp[L_0^\beta]$ and $|E - E_0| \leq \frac{1}{2}\exp[-L_0^\beta]$, then it follows from (4.6) that $||G_{\Lambda_{L_0}(0)}(E)|| \leq 2\exp[L_0^\beta]$. Moreover,

$$\begin{aligned}\langle 1_{\Lambda_1(0)}\,|\,|G_{\Lambda_{L_0}(0)}(E)1_{\tilde{\Lambda}_{L_0}(0)}\phi|\rangle &\leq \langle 1_{\Lambda_1(0)}\,|\,|G_{\Lambda_{L_0}(0)}(E_0)1_{\tilde{\Lambda}_{L_0}(0)}\phi|\rangle \\ &\quad + |E - E_0|\,||G_{\Lambda_{L_0}(0)}(E)||\,||G_{\Lambda_{L_0}(0)}(E_0)||\,||1_{\tilde{\Lambda}_{L_0}(0)}\phi||.\end{aligned} \tag{4.7}$$

Thus, if $\Lambda_{L_0}(0)$ is (m_0, E_0)-regular then

$$\langle 1_{\Lambda_1(0)}\,|\,|G_{\Lambda_{L_0}(0)}(E)1_{\tilde{\Lambda}_{L_0}(0)}\phi|\rangle \leq (e^{-m_0 L_0} + 2|E - E_0|e^{2L_0^\beta})\,||1_{\tilde{\Lambda}_{L_0}(0)}\phi||. \tag{4.8}$$

Given $m_0' \in (0, m_0)$ and $p' \in (2, p)$ define

$$\delta = \frac{1}{2}e^{-2L_0^\beta}(e^{-m_0' L_0} - e^{-m_0 L_0}). \tag{4.9}$$

Then, if $|E - E_0| < \delta$,

$$\langle 1_{\Lambda_1(0)}\,|\,|G_{\Lambda_{L_0}(0)}(E)1_{\tilde{\Lambda}_{L_0}(0)}\phi|\rangle \leq e^{-m_0' L}||1_{\tilde{\Lambda}_{L_0}(0)}\phi||. \tag{4.10}$$

This is the second regularity condition (RB). If we also assume that $d\left(E_0, \sigma\left(H_{\Lambda_{L_0}(0)}\right)\right) \geq \exp[-L_0^\beta]$ then it follows, since $\delta < \frac{1}{2}\exp[-L_0^\beta]$, that $\Lambda_{L_0}(0)$ is (m_0', E)-regular. We may now conclude that

$$\begin{aligned}&\mathbb{P}\left\{\forall E \in (E_0 - \delta, E_0 + \delta) : \Lambda_{L_0}(0) \text{ is } (m_0', E)\text{-regular}\right\} \\ &\geq \mathbb{P}\left\{\Lambda_{L_0}(0) \text{ is } (m_0, E_0)\text{-regular and } d\left(E_0, \sigma\left(H_{\Lambda_{L_0}(0)}\right)\right) \geq e^{-L_0^\beta}\right\} \\ &\geq 1 - L_0^{-p} - L_0^{-q} > 1 - L_0^{-p'}\end{aligned} \tag{4.11}$$

provided L_0 is large enough: $L_0 > Q'$. (Here we used (P1) and (P2).) Now let $x, y \in \mathbb{Z}^2$ with $|x-y|_\infty > L_0 + 1$. Then

$$\begin{aligned}&\mathbb{P}\left\{\forall E \in (E_0-\delta, E_0+\delta): \text{ either } \Lambda_{L_0}(x) \text{ or } \Lambda_{L_0}(y) \text{ is } (m_0', E)\text{-regular}\right\}\\ &\geq 1 - \left[\mathbb{P}\left(\{\forall E \in (E_0-\delta, E_0+\delta): \Lambda_{L_0}(0) \text{ is } (m_0', E)\text{-regular}\}^c\right)\right]^2 \\ &\geq 1 - L_0^{-2p'}.\end{aligned} \tag{4.12}$$

Thus (K1) of Theorem 4.3 is satisfied with p replaced by p', $I = (E_0-\delta, E_0+\delta)$ and m_0 replaced by m_0'. (P2) implies that (K2) is also satisfied provided $\delta < \eta$, so that Theorem 4.3 applies. Therefore the conditions for Theorem 4.2 are satisfied and the conclusion combined with Theorem 2.3 implies Theorem 4.1 with $Q_0 = Q \vee Q'$.

5. Proof of the Conditions (P1) and (P2) for Large Energies

In this section we prove the conditions (P1) and (P2) in the case of a probability measure μ with $\text{supp}(\mu) = \mathbb{R}$ satisfying the additional condition that there exists a function $\tilde{\rho}(v) \geq 0$ such that, for $0 < \lambda < 1/2$, $\int_{\mathbb{R}} v^4 \tilde{\rho}(v)\, dv < \infty$ and

$$|\rho((1+\lambda)v) - \rho((1-\lambda)(v))| \leq \lambda |v|^2 \tilde{\rho}(v). \tag{5.1}$$

We will prove localisation for large energies: $|E| \gg 1$. This is the simplest case because, for large $|E|$, the density of states is small. (We do not actually use this fact explicitly.) For the more physical case of a probability distribution with bounded support the proof of (P1) when E is near the band edge is much more delicate and will be presented in a separate publication [DoMaPu1].

Lemma 5.1 *Fix $\epsilon > 0$, $L > 0$ and $B_0 > 0$. One can find W large enough, but independent of $B > B_0$, such that*

$$\mathbb{P}\Big[\langle 1_{\Lambda_1(0)}|\, |G_{\Lambda_L(0)} 1_{\tilde{\Lambda}_L(0)}\phi|\rangle \leq 8A_1\epsilon^{-1}\exp(-\epsilon A_2 L)||1_{\Lambda_{\tilde{L}(0)}}||,\\ E \notin \sigma(H_{\Lambda_{L(0)}}), |E| \geq E_0\Big] \geq 1-\epsilon \tag{5.2}$$

with $E_0 = 2W$ and $A_1 = \exp(\frac{\epsilon}{2\sqrt{2}A_2})$, $A_2 = W\exp(\frac{4}{B})$.

Proof. We use a Combes-Thomas type of argument [CT]. Let U be the operator on $H_{\Lambda_{L(0)}}$ defined by $(Uf)(x) = e^{x_0\cdot x} f(x)$ where $x_0 \in \mathbb{R}^2$ with $|x_0| < 1$ and let

$$Q = UH_{\Lambda_{L(0)}}U^{-1} - H_{\Lambda_{L(0)}} \tag{5.3}$$

Then Q has a kernel $Q(x, y)$ where

$$Q(x,y) = (e^{x_0\cdot(x-y)} - 1)H_{\Lambda_{L(0)}}(x,y), \tag{5.4}$$

$H_{\Lambda_{L(0)}}(x,y)$ being the kernel of $H_{\Lambda_{L(0)}}$. Now we suppose that the potential satisfies $||V_{\Lambda_L(0)}|| = \sup_{\Lambda_L(0)} |v_x| < W$, for some W that we choose later. We have (c is a positive constant)

$$|(Q\phi)(x)| \leq cBW \int |e^{x_0\cdot(x-y)} - 1| e^{-\frac{B}{2}|x-y|^2} |\phi(y)| dy. \tag{5.5}$$

and

$$\begin{aligned} |e^{x_0\cdot(x-y)} - 1| e^{-\frac{B}{4}|x-y|^2} &\leq |x_0\cdot(x-y)| e^{|x_0\cdot(x-y)|} e^{-\frac{B}{4}|x-y|^2} \\ &\leq |x_0||x-y| e^{|x_0||x-y|} e^{-\frac{B}{4}|x-y|^2} \\ &\leq |x_0| e^{2|x-y|} e^{-\frac{B}{4}|x-y|^2} \leq e^{4/B}|x_0|, \end{aligned} \tag{5.6}$$

>From (5.5) and (5.6)

$$|(Q\phi)(x)| \leq c|x_0|e^{4/B}(T|\phi|)(x) \tag{5.7}$$

where T is the operator with kernel $T(x,y) = BWe^{-\frac{B}{4}|x-y|^2}$. Thus

$$\|Q\phi\| \leq \|T|\phi|\| \ \ |x_0| \leq \|T\| \ \ |x_0| \ \ \|\phi\| \tag{5.8}$$

The operator norm of T is $||T|| = \sup_k |\hat{T}(k)|$ where $\hat{T}(k)$ is the Fourier transform of the kernel $T(x,y)$ as a function of $(x-y)$. Now $\hat{T}(k) = We^{-k^2/B} \leq W$. Thus $||T|| = W$, which gives $\|Q\| \leq A_2|x_0|$ because of (5.8). Since ϵ is fixed (small), for W large and $E \geq E_0 = 2W$ we have that $d(E, \sigma(H_L)) \geq \epsilon$. We choose x_0 such that $|x_0| < \epsilon/(2A_2)$ so that $\|Q\| \leq \frac{1}{2}\epsilon$. Then, by (5.3),

$$\|UG_{\Lambda_L(0)}(E)U^{-1}\| = \|(H_{\Lambda_L(0)} + Q - E)^{-1}\| < \frac{2}{\epsilon}. \tag{5.9}$$

Now we split up $\tilde{\Lambda}_L$ into four parts:

$$\tilde{\Lambda}_L = \bigcup_{i=1}^{4} \tilde{\Lambda}_L^{(i)},$$

where $\tilde{\Lambda}_L^{(i)} = \{x \ : \ x \in \tilde{\Lambda}, \ \ e_i\cdot x \geq |x|/\sqrt{2}\}$ and $e_1 = (1,0)$, $e_2 = (-1,0)$, $e_3 = (0,1)$, $e_4 = (0,-1)$. We have

$$\langle 1_{\Lambda_1}, |G_L 1_{\tilde{\Lambda}_L}\phi|\rangle \leq \sum_{i=1}^{4} \langle 1_{\Lambda_1}, |G_L 1_{\tilde{\Lambda}_L^{(i)}}\phi|\rangle. \tag{5.10}$$

Now

$$\begin{aligned} \langle 1_{\Lambda_1}, |G_L 1_{\tilde{\Lambda}_L^{(1)}}\phi|\rangle &= \langle 1_{\Lambda_1}, U^{-1}|UG_LU^{-1}U1_{\tilde{\Lambda}_L^{(1)}}\phi|\rangle \\ &\leq \|U^{-1}1_{\Lambda_1}\| \|UG_LU^{-1}\| \|U1_{\tilde{\Lambda}_L^{(1)}}\| \|1_{\tilde{\Lambda}_L}\phi\| \\ &\leq \frac{2}{\epsilon} \|U^{-1}1_{\Lambda_1}\| \|U1_{\tilde{\Lambda}_L^{(1)}}\| \|1_{\tilde{\Lambda}_L}\phi\|. \end{aligned} \tag{5.11}$$

Clearly

$$\|U^{-1}1_{\Lambda_1}\| \leq A_1 \tag{5.12}$$

and by choosing $x_0 = (-\frac{\epsilon}{2\sqrt{2}A_2}, 0)$ we get

$$\|U1_{\tilde{\Lambda}_L^{(1)}}\psi\|^2 = \int_{\tilde{\Lambda}_L^{(1)}} e^{2x_0\cdot x}|\psi(x)|^2 dx \le e^{-\frac{\epsilon}{4A_2}(L-L^s)}\|\psi\|^2, \tag{5.13}$$

from which it follows that

$$\|U1_{\tilde{\Lambda}_L^{(1)}}\| \le e^{-\frac{\epsilon}{8A_2}(L-L^s)} < e^{-\frac{\epsilon}{9A_2}L} \tag{5.14}$$

for L sufficiently large. Thus using (5.11) to (5.14) we get

$$\langle 1_{\Lambda_1}, |G_L 1_{\tilde{\Lambda}_L^{(1)}}\phi|\rangle < \frac{2A_1}{\epsilon}e^{-\frac{\epsilon}{9A_2}L}\|1_{\tilde{\Lambda}_L}\phi\|. \tag{5.15}$$

and similarly for $i = 2, 3, 4$. Therefore

$$\langle 1_{\Lambda_1}, \ |G_L 1_{\tilde{\Lambda}_L}\phi|\rangle < \frac{8A_1}{\epsilon}e^{-\frac{\epsilon}{9A_2}L}\|1_{\tilde{\Lambda}_L}\phi\|. \tag{5.16}$$

So far we have shown that for $||V_{\Lambda_L(0)}|| < W$ and $E \ge E_0$, (5.16) holds. Thus the probability in the left hand side of the inequality (5.2) is greater than $\mathbb{P}\Big[||V_{\Lambda_L(0)}|| < W\Big]$. The Lemma follows from

$$\mathbb{P}\Big[||V_{\Lambda_{L_0}(0)}|| < W\Big] = \left[\int_{-W}^{W} \rho(v)\, dv\right]^N \tag{5.17}$$

where N is the number of unit squares intersecting $\Lambda_{L_0}(0)$, since for any $\epsilon > 0$, we can make the right-hand side of (5.17) larger than $1-\epsilon$ by taking W large enough, but independent of B. QED

>From the Lemma we conclude that (P1) holds for E_0 large enough, independent of $B > B_0 > 0$.

It remains to prove (P2). The main idea goes back to Wegner [We2] who reduced the estimate to one on the difference of the density of states for energies $E-\epsilon$ and $E+\epsilon$. In the present case, however, H_Λ is a compact operator so that there is an infinite number of energy eigenvalues accumulating at zero. We are therefore forced to define the density of states in a non-standard way: for $E > 0$ we define

$$N_\Lambda^>(V, E) = \sharp\{\lambda > E |\ \lambda \text{ eigenvalue of } H_\Lambda(V)\}. \tag{5.18}$$

Note that, since $H_\Lambda(V)$ is compact for bounded regions Λ, $N_\Lambda^>(V, E)$ is finite. Moreover, it is a decreasing function of E and an increasing function of V. Also, it is easy to verify that, for any $a > 0$, $N_\Lambda^>(V, E) = N_\Lambda^>(aV, aE)$. So, replacing E by $E + \epsilon$ and choosing $a = E/(E+\epsilon)$ we get

$$N_\Lambda^>(V, E+\epsilon) = N_\Lambda^>\left(\frac{E}{E+\epsilon}V, E\right) \tag{5.19}$$

and similarly, with E replaced by $E-\epsilon$ and $a = E/(E-\epsilon)$,

$$N_\Lambda^>(V, E-\epsilon) = N_\Lambda^>\left(\frac{E}{E-\epsilon}V, E\right). \tag{5.20}$$

If $E < 0$ we define

$$N_\Lambda^<(V,E) = \sharp\{\lambda < E \big| \, \lambda \text{ eigenvalue of } H_\Lambda(V)\} \tag{5.21}$$

and similar inequalities hold true.

Lemma 5.2 *There exists a constant K, independent of V, such that for $E > 0$,*

$$N_\Lambda^>(V,E) \leq KE^{-2}|\Lambda| \left(\sum_{x:\, B(x)\cap\Lambda\neq\emptyset} |v_x|\right)^2. \tag{5.22}$$

The same inequality holds for $N_\Lambda^<(V,E)$ if $E < 0$.

Proof. By definition, $N_\Lambda^>(V,E)$ is the number of eigenvalues λ such that $\lambda/E > 1$. It is therefore smaller than the sum of $(\lambda_i/E)^2$ over all eigenvalues λ_i. This is just Trace(H_Λ^2):

$$\begin{aligned} N_\Lambda^>(V,E) &\leq \frac{1}{E^2}\text{Trace}\,(H_\Lambda^2) \\ &= \frac{1}{E^2}\text{Trace}\,(1_\Lambda P_0 V_\Lambda P_0 1_\Lambda P_0 V_\Lambda P_0 1_\Lambda) \\ &= \frac{1}{E^2}\text{Trace}\,[(V_\Lambda P_0 1_\Lambda P_0 V_\Lambda) P_0 1_\Lambda P_0] \\ &\leq \frac{1}{E^2}||V_\Lambda||^2 \text{Trace}\,(P_0 1_\Lambda P_0). \end{aligned} \tag{5.23}$$

Now,

$$\text{Trace}\,(P_0 1_\Lambda P_0) = \text{Trace}\,(1_\Lambda P_0 1_\Lambda) = \int_\Lambda d^2z' K_B(z,z)^2 = \frac{B}{2\pi}|\Lambda|. \tag{5.24}$$

>From (5.23), (5.24) and the fact that $||V_\Lambda|| \leq \sum_{x:\, B(x)\cap\Lambda\neq\emptyset} |v_x|$ we get the final inequality. QED

For the rest of the argument we just imitate the usual proofs. Using (5.19) and (5.20) we have for the expectation value of the density of states

$$\begin{aligned} &\mathbb{E}\left(N_\Lambda^>(V,E-\epsilon) - N_\Lambda^>(V,E+\epsilon)\right) \\ &= \int N_\Lambda^>(V,E) \prod_{x\in\Lambda}(1-\frac{\epsilon}{E})\rho((1-\frac{\epsilon}{E})v_x)dv_x \\ &\quad - \int N_\Lambda^>(V,E) \prod_{x\in\Lambda}(1+\frac{\epsilon}{E})\rho((1+\frac{\epsilon}{E})v_x)dv_x \\ &\leq \sum_{y\in\Lambda}\int \prod_{x\in\Lambda;\, x<y}\left\{\left(1-\frac{\epsilon}{E}\right)\rho\left(\left(1-\frac{\epsilon}{E}\right)v_x\right)dv_x\right\} \\ &\quad \prod_{x\in\Lambda;\, x>y}\left\{\left(1+\frac{\epsilon}{E}\right)\rho\left(\left(1+\frac{\epsilon}{E}\right)v_x\right)dv_x\right\} \\ &\quad \times \int dv_y \left|\rho\left(\left(1-\frac{\epsilon}{E}\right)v_y\right) - \rho\left(\left(1+\frac{\epsilon}{E}\right)v_y\right)\right| N_\Lambda^>(V,E). \end{aligned} \tag{5.25}$$

To get the second inequality we have ordered the unit squares in Λ. Using Lemma 5.2 and the hypothesis (5.1) on the probability distribution we find

$$\begin{aligned}&\mathbb{E}\left(N_\Lambda^>(V,E-\epsilon)-N_\Lambda^>(V,E+\epsilon)\right)\\&\le K\frac{\epsilon}{E^3}|\Lambda|\sum_{y\in\Lambda}\int\left\{\prod_{x<y}dv_x\rho(v_x)\right\}\left\{\prod_{x>y}dv_x\rho(v_x)\right\}\\&\times\int dv_y\,v_y^2\tilde\rho(v_y)\left(\left(1-\frac{\epsilon}{E}\right)^{-1}\sum_{x<y}v_x+\left(1+\frac{\epsilon}{E}\right)^{-1}\sum_{x>y}v_x+v_y\right)^2.\end{aligned}\tag{5.26}$$

By the Cauchy-Schwarz inequality and assuming $\epsilon/E<\frac{1}{2}$,

$$\begin{aligned}\left(\left(1-\frac{\epsilon}{E}\right)^{-1}\sum_{x<y}v_x+\left(1+\frac{\epsilon}{E}\right)^{-1}\sum_{x>y}v_x+v_y\right)^2&\le|\Lambda|\left\{4\sum_{x<y}v_x^2+\sum_{x>y}v_x^2+v_y^2\right\}\\&\le 4|\Lambda|\sum_{x\in\Lambda}v_x^2\end{aligned}\tag{5.27}$$

so

$$\begin{aligned}&\mathbb{E}\left(N_\Lambda^>(V,E-\epsilon)-N_\Lambda^>(V,E+\epsilon)\right)\le\\&4K\frac{\epsilon}{E^3}|\Lambda|^2\left(2|\Lambda|\mathbb{E}(v_x^2)\int\tilde\rho(v)v^2\,dv+\int\tilde\rho(v)v^4\,dv\right)\le K'\frac{\epsilon}{E^3}|\Lambda|^3.\end{aligned}\tag{5.28}$$

Finally we use the fact [W] that $\mathbb{P}[d(E,\sigma(H_\Lambda))<\epsilon]$ is bounded by the left-hand side of (5.28) and we set $\Lambda=\Lambda_L(0)$ and $\epsilon=\exp[-L^\beta]$ to get

$$\mathbb{P}\left[d(E,\sigma(H_{\Lambda_L(0)}))<\exp[-L^\beta]\right]<K'\frac{L^6\exp[-L^\beta]}{E^3}<L^{-q}\tag{5.29}$$

provided L and E are large enough. This proves condition (P2).

We can now apply Theorem 4.1 to conclude that

Theorem 5.1 *Suppose that the probability distribution μ has support $\mathbb{R}$ and satisfies the conditions 1,2,3 and 4 of Section 2 as well as (5.1). Then there exists E_0 such that, almost surely with respect to the product measure $\mathbb{P}$, the spectrum $\sigma(H)\cap\{E\,|\,|E|>E_0\}$ is pure-point and the corresponding eigenfunctions decay exponentially. Moreover one can choose B_0 such that for all $B\ge B_0$, E_0 is independent of B.*

Remark: Here we have restricted ourselves to the first Landau band so that we get pure localisation for all $|E|>E_0$. However for the full hamiltonian (1.1), when the magnetic field is large, there are many Landau bands separated by an energy of order B. Therefore the physically relevant range of energies where the theorem should be applied is $E<E_0$, $E_0<E<<B$. Theorem 5.1 covers this case since for B large enough ($B\ge B_0$) we can choose E_0 independent of B, i.e. of order one with respect to the magnetic field strenght.

6. Proof of Theorem 4.2

In this section we prove Theorem 4.2. As part of this proof is analogous to the one of Ref. [DrKl], we shall abbreviate some of the arguments which need no modification. We first need some technical lemmas:

Lemma 6.1 *The kernel $H(z,z')$ of the Hamiltonian H satisfies the following bounds. For every double-index α,*

$$|\partial_z^\alpha H(z,z')| \le A_\alpha (1+|z|^2)^{|\alpha|/2+1} e^{-B|z-z'|^2/16} \tag{6.1}$$

for some constants A_α depending on V.

Proof. By formulas (2.28) and (2.31),

$$\begin{aligned}|\partial_z^\alpha H(z,z')| \le C_V a_\alpha \int d^2z''(1+|z_1-z''|^2)^{\alpha_1/2}(1+|z_2+iz''|^2)^{\alpha_2/2} \\ K_B(z,z'')(1+|z''|^2)K_B(z'',z').\end{aligned} \tag{6.2}$$

In the product $K_B(z,z'')K_B(z'',z')$ we use the identity $|z-z''|^2+|z'-z''|^2 = 2|z''-(z+z')/2|^2+|z-z'|^2/2$. Next we change variables to $w = z''-(z+z')/2$. This leads to

$$\begin{aligned}&(1+|z_1-z''|^2)^{\alpha_1/2}(1+|z_2+iz''|^2)^{\alpha_2/2} \\ &\quad = (1+|\tfrac{1}{2}(\bar z - z') - w|^2)^{\alpha_1/2}(1+|\tfrac{1}{2}i(\bar z + z') + iw|^2)^{\alpha_2/2} \\ &\quad \le (2(1+|w|^2))^{|\alpha|/2}(1+|(\bar z - z')/2|^2)^{\alpha_1/2}(1+|(\bar z + z')/2|^2)^{\alpha_2/2} \\ &\quad \le \left(2(1+|w|^2)(1+(|z|\vee|z'|)^2)\right)^{|\alpha|/2}.\end{aligned} \tag{6.3}$$

Inserting this we obtain

$$\begin{aligned}|\partial_z^\alpha H(z,z')| &\le 2^{|\alpha|/2+1} C_V a_\alpha (1+(|z|\vee|z'|)^2)^{|\alpha|/2+1} e^{-B|z-z'|^2/8} \\ &\quad \left(\frac{B}{2\pi}\right)^2 \int d^2w (1+|w|^2)^{|\alpha|/2+1} e^{-B|w|^2/2} \\ &\le A_\alpha (1+|z|^2)^{|\alpha|/2+1} e^{-B|z-z'|^2/16}\end{aligned} \tag{6.4}$$

for some constant A_α. QED

Lemma 6.2 *Suppose that $\xi \in \mathcal{S}'(\mathbb{R}^2)$ is a generalised eigenvector of H with eigenvalue $E \neq 0$. Then $\xi \in \mathcal{C}^\infty(\mathbb{R}^2)$ and ξ is polynomially bounded, that is,*

$$|\xi(z)| \le K(1+|z|^2)^p \tag{6.5}$$

for certain constants K and $p > 0$.

Proof. As $\xi \in \mathcal{S}'(\mathbb{R}^2)$, there exists a constant M and integers r and n such that

$$|\langle \phi \,|\, \xi \rangle| \le M\, p_{r,n}(\phi). \tag{6.6}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^2)$. Now,

$$(H(V)\xi)(z) = \langle H(\cdot, z) \,|\, \xi \rangle, \tag{6.7}$$

so to prove that $H(V)\xi$ is $\mathcal{C}^\infty$ it suffices that

$$\sup_{z' \in \mathbb{C}} p_{r,n} \left(\partial^\beta_{z'} H(\cdot, z') \right) < \infty. \tag{6.8}$$

This is analogous to Lemma 6.1. Since $H(V)\xi = E\xi$ it follows that ξ is itself a $\mathcal{C}^\infty$-function. Moreover,

$$\begin{aligned} |\xi(z)| &\leq \frac{1}{|E|} |\langle H(\cdot, z) \,|\, \xi \rangle| \\ &\leq \frac{M}{|E|} p_{r,n} \left(H(\cdot, z) \right) \\ &\leq \frac{MA}{|E|} \sup_{z'} (1 + |z'|^2)^{n+|\alpha|+1} e^{-B|z-z'|^2/16} \\ &\leq 2^{n+|\alpha|+1} \frac{MA}{|E|} (1 + |z|^2)^{n+|\alpha|+1} \sup_{t \in \mathbb{R}} (1+t)^{n+|\alpha|+1} e^{-Bt/16} \\ &\leq K (1 + |z|^2)^{n+|\alpha|+1} \end{aligned} \tag{6.9}$$

This proves the lemma. QED

Lemma 6.3 *Suppose that N is an operator on $L^2(\mathbb{R}^2)$ with kernel $N(z, z')$ satisfying*

$$|N(z, z')| \leq A_0 (1 + |z|^2) e^{-\kappa |z - z'|^2} \tag{6.10}$$

for some constants A_0 and k_0 and suppose that $\psi \in \mathcal{C}(\mathbb{R}^2)$ is polynomially bounded: $|\psi(z)| \leq K(1 + |z|^2)^p$. Then, for any $m > 0$, there exist constants A_0' and $L_0 > 0$ such that, if $L > L_0$ and $\Lambda_L(x)$ is (m, E)-regular then

$$\begin{aligned} &\langle 1_{\Lambda_1(x)} \big| \, |G_{\Lambda_L(x)} N 1_{\Lambda_L(x)^c} \psi| \rangle \\ &\qquad \leq A_0' L^3 (1 + |x|^2)^{p+1} \left(e^{-\kappa L^2/8} + e^{-mL} \int_{\Lambda_{2L}(x)} |\psi(z')| d^2 z'. \right) \end{aligned} \tag{6.11}$$

Proof. We write

$$\begin{aligned} &\langle 1_{\Lambda_1(x)} \big| \, |G_{\Lambda_L(x)} N 1_{\Lambda_L(x)^c} \psi| \rangle \\ &\quad = \langle 1_{\Lambda_1(x)} \big| \, |G_{\Lambda_L(x)} N 1_{\Lambda_{2L}(x) \setminus \Lambda_L(x)} \psi| \rangle + \langle 1_{\Lambda_1(x)} \big| \, |G_{\Lambda_L(x)} N 1_{\Lambda_{2L}(x)^c} \psi| \rangle. \end{aligned} \tag{6.12}$$

We start by considering the first term: Suppose first that $z \in \Lambda_{\tilde{L}}(x)$. In that case $|z - z'| > L^s/2$ and, moreover, since $z' \in \Lambda_{2L}(x)$, $|z'|^2 \leq 2(|x|^2 + |z' - x|^2) \leq 2(|x|^2 + 2L^2)$ and hence certainly $1 + |z'|^2 \leq 5L^2(1 + |x|^2)$. Thus we have

$$\begin{aligned} \left| \left(N 1_{\Lambda_{2L} \setminus \Lambda_L(x)} \psi \right)(z) \right| &\leq A_0 \int_{\Lambda_{2L}(x) \setminus \Lambda_L(x)} (1 + |z'|^2) e^{-\kappa |z - z'|^2} |\psi(z')| d^2 z' \\ &\leq 5 A_0 L^2 (1 + |x|^2) e^{-\kappa L^{2s}/4} \int_{\Lambda_{2L}(x)} |\psi(z')| d^2 z'. \end{aligned} \tag{6.13}$$

Using the inequality $||fg||_2 \leq ||f||_\infty ||g||_2$ this implies that

$$||1_{\Lambda_L(x)} N 1_{\Lambda_{2L}(x)\setminus\Lambda_L(x)}\psi|| \\ \leq 5A_0 L^3 (1+|x|^2) e^{-\kappa L^{2s}/4} \int_{\Lambda_{2L}(x)} |\psi(z')| d^2 z'. \tag{6.14}$$

Using the regularity condition we now find that

$$\begin{aligned}\langle 1_{\Lambda_1(x)} | \, |G_{\Lambda_L(x)} 1_{\Lambda_{\bar{L}}(x)} N 1_{\Lambda_{2L}(x)\setminus\Lambda_L(x)} \psi|\rangle &\\ \leq 5A_0 L^3 e^{L^\beta} (1+|x|^2) e^{-\kappa L^{2s}/4} \int_{\Lambda_{2L}(x)} |\psi(z')| d^2 z' &\\ \leq 5A_0 L^3 e^{-mL} (1+|x|^2) \int_{\Lambda_{2L}(x)} |\psi(z')| d^2 z' &\end{aligned} \tag{6.15}$$

provided L is so large that $\kappa L^{2s}/4 > L^\beta + mL$.

Next suppose that $z \in \tilde{\Lambda}_L(x)$. In that case we can simply write $|N(z,z')| \leq A_0(1+|z'|^2) \leq 5A_0(1+|x|^2)L^2$ and hence, using the regularity condition,

$$\langle 1_{\Lambda_1(x)} | \, |G_{\Lambda_L(x)} 1_{\tilde{\Lambda}_L(x)} N 1_{\Lambda_{2L}(x)\setminus\Lambda_L(x)} \psi|\rangle \leq 5A_0 L^3 (1+|x|^2) e^{-mL} \int_{\Lambda_{2L}(x)} |\psi(z')| d^2 z'. \tag{6.16}$$

Next we consider the second term of (6.12). For $z \in \Lambda_L(x)$ and $z' \in \Lambda_{2L}(x)^c$,

$$\begin{aligned} 1+|z'|^2 &\leq 1 + 3(|x|^2 + L^2/2 + |z-z'|^2) \\ &\leq 2L^2(1+|x|^2)(1+|z-z'|^2) \\ &\leq 8(1+|x|^2)(1+|z-z'|^2)^2 \end{aligned} \tag{6.17}$$

assuming $L^2 > 2$. Therefore, if $|\psi(z)| \leq K(1+|z|^2)^p$ then

$$\begin{aligned} |(N 1_{\Lambda_{2L}(x)^c}\psi)(z)| &\leq A_0 K \int_{\Lambda_{2L}(x)^c} d^2 z' (1+|z'|^2)^{p+1} e^{-\kappa|z-z'|^2} \\ &\leq 8A_0 K (1+|x|^2)^{p+1} \int_{|z'|>L/2} d^2 z' (1+|z'|^2)^{p+1} e^{-\kappa|z'|^2} \\ &\leq 8\pi A_0 K (1+|x|^2)^{p+1} \int_{L^2/4} dt (1+t)^{p+1} e^{-\kappa t} \\ &\leq 8\pi A_0 K (p+1)! (2/\kappa)^{p+2} e^{\kappa/2} (1+|x|^2)^{p+1} e^{-\kappa L^2/8}. \end{aligned} \tag{6.18}$$

(6.15),(6.16) and (6.18) prove the lemma with A_0' equal to the minimum of $10A_0$ and $8\pi A_0 K(p+1)!(2/\kappa)^{p+2} e^{\kappa/2}$, and with $L_0 = (4(m+1)/\kappa)^{1/(2s-1)}$. QED

Proof of Theorem 4.2

Let b be a positive integer to be determined below. As in [DrKl] we define for $x_0 \in \mathbb{Z}^2$,

$$\mathcal{A}_{k+1} = \Lambda_{2bL_{k+1}}(x_0) \setminus \Lambda_{2L_k}(x_0). \quad (k = 0,1,2,\ldots) \tag{6.19}$$

We also define the events

$$\begin{aligned}\mathcal{E}_k(x_0) = \{V \mid \Lambda_{L_k}(x_0) \text{ and } \Lambda_{L_k}(x) \text{ are} \\ (m,E)\text{-singular for some } E \in I \text{ and some } x \in \mathcal{A}_{k+1}(x_0) \cap \mathbb{Z}^2\}.\end{aligned} \tag{6.20}$$

If

$$\Omega_0 = \left\{V \mid \forall x_0 \in \mathbb{Z}^2 : \sharp\{k \mid V \in \mathcal{E}_k(x_0)\} < \infty\right\} \tag{6.21}$$

then it follows from the Borel-Cantelli Lemma that $\mathbb{P}(\Omega_0) = 1$. (See [DrKl]).

Now let $V \in \Omega_0$ and $E \in I$ and suppose that $\xi \in \mathcal{S}'(\mathbb{R}^2)$ is a generalised eigenvector of $H(V)$. It follows that $\xi \in \mathcal{C}^\infty(\mathbb{R}^2)$. Choose $x_0 \in \mathbb{Z}^2$ such that $\langle 1_{\Lambda_1(x_0)} \,|\, |\xi|\rangle \neq 0$. Suppose that $\Lambda_{L_k}(x_0)$ is (m,E)-regular. Then, by formula (3.12),

$$\begin{aligned}\langle 1_{B(x_0)} \,|\, |\xi|\rangle \le \langle 1_{B(x_0)} \,|\, |G_{\Lambda_{L_k}(x_0)} H 1_{\Lambda_{L_k}(x_0)^c}\xi|\rangle \\ + \langle 1_{B(x_0)} \,|\, |G_{\Lambda_{L_k}(x_0)} P_0 V_{\Lambda_{L_k}(x_0)^c} P^*_{\Lambda_{L_k}(x_0)}\xi|\rangle.\end{aligned} \tag{6.22}$$

By Lemma 6.3 with $N = H$ the first term on the right-hand side is bounded by

$$\begin{aligned}&\langle 1_{B(x_0)} \,|\, |G_{\Lambda_{L_k}(x_0)} H 1_{\Lambda_{L_k}(x_0)^c}\xi|\rangle \\ &\quad\le A' L_k^3 (1+|x_0|^2)^{p+1} \left(e^{-BL_k^2/16} + e^{-mL_k} \int_{\Lambda_{L_k}(x_0)} |\xi(z')| d^2 z'\right).\end{aligned} \tag{6.23}$$

Notice that, by Lemma 6.1, the kernel of H satisfies (6.10) and $B(x_0) = \Lambda_1(x_0)$.

As to the second term of (6.22), we can use Lemma 6.3 with $N = P_0 V$ and $\psi = P^*_{\Lambda_{L_k}(x_0)}\xi$ to conclude

$$\begin{aligned}&\langle 1_{B(x_0)} \,|\, |G_{\Lambda_{L_k}(x_0)} P_0 V_{\Lambda_{L_k}(x_0)^c} P^*_{\Lambda_{L_k}(x_0)}\xi|\rangle \\ &\quad\le A L_k^3 (1+|x|^2)^{p+1} \left(e^{-BL^2/32} + e^{-mL_k} \int_{\Lambda_{2L_k}(x_0)} |\psi(z)| d^2 z\right)\end{aligned} \tag{6.24}$$

Notice that ψ is also polynomially bounded:

$$\begin{aligned}|\psi(z)| &\le 2^p K (B/2\pi)(1+|z|^2)^p \int d^2 z' (1+|z'|^2)^p e^{-B|z'|^2/4} \\ &\le \frac{1}{4} K B (p+1)! (2/\kappa)^{p+1} e^\kappa (1+|z|^2)^p.\end{aligned} \tag{6.25}$$

Moreover, since

$$\begin{aligned}\int \Lambda_{2L_k(x_0)} |\psi(z)| d^2 z &\le \int_{\Lambda_{2L_k}(x_0)} d^2 z \int_{\Lambda_{L_k}(x_0)} d^2 z' K_B(z,z') |\xi(z')| \\ &\le 2 \int_{\Lambda_{2L_k}(x_0)} d^2 z' |\xi(z')|,\end{aligned} \tag{6.26}$$

both terms in (6.22) satisfy the same bound and we have

$$\langle 1_{B(x_0)} | \, |\xi| \rangle \le A_1 L_k^3 (1+|z_0|^2)^{p+1} \left(e^{-BL_k^2/32} + e^{-mL_k} \int_{\Lambda_{2L_k}(x_0)} |\xi(z)| d^2 z \right). \tag{6.27}$$

Combining this with the polyniomial bound on ξ (Lemma 6.2) we find

$$\langle 1_{B(x_0)} | \, |\xi| \rangle \le A_1 L_k^3 (1+|x_0|^2)^{p+1} \left(e^{-BL_k^2/32} + K L_k^2 (1+2|x_0|^2 + 4L_k^2)^{p+1} e^{-mL_k} \right). \tag{6.28}$$

If there were to exist an infinite number of k's such that $\Lambda_{L_k}(x_0)$ was (m,E)-regular, then it would follow from (6.28) that $\langle 1_{B(x_0)} | \, |\xi| \rangle = 0$, a contradiction. Therefore, there exists k_1 such that for all $k \ge k_1$, $\Lambda_{L_k}(x_0)$ is (m,E)-singular. By the definition of Ω_0 this implies, as in [DrKl], that there is k_2 such that $\Lambda_{L_k}(x)$ is regular for all $x \in \mathcal{A}_{k+1}(x_0) \cap \mathbb{Z}^2$.

Now let $\rho \in (0,1)$ and choose $b > (1+\rho)/(1-\rho)$. Define

$$\tilde{\mathcal{A}}_{k+1} = \Lambda_{2bL_{k+1}/(1+\rho)}(x_0) \setminus \Lambda_{2L_k/(1-\rho)}(x_0). \tag{6.29}$$

Let $k \ge k_2$ so that $\Lambda_{L_k}(y)$ is (m,E)-regular for all $y \in \mathcal{A}_{k+1}(x_0) \cap \mathbb{Z}^2$. Inserting in (6.27) the obvious bound

$$\int_{\Lambda_{2L_k}(y)} |\xi(z)| d^2 z \le L_k^2 \sup_{y' \in \Lambda_{2L_k}(y) \cap \mathbb{Z}^2} \langle 1_{B(y')} | \, |\xi| \rangle \tag{6.30}$$

we have

$$\langle 1_{B(y)} | \, |\xi| \rangle \le A_1 L_k^3 (1+|y|^2)^{p+1} \left(e^{-BL_k^2/32} + L_k^2 e^{-mL_k} \sup_{y' \in \Lambda_{2L_k}(y) \cap \mathbb{Z}^2} \langle 1_{B(y')} | \, |\xi| \rangle \right). \tag{6.31}$$

>From now on we shall keep x_0 fixed and all constants may depend on x_0. If $y \in \mathcal{A}_{k+1} \cap \mathbb{Z}^2$ then $|y| \le |x_0| + bL_{k+1}\sqrt{2} \le cL_k^\alpha$ so

$$\langle 1_{B(y)} | \, |\xi| \rangle \le C L_k^r \left(e^{-BL_k^2/32} + e^{-mL_k} \sup_{y' \in \Lambda_{2L_k}(y) \cap \mathbb{Z}^2} \langle 1_{B(y')} | \, |\xi| \rangle \right) \tag{6.32}$$

where $r = 5 + (1+p)\alpha$. As $\tilde{\mathcal{A}}_{k+1}(x_0) \subset \mathcal{A}_{k+1}(x_0)$ and $d(x, \partial\mathcal{A}_{k+1}(x_0)) \ge \rho|x - x_0|$ for $x \in \tilde{\mathcal{A}}_{k+1}(x_0)$, we can iterate (6.32) at least $n = \rho|x-x_0|/L_k$ times to obtain

$$\begin{aligned}
\langle 1_{B(x)} | \, |\xi| \rangle &\le C^n L_k^{rn} \left(n e^{-BL_k^2/32} + K e^{-mL_k n} L_k^{\alpha p} \right) \\
&\le (C^{1/L_k})^{\rho|x-x_0|} \left(L_k^{r/L_k} \right)^{\rho|x-x_0|} e^{-m\rho|x-x_0|} \times \\
&\quad \left(\frac{\rho|x-x_0|}{L_k} \exp\left[-\frac{B}{32} L_k^{2-\alpha} \frac{(1+\rho)}{2b\sqrt{2}} |x-x_0| + m\rho|x-x_0| \right] + \right. \\
&\quad \left. K(1-\rho)^{\alpha p} |x-x_0|^{\alpha p} \right)
\end{aligned} \tag{6.33}$$

If $\rho' < \rho$ then for k sufficiently large, this yields

$$\langle 1_{B(x)} | \, |\xi| \rangle \leq M e^{-m\rho'|x-x_0|} \tag{6.34}$$

for some constant M. It then follows from Lemma 2.6 that ξ is exponentially decaying with rate $m\rho'$. QED

7. Proof of Theorem 4.3

As in [DrKl], the proof is by induction on k. The induction step is

Lemma 7.1 *Let $p > 2$, $\beta \in (0,1)$ and $s \in (\frac{1}{2}, 1)$ be given. Let J_p be the smallest odd integer $> (p+2)/(p-2)$ and define*

$$\alpha_0 = \frac{(J_p+1)p}{2(p+J_p+1)}. \tag{7.1}$$

Pick $\alpha \in (1, \alpha_0 \wedge (2s))$.
There exists $Q = Q(\alpha, \beta, s, p)$ such that if $l > Q$ and $m_l > 4J/l^{1-\beta}$ then if R(l,m_l) holds and (K2) holds for all $L > l$ and for some $q > 4p+12$ then R(L,m_L) holds with $L = l^\alpha$ and

$$\begin{aligned} m_L &\geq m_l - \left[\left(2l^{-(2-\alpha)/4} + l^{-\alpha(1-s)} \right) m_l + \alpha(\nu+4) l^{-1} \ln l + l^{-\alpha(1-\beta)} \right] \\ &\geq 4J_p l^{-1+\beta}. \end{aligned} \tag{7.2}$$

Again, the proof of this lemma has a probabilistic part and a deterministic part. The latter is much more complicated than in [DrKl] and will be split into several lemmas.

Lemma 7.2 *Suppose that N is an operator on $L^2(\mathbb{R}^2)$ with kernel $N(z,z')$ satisfying*

$$|N(z,z')| \leq A_0 e^{-\kappa|z-z'|^2}. \tag{7.3}$$

Then, if $\Lambda_1, \Lambda_2 \subset \mathbb{R}^2$ are disjoint regions with distance $d = d(\Lambda_1, \Lambda_2)$,

$$||1_{\Lambda_1} N 1_{\Lambda_2}|| \leq A_0 \left(\frac{\pi}{2\kappa} \right)^{1/2} |\Lambda_1|^{1/2} e^{-\kappa d^2}. \tag{7.4}$$

Proof. Since the operator norm is bounded by the Hilbert-Schmidt norm we have

$$\begin{aligned} ||1_{\Lambda_1} N 1_{\Lambda_2}|| &\leq \left[\int_{\Lambda_1} d^2 z \int_{\Lambda_2} d^2 z' |N(z,z')|^2 \right]^{1/2} \\ &\leq A_0 \left[\int_{\Lambda_1} d^2 z \int_{\Lambda_2} d^2 z' e^{-2\kappa|z-z'|^2} \right]^{1/2} \\ &\leq A_0 (\pi/2\kappa)^{1/2} |\Lambda_1|^{1/2} e^{-\kappa d^2} \end{aligned} \tag{7.5}$$

since, for any $z \in \Lambda_1$,

$$\int_{\Lambda_2} d^2z' e^{-2\kappa|z-z'|^2} \le \int_{|z'-z|\ge d} d^2z' e^{-2\kappa|z-z'|^2} = (\pi/2\kappa)e^{-2\kappa d^2}. \tag{7.6}$$

QED

Lemma 7.3 *Let $\Lambda_1 \subset \Lambda = \Lambda_L(x)$ with $L \in \mathbb{N}$ and $x \in \mathbb{Z}^2$ and let $\psi \in L^2(\Lambda)$ with $||\psi|| \neq 0$. Then*

$$||1_\Lambda P^*_{\Lambda_1} G_\Lambda \psi|| \le \frac{B}{2\pi} L^3 ||\psi|| \sup_{u\in\Lambda_L(x)\cap\mathbb{Z}^2} g_\Lambda(u) \tag{7.7}$$

where $g_\Lambda(u)$ is defined by

$$g_\Lambda(u) = \frac{\langle 1_{B(u)}|\,|G_\Lambda\psi|\rangle}{||\psi||}. \tag{7.8}$$

Proof. For any $z \in \Lambda$,

$$\begin{aligned} |\left(P^*_{\Lambda_1} G_\Lambda \psi\right)(z)| &\le \int_{\Lambda_1} d^2z' K_B(z,z')|(G_\Lambda\psi)(z')| \\ &\le \frac{B}{2\pi}\int_{\Lambda_1} |(G_\Lambda\psi)(z')|d^2z' \\ &= \frac{B}{2\pi}\sum_{u\in\Lambda_1\cap\mathbb{Z}^2} \langle 1_{B(u)}|\,|G_\Lambda\psi|\rangle \\ &\le \frac{B}{2\pi} L^2\, ||\psi|| \sup_{u\in\Lambda_1\cap\mathbb{Z}^2} g_\Lambda(u). \end{aligned} \tag{7.9}$$

The Lemma follows by integration over $z \in \Lambda$. QED

We now come to the main deterministic lemma, the analogue of Lemma 4.2 in [DrKl].

Lemma 7.4 *Let $J \in \mathbb{N}$, $\beta \in (0,1)$, $s \in (\frac{1}{2}, 1)$, $\alpha \in (1, 2s)$, and $E \in \mathbb{R}$ be given. Fix $x \in \mathbb{Z}^2$ and assume that for all $L \ge 1$, $|V(z)| \le L^\nu$ for all $z \in \Lambda_L(x)$, where $\nu < \infty$. Then there exists $Q' = Q'(\alpha, \beta, s, J, \nu)$ such that the following holds:*
If $l > Q'$ and $m_l > 4Jl^{-1+\beta}$ then the three conditions below imply that $\Lambda_L(x)$ is (m_L, E)-regular with

$$m_L \ge m_l\left(1 - 2l^{-\alpha(1-s)} - l^{-(2-\alpha)/4}\right) - \alpha(\nu+4)l^{-1}\ln l - l^{-\alpha(1-\beta)} > \frac{4J}{L^{1-\beta}}. \tag{7.10}$$

(i) $\Lambda_L(x)$ satisfies (RA) *in the definition of regularity.*

(ii) $\Lambda_{l'}(y)$ satisfies (RA) *for all $l' \in \mathcal{J}_l = \{l, l+\hat{l}+1, 2(l+\hat{l}+1), \ldots, J(l+\hat{l}+1)\}$, where $\hat{l} = l + l^{(\alpha+2)/4}$, and for all $y \in \Lambda_L(x) \cap \mathbb{Z}^2$ such that $\Lambda_{l'}(y) \subset \Lambda_{\tilde{L}}(x)$.*

(iii) There are at most J squares $\Lambda_l(u_i) \subset \Lambda_{\tilde{L}}(x)$ with centres $u_i \in \mathbb{Z}^2$ and with $d(u_i, u_j) \ge l+1\,(i \neq j)$, which are (m_l, E)-singular.

Proof. We want to prove that, provided l is large enough, conditions (i), (ii) and (iii) above imply that $\Lambda = \Lambda_L(x)$ is (m_L, E)-regular. Suppose, therefore, that (i), (ii) and (iii) are satisfied and choose a maximal set $\Lambda_l(u_i)\,(i = 1, \ldots, r)$ of non-overlapping squares with centres $u_i \in \mathbb{Z}^2$ contained in $\Lambda_{\tilde{L}}(x)$ which are (m_l, E)-singular. By (iii), $r \leq J$ and if $\Lambda_l(u) \subset \Lambda_{\tilde{L}}(x)$ and $u \notin \bigcup_{i=1}^r \Lambda_{2l}(u_i)$ then $\Lambda_l(u)$ is (m_l, E)-regular. An easy geometric induction argument shows that this impies that there are squares $\Lambda_{l_i} \subset \Lambda_{\tilde{L}}(x)$ $(i = 1, \ldots, t;\ t \leq r)$ with centres on $\mathbb{Z}^2$ and with $l_i \in \mathcal{J}_l$ such that $\Lambda_{l_i} \cap \Lambda_{l_j} \neq \emptyset$ for $i \neq j$,

$$\sum_{i=1}^t l_i \leq J(l + \hat{l} + 1) \text{ and } \bigcup_{i=1}^r \Lambda_{l+\hat{l}+1}(u_i) \subset \bigcup_{j=1}^t \Lambda_{l_j}. \tag{7.11}$$

Clearly, if $u \notin \Lambda_{\check{l}_j}$ for any $j = 1, \ldots, t$, where $\check{l}_j = l_j - l^{(\alpha+2)/4}$ and $u \in \mathbb{Z}^2$ and $\Lambda_l(u) \subset \Lambda_{\tilde{L}}(x)$ then $u \notin \Lambda_{2l}(u_i)\,(i = 1, \ldots, r)$ and hence $\Lambda_l(u)$ is (m_l, E)-regular.

To prove that $\Lambda = \Lambda_L(x)$ is regular, choose an arbitrary $\phi \in L^2(\Lambda)$. From (3.6) we have that, if $\Lambda_1, \Lambda_2 \subset \Lambda$ are disjoint and $\Lambda_1' \subset \Lambda_1$ and $\Lambda_2' \subset \Lambda_2$ then

$$G_{\Lambda_1\cup\Lambda_2} 1_{\Lambda_2'}\phi = G_{\Lambda_2} 1_{\Lambda_2'}\phi - (G_{\Lambda_1} \oplus G_{\Lambda_2})\Gamma_{\Lambda_1,\Lambda_2} G_{\Lambda_1\cup\Lambda_2} 1_{\Lambda_2'}\phi. \tag{7.12}$$

Using (3.7) this yields

$$\begin{aligned}\langle 1_{\Lambda_1'}|\,|G_{\Lambda_1\cup\Lambda_2} 1_{\Lambda_2'}\phi|\rangle &= \langle 1_{\Lambda_1'}|\,|G_{\Lambda_1}\Gamma_{\Lambda_1,\Lambda_2} G_{\Lambda_1\cup\Lambda_2} 1_{\Lambda_2'}\phi|\rangle \\ &\leq \langle 1_{\Lambda_1'}|\,|G_{\Lambda_1} P_0 V_{\Lambda_1\cup\Lambda_2} P_{\Lambda_2}^* G_{\Lambda_1\cup\Lambda_2} 1_{\Lambda_2'}\phi|\rangle \\ &\quad + \langle 1_{\Lambda_1'}|\,|G_{\Lambda_1} P_0 V_{\Lambda_2} P_{\Lambda_1}^* G_{\Lambda_1\cup\Lambda_2} 1_{\Lambda_2'}\phi|\rangle.\end{aligned} \tag{7.13}$$

We now specialise to $\Lambda_1 = \Lambda_l(u)$, $\Lambda_2 = \Lambda_L(x) \setminus \Lambda_l(u)$, $\Lambda_1' = \Lambda_1(u)$ and $\Lambda_2' = \tilde{\Lambda}_L(x)$, assuming that $\Lambda_l(u)$ is (m_l, E)-regular. (Notice that $\Lambda_2' \subset \Lambda_2$ because $u \in \Lambda_{\tilde{L}}(x)$.) Denoting $1_{\tilde{\Lambda}_L(x)}\phi = \psi$, we find

$$||\psi||\, g_\Lambda(u) \leq I_1(u) + I_2(u), \tag{7.14}$$

where

$$I_1(u) = \langle 1_{B(u)}|\,|G_{\Lambda_l(u)} P_0 V_\Lambda P_{\Lambda\setminus\Lambda_l(u)}^* G_\Lambda\psi|\rangle \tag{7.14a}$$

and

$$I_2(u) = \langle 1_{B(u)}|\,|G_{\Lambda_l(u)} P_0 V_{\Lambda\setminus\Lambda_l(u)} P_{\Lambda_l(u)}^* G_\Lambda\psi|\rangle \tag{7.14b}$$

and where $g_\Lambda(u)$ was defined in (7.8). We will now bound the two terms $I_1(u)$ and $I_2(u)$ in similar fashions using Lemmas 7.2 and 7.3. We split $I_1(u)$ into three terms:

$$\begin{aligned}I_1(u) \leq\ & \langle 1_{B(u)}|\,|G_{\Lambda_l(u)} P_0 V_\Lambda P_{\Lambda\setminus\Lambda_{\hat{l}}(u)}^* G_\Lambda\psi|\rangle \\ &+ \langle 1_{B(u)}|\,|G_{\Lambda_l(u)} 1_{\Lambda_{\bar{l}}(u)} P_0 V_\Lambda P_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}^* G_\Lambda\psi|\rangle \\ &+ \langle 1_{B(u)}|\,|G_{\Lambda_l(u)} 1_{\tilde{\Lambda}_l(u)} P_0 V_\Lambda P_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}^* G_\Lambda\psi|\rangle\end{aligned} \tag{7.15}$$

The first term on the right-hand side can be bounded with the help of Lemma 7.2. By (i) and (ii), $||G_\Lambda|| \leq 2e^{L^\beta}$ and $||G_{\Lambda_l(u)}|| \leq 2e^{l^\beta}$. Moreover, since $|V_{\Lambda_L(x)}| \leq L^\nu$,

$$|P_0 V_\Lambda P_0^*(z, z')| \leq L^\nu \int K_B(z, z_1) K_B(z_1, z') d^2 z_1 = L^\nu \frac{B}{2\pi} e^{-B|z-z'|^2/8} \tag{7.16}$$

using (2.16). With Lemma 7.2 this yields

$$\begin{aligned}\langle 1_{B(u)}|\,|G_{\Lambda_l(u)}P_0V_\Lambda P^*_{\Lambda\setminus\Lambda_{\hat{l}}(u)}G_\Lambda\psi|\rangle &\le 4(B/\pi)^{1/2}L^\nu l e^{l^\beta}e^{L^\beta}e^{-Bl^{(\alpha+2)/2}/8}||\psi|| \\ &= 4(B/\pi)^{1/2}l^{\alpha\nu+1}e^{l^\beta+l^{\alpha\beta}-Bl^{(\alpha+2)/2}/8}||\psi|| \\ &\le c_1e^{-Bl^{(\alpha+2)/2}/16}||\psi||\end{aligned}\tag{7.17}$$

provided $l > Q_1 = (48/B)^{1/\gamma_1}$ with $\gamma_1 = 1-\alpha/2$, where $c_1 = 4(B/\pi)^{1/2}((\alpha\nu+1)/e)^{\alpha\nu+1}$. (We use that $\beta < \beta\alpha < \alpha < (\alpha+2)/2$ and also $l^\gamma \le (\gamma/e)^\gamma e^l$) The second term is estimated in the same way:

$$\langle 1_{B(u)}|\,|G_{\Lambda_l(u)}1_{\Lambda_{\tilde{l}}(u)}P_0V_\Lambda P^*_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}G_\Lambda\psi|\rangle \le c_1e^{-Bl^{2s}/16}\tag{7.18}$$

provided $l > Q_2 = (48/B)^{1/\gamma_2}$ with $\gamma_2 = (2s-\alpha\beta)\wedge(2s-1)$. (The wedge denotes the minimum.) To estimate the third term of (7.15) we use the assumption that $\Lambda_l(u)$ is (m_l, E)-regular. This gives

$$\langle 1_{B(u)}|\,|G_{\Lambda_l(u)}1_{\tilde{\Lambda}_l(u)}P_0V_\Lambda P^*_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}G_\Lambda\psi|\rangle \le 2L^\nu e^{-m_ll}e^{l^{\alpha\beta}}||\psi||.\tag{7.19}$$

On the other hand we can also use Lemma 7.3 with $\Lambda_1 = \Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)$:

$$\begin{aligned}&\langle 1_{B(u)}|\,|G_{\Lambda_l(u)}1_{\tilde{\Lambda}_l(u)}P_0V_\Lambda P^*_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}G_\Lambda\psi|\rangle \\ &\quad\le \frac{B}{2\pi}L^{\nu+3}e^{-m_ll}||\psi|| \sup_{w\in\Lambda_1\cap\mathbb{Z}^2} g_\Lambda(w).\end{aligned}\tag{7.20}$$

Collecting terms we find

$$\frac{I_1(u)}{||\psi||} \le 2c_1e^{-Bl^\delta/16} + l^{\alpha\nu}e^{-m_ll}\left[2e^{l^{\alpha\beta}} \bigwedge \frac{B}{2\pi}l^{3\alpha} \sup_{w\in\Lambda\cap\Lambda_{\hat{l}}(u)\cap\mathbb{Z}^2} g_\Lambda(u)\right].\tag{7.21}$$

The second term of (7.14) is estimated in the same way:

$$\begin{aligned}I_2(u) &\le \langle 1_{B(u)}|\,|G_{\Lambda_l(u)}P_0V_{\Lambda\setminus\Lambda_{\hat{l}}(u)}P^*_{\Lambda_l(u)}G_\Lambda\psi|\rangle \\ &\quad+ \langle 1_{B(u)}|\,|G_{\Lambda_l(u)}1_{\tilde{\Lambda}_l(u)}P_0V_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}P^*_{\Lambda_l(u)}G_\Lambda\psi|\rangle \\ &\quad+ \langle 1_{B(u)}|\,|G_{\Lambda_l(u)}1_{\tilde{\Lambda}_l(u)}P_0V_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}P^*_{\Lambda_l(u)}G_\Lambda\psi|\rangle.\end{aligned}\tag{7.22}$$

In the first term we use Lemma 7.2 with $N = P_0$:

$$\langle 1_{B(u)}|\,|G_{\Lambda_l(u)}P_0V_{\Lambda\setminus\Lambda_{\hat{l}}(u)}P^*_{\Lambda_l(u)}G_\Lambda\psi|\rangle \le 4\left(\frac{B}{2\pi}\right)^{1/2} l^{\alpha\nu+1}e^{l^\beta+l^{\alpha\beta}}e^{-Bl^{(\alpha+2)/2}/4}||\psi||.\tag{7.23}$$

Similarly, for the second term,

$$\begin{aligned}&\langle 1_{B(u)}|\,|G_{\Lambda_l(u)}1_{\tilde{\Lambda}_l(u)}P_0V_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}P^*_{\Lambda_l(u)}G_\Lambda\psi|\rangle \\ &\quad\le 4(2\pi/B)^{1/2}l^{\alpha\nu+1}e^{l^\beta+l^{\alpha\beta}}e^{-Bl^{2s}/4}||\psi||.\end{aligned}\tag{7.24}$$

In the third term we have as before

$$\langle 1_{B(u)}|\,|G_{\Lambda_l(u)}1_{\tilde{\Lambda}_l(u)}P_0V_{\Lambda\cap\Lambda_{\hat{l}}(u)\setminus\Lambda_l(u)}P^*_{\Lambda_l(u)}G_\Lambda\psi|\rangle \\ \leq l^{\alpha\nu}e^{-m_l l}||\psi||\left[2e^{l^{\alpha\beta}}\bigwedge\frac{B}{2\pi}l^{3\alpha}\sup_{w\in\Lambda_l(u)\cap\mathbb{Z}^2}g_\Lambda(w)\right]. \tag{7.25}$$

Collecting terms,

$$\frac{I_2(u)}{||\psi||}\leq 2c_1e^{-Bl^\delta/16}+l^{\alpha\nu}e^{-m_l l}\left[2e^{l^{\alpha\beta}}\bigwedge\frac{B}{2\pi}l^{3\alpha}\sup_{w\in\Lambda_l(u)\cap\mathbb{Z}^2}g_\Lambda(w)\right]. \tag{7.26}$$

It now follows from (7.21), (7.26) and (7.14) that, if $\Lambda_l(u)$ is (m_l,E)-regular, then

$$g_\Lambda(u)\leq 4c_1e^{-Bl^\delta/16}+2l^{\alpha\nu}e^{-m_l l}\left[2e^{l^{\alpha\beta}}\bigwedge\frac{B}{2\pi}l^{3\alpha}\sup_{w\in\Lambda_{\hat{l}}(u)\cap\mathbb{Z}^2}g_\Lambda(w)\right]. \tag{7.27}$$

Next, suppose that $\Lambda_l(u)$ is not (m_l,E)-regular. Then $u\in\Lambda_{l_j}$ for some $j\in\{1,\ldots,t\}$. We use (7.13) with $\Lambda_1=\Lambda_{l_j}$, $\Lambda_2=\Lambda\setminus\Lambda_1$, $\Lambda_1'=B(u)$ and $\Lambda_2'=\tilde{\Lambda}_L(u)$ to write an analogue of (7.14):

$$||\psi||\,g_\Lambda(u)\leq I_1'(u)+I_2'(u) \tag{7.28}$$

with

$$I_1'(u)=\langle 1_{B(u)}|\,|G_{\Lambda_{l_j}}P_0V_\Lambda P^*_{\Lambda\setminus\Lambda_{l_j}}G_\Lambda\psi|\rangle \tag{7.28a}$$

and

$$I_2'(u)=\langle 1_{B(u)}|\,|G_{\Lambda_{l_j}}P_0V_{\Lambda\setminus\Lambda_{l_j}}P^*_{\Lambda_{l_j}}G_\Lambda\psi|\rangle. \tag{7.28b}$$

Notice that the only difference between $I_1(u)$ and $I_1'(u)$ is that $\Lambda_l(u)$ has been replaced by Λ_{l_j}. We can split $I_1'(u)$ into three terms analogous to (7.15), where the first two terms are bounded as in (7.23) and (7.24) with l replaced by $l_j\leq(l+\hat{l})J\leq 3Jl$. For the third term there is no analogue of (7.25) but (7.26) still holds. We thus obtain

$$\frac{I_1'(u)}{||\psi||}\leq 2c_1'e^{-Bl^\delta/16}+2\frac{B}{2\pi}l^{\alpha(\nu+3)}e^{l_j^\beta}\sup_{\substack{w\in\Lambda\cap\Lambda_{\hat{l}_j}\setminus\Lambda_{l_j}\\ w\in\mathbb{Z}^2}}g_\Lambda(w), \tag{7.29}$$

where $c_1'=3Jc_1$. We split up $I_2'(u)$ differently:

$$I_2'(u)\leq\langle 1_{B(u)}|\,|G_{\Lambda_{l_j}}P_0V_{\Lambda\setminus\Lambda_{l_j}}P^*_{\Lambda_{\hat{l}_j}}G_\Lambda\psi|\rangle \\ +\langle 1_{B(u)}|\,|G_{\Lambda_{l_j}}P_0V_{\Lambda\setminus\Lambda_{l_j}}P^*_{\Lambda_{l_j}\setminus\Lambda_{\hat{l}_j}}G_\Lambda\psi|\rangle. \tag{7.30}$$

We obtain, using Lemma 7.2 and 7.3 respectively,

$$\frac{I_2'(u)}{||\psi||}\leq c_1'e^{-Bl^\delta/16}+2\frac{B}{2\pi}l^{\alpha(\nu+3)}e^{l_j^\beta}\sup_{w\in(\Lambda_{l_j}\setminus\Lambda_{\hat{l}_j})\cap\mathbb{Z}^2}g_\Lambda(w). \tag{7.31}$$

In all we have, if $\Lambda_l(u)$ is (m_l, E)-singular,

$$g_\Lambda(u) \leq 3c_1' e^{-Bl^\delta/16} + 2\frac{B}{\pi} l^{\alpha(\nu+3)} e^{l_j^\beta} g_L(w) \tag{7.32}$$

for some $w \in \left(\Lambda \cap \Lambda_{\hat{l}_j} \setminus \Lambda_{\tilde{l}_j}\right) \cap \mathbb{Z}^2$. But, if $w \in \Lambda_{\hat{l}_j} \setminus \Lambda_{\tilde{l}_j}$ then $w \notin \Lambda_{\tilde{l}_i}$ for any $i = 1, \ldots, t$. If, therefore, $d(\Lambda_{l_j}, \partial\Lambda_{\tilde{L}}(x)) \geq \hat{l}$ then $\Lambda_l(w) \subset \Lambda_{\tilde{L}}(x)$ and hence $\Lambda_l(w)$ is (m_l, E)-regular. We can then insert (7.27) into (7.32) and conclude that

$$g_\Lambda(u) \leq \left(3c_1' + \frac{8B}{\pi} c_1 l^{\alpha(\nu+3)} e^{l_j^\beta}\right) e^{-Bl^\delta/16} + \frac{2B^2}{\pi^2} l^{2\alpha(\nu+3)} e^{l_j^\beta} e^{-m_l l} g_\Lambda(w') \tag{7.33}$$

for some $w' \in \Lambda \cap \Lambda_{\hat{l}}(w) \cap \mathbb{Z}^2$. We are now going to iterate the inequalities (7.27) and (7.33). We first simplify these inequalities as follows. There exist $Q_3(\alpha, \beta, J, \nu)$ such that if $l > Q_3$ then

$$l^{2\alpha(\nu+3)} e^{-Jl^\beta} < \frac{\pi^2}{4B^2} \wedge \frac{\pi}{2B} \tag{7.34}$$

and there exists $Q_4(\alpha, \beta, s, J, \nu)$ such that if $l > Q_4$ then

$$\left(3J + \frac{8B}{\pi} l^{\alpha(\nu+3)} e^{3Jl^\beta}\right) c_1 < \frac{1}{4} e^{Bl^\delta/32}. \tag{7.35}$$

Using the fact that $m_l > 4Jl^{-1+\beta}$ we now have

$$g_\Lambda(u) \leq \frac{1}{4} e^{-Bl^\delta/32} + Z(u) g_\Lambda(w) \tag{7.36}$$

where

$$Z(u) = \begin{cases} \frac{B}{\pi} l^{\alpha(\nu+3)} e^{-m_l l} & \text{if } \Lambda_l(u) \text{ is } (m_l, E)\text{-regular} \\ \frac{1}{2} & \text{if } \Lambda_l(u) \text{ is } (m_l, E)\text{-singular} \end{cases} \tag{7.37}$$

and where $w \in \Lambda \cap \Lambda_{\hat{l}}(u)$ if $\Lambda_l(u)$ is regular and $w \in \Lambda \cap \Lambda_{\hat{l}}(w')$ for some $w' \in (\Lambda \cap \Lambda_{\hat{l}_j}) \setminus \Lambda_{\tilde{l}_j}$ if $\Lambda_l(u)$ is singular. If we can iterate (7.36) N times, starting at $u = x$ then we get

$$\begin{aligned} g_\Lambda(x) &\leq e^{-Bl^\delta/32} \left\{1 + Z(x) \sum_{n=0}^{N-2} Z(w_1) Z(w_2) \cdots Z(w_n)\right\} \\ &\quad + 2Z(x) Z(w_1) \cdots Z(w_{N-1}) e^{l^{\alpha\beta}} \\ &\leq \frac{1}{2} e^{-Bl^\delta/32} + 2Z(x) Z(w_1) \cdots Z(w_{N-1}) e^{l^{\alpha\beta}}. \end{aligned} \tag{7.38}$$

(In the final step we have used the other bound on g_Λ in (7.27).) If N_1 is the number of n such that $\lambda_l(w_n)$ is regular ($w_0 = x$) then the procedure can be repeated as long as

$$\tilde{L} - \left(N_1 \hat{l} + \sum_{i=1}^t \hat{l}_i + t\hat{l}\right) > \bigvee_{i=1}^t \hat{l}_i + l. \tag{7.39}$$

For this it suffices that $N_1 < (\tilde{L} - (9J+1)l)/\hat{l}$. Let, therefore, N_1 be the integer satisfying

$$\frac{\tilde{L} - (9J+1)l}{\hat{l}} < N_1 \le \frac{\tilde{L} - (9J+1)l}{\hat{l}} \tag{7.40}$$

Then

$$g_\Lambda(x) \le \frac{1}{2} e^{-Bl^\delta/32} + 2(B/\pi)^{N_1} l^{\alpha(\nu+3)N_1} e^{-m_l N_1 l} e^{l^{\alpha\beta}}. \tag{7.41}$$

Define

$$\tilde{m}_L = \frac{N_1}{L}\left(m_l l - \ln\frac{B}{\pi} - \alpha(\nu+3)\ln l\right) - \frac{2}{L}\ln 2 - l^{\alpha(\beta-1)}. \tag{7.42}$$

Inserting the first inequality (7.40) and the relation

$$\frac{\tilde{L}l}{\hat{l}L} = \frac{1 - l^{-\alpha(1-s)}}{1 + l^{(\alpha-2)/4}} > 1 - l^{-\alpha(1-s)} - l^{(\alpha-2)/4} \tag{7.43}$$

it easily follows that

$$\tilde{m}_L > m_l\left(1 - 2l^{-\alpha(1-s)} - l^{-(2-\alpha)/4}\right) - \alpha(\nu+4)l^{-1}\ln l - l^{-\alpha(1-\beta)} \tag{7.44}$$

provided $l > (9J+1)^2 \vee (B/\pi)$. Therefore

$$\begin{aligned} L^{1-\beta}\tilde{m}_L > {} & 4Jl^{(\alpha-1)(1-\beta)}\left(1 - 2l^{-\alpha(1-s)} - l^{-(2-\alpha)/4}\right) \\ & - \alpha(\nu+4)l^{\alpha(1-\beta)-1}\ln l - 1 > 4J \end{aligned} \tag{7.45}$$

if l is large enough: $l > Q_5(\alpha, \beta, s, J, \nu)$. Finally, we define $m_L = \tilde{m}_L \wedge \frac{1}{32}Bl^{\delta-\alpha}$. As $\delta > \alpha$, we then have $g_\Lambda(x) < e^{-m_L L}$ and $m_L > 4JL^{-1+\beta}$ provided $l > Q' = Q_1 \vee Q_2 \vee Q_3 \vee Q_4 \vee Q_5$, which proves the lemma QED

To prove Lemma 7.1 we introduce another definition. Given an interval $I \subset \mathbb{R}$, let

$$\sigma'(H_{\Lambda_r}) = \sigma(H_{\Lambda_r}) \cap \{E \,|\, d(E, I) \le \frac{1}{2}e^{-r^\beta}\}. \tag{7.46}$$

We shall assume that $d(I, 0) > \epsilon_0$ and $e^{-r^\beta} < \epsilon_0$. Then $\sigma'(H_{\Lambda_r}) \subset \mathbb{R} \setminus [-\frac{1}{2}\epsilon_0, \frac{1}{2}\epsilon_0]$. As in [DrKl] we now have

Lemma 7.5 *Suppose that* (K2) *is satisfied and that* $l_1 > l_2 > L_0 \vee (-\frac{1}{\beta}\ln\epsilon_0)$. *Let the squares* Λ_{l_1} *and* Λ_{l_2} *have centres on* $\mathbb{Z}^2$ *and distance* $d(\Lambda_{l_1}, \Lambda_{l_2}) \ge 1$. *Then there exists a constant* $c > 0$ *such that*

$$\mathbb{P}\left[d\left(\sigma'(H_{\Lambda_{l_1}}), \sigma'(H_{\Lambda_{l_2}})\right) < e^{-l_2^\beta}\right] \le c\frac{l_1^2}{l_2^q}. \tag{7.47}$$

Proof. Let $V_1 = V_{\Lambda_{l_1}}$ and $V_2 = V_{\Lambda_{l_2}}$. Since $d(\Lambda_{l_1}, \Lambda_{l_2}) \geq 1$, V_1 and V_2 are independent random variables. We denote the corresponding probability measures by $\mathbb{P}_1$ and $\mathbb{P}_2$. Let $\lambda_1(V_1), \ldots, \lambda_{n(V_1)}(V_1)$ be the eigenvalues of $H_{\Lambda_{l_1}}(V_1)$ in $\sigma'(H_{\Lambda_{l_1}}(V_1))$. Then $d(\lambda_i(V_1), I) \leq \frac{1}{2}e^{-l_1^\beta} \leq \frac{1}{2}e^{-l_2^\beta}$ and hence by (K2),

$$\begin{aligned}
&\mathbb{P}\left[d\left(\sigma'(H_{\Lambda_{l_1}}(V_1)), \sigma'(H_{\Lambda_{l_2}}(V_2))\right) < e^{-l_2^\beta}\right] \\
&= \int \mathbb{P}_1(dV_1)\, \mathbb{P}_2\left[\bigcup_{i=1}^{n(V_1)} \left\{V_2 \,|\, d\left(\lambda_i(V_1), \sigma'(H_{\Lambda_{l_2}}(V_2))\right) < e^{-l_2^\beta}\right\}\right] \\
&= \int \mathbb{P}_1(dV_1) \sum_{i=1}^{n(V_1)} \mathbb{P}_2\left[\left\{V_2 \,|\, d\left(\lambda_i(V_1), \sigma'(H_{\Lambda_{l_2}}(V_2))\right) < e^{-l_2^\beta}\right\}\right] \\
&\leq l_2^{-q}\mathbb{E}_1\left(n(V_1)\right) \qquad (7.48)\\
&\leq l_2^{-q}\mathbb{E}\left(N^>_{\Lambda_{l_1}}(V, \epsilon_0/2) + N^<_{\Lambda_{l_1}}(V, -\epsilon_0/2)\right) \\
&\leq 2l_2^{-q}\mathbb{E}\left(N^>_{\Lambda_{l_1}}(|V|, \epsilon_0/2)\right) \\
&\leq 4l_2^{-q}\epsilon_0^{-1}\mathbb{E}\left(\mathrm{Trace}\, H_{\Lambda_{l_1}}(|V_1|)\right) \\
&\leq \frac{4|\Lambda_{l_1}|}{l_2^q\epsilon_0}\mathbb{E}(|v_x|) \sup_{z\in\Lambda_{l_1}} \int_{\Lambda_{l_1}} d^2z' K_B(z,z')^2 \leq \frac{4B|\Lambda_{l_1}|}{\pi\epsilon_0}\mathbb{E}(|v_x|)l_2^{-q}.
\end{aligned}$$

This proves the lemma. QED

Proof of Lemma 7.1

The proof now proceeds exactly as in [DrKl]. Taking $Q \geq Q'$ of Lemma 7.2, it suffices to show that if Q is large enough and $l > Q$ then, for any $x, y \in \mathbb{Z}^2$ with $|x-y|_\infty > L+1$,

$$\mathbb{P}\left[\forall E \in I : \text{(i), (ii) and (iii) of Lemma 7.4 hold either for } x \text{ or for } y\right] > 1 - L^{-2p}. \quad (7.49)$$

Indeed, one proves as in Lemma 2.1 that, for given $x \in \mathbb{Z}^2$, $|V(z)| \leq L^2$ for all $z \in \Lambda_L(x)$ provided L is large enough (independently of x).

Fix, therefore, $x, y \in \mathbb{Z}^2$ with $|x-y|_\infty > L+1$. We may assume that

$$d\left(\sigma'\left(H_{\Lambda_{l_1}(x')}, \sigma'(H_{\Lambda_{l_2}(y')})\right)\right) > e^{-(l_1\wedge l_2)^\beta} \quad (7.50)$$

for all $x' \in \Lambda_L(x) \cap \mathbb{Z}^2$ and $y' \in \Lambda_L(y) \cap \mathbb{Z}^2$ and all $l_1, l_2 \in \mathcal{J}_l \cup \{L\}$ with $\Lambda_{l_1}(x') \subset \Lambda_L(x)$ and $\Lambda_{l_2}(y') \subset \Lambda_L(y)$. Indeed, by Lemma 7.5, the probability that this is not the case is less than $L^4(J+2)^2cL^2l^{-q} = c'L^6l^{-q}$. It is easy to see that if (7.50) holds for all x', y', l_1 and l_2 as above then, for all $E \in I$, (i) and (ii) of Lemma 7.4 hold either for x or for y. Hence,

$$\mathbb{P}\left[\forall E \in I : \text{(i) and (ii) hold either for } x \text{ or for } y\right] \geq 1 - c'L^6l^{-q}. \quad (7.51)$$

As to condition (iii) we have, moreover,

$$\begin{aligned}
&\mathbb{P}\big[\exists E \in I : \text{there are at least } J+1 (m_l,E)\text{-singular squares } \Lambda_l(u) \text{ with}\\
&\quad \text{distance } \geq 1 \text{ contained in } \Lambda_L(x)\big]\\
&\quad \leq \mathbb{P}\big[\exists E \in I : \text{there are at least two } (m_l,E)\text{-singular squares } \Lambda_l(u), \Lambda_l(v) \subset \\
&\qquad \Lambda_L(x) \text{ with } |u-v|_\infty \geq 1\big]^{(J+1)/2}\\
&\quad \leq (L^2 l^{-p})^{J+1}
\end{aligned} \tag{7.52}$$

and similarly for the square $\Lambda_L(y)$. It follows that (7.49) holds since $c' L^6 l^{-q} + 2L^{2(J+1)} l^{-p(J+1)} < L^{-2p}$ for l large enough. (Remember that $q > 4p + 12 > 2\alpha p + 6\alpha$ and $\alpha < \alpha_0 < 2$.) QED

Proof of Theorem 4.3
Let α_0 and J_p be as in Lemma 7.1. Starting from $l = L_0$ and $m_l = m_0$ we can iterate Lemma 7.1 to find that $\mathrm{R}(L_k, m_k)$ holds for $L_{k+1} = L_k^\alpha$ and a sequence $\{m_k\}$ satisfying

$$m_{k+1} \geq m_k - \left[\left(2L_k^{-(2-\alpha)/4} + L_k^{-\alpha(1-s)}\right) m_k + \alpha(\nu+4) L_k^{-1} \ln L_k + L_k^{-\alpha(1-\beta)}\right]. \tag{7.53}$$

Given $m < m_0$ it remains to show that $\sum_{k=1}^{\infty}(m_k - m_{k+1}) \leq m_0 - m$ for L_0 large enough. But this follows immediately from the inequality (7.53). QED

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