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On the Difference Between Conformal and Minimal Couplings in General Relativity

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Abstract. Various aspects of conformal and minimal couplings in general relativity are studied in detail. It is shown that the free minimally coupled scalar field has a pathological and unexpected quasiclassical behavior. The equivalence principle of general relativity rejects the free minimally coupled scalar field as is shown by studying the behavior of the Feynman propagator in curved space-time. The self-interaction can "cure" the minimally coupled scalar field from pathological behavior via the mechanism of spontaneous symmetry breaking. Some physical consequences of obtained results are discussed in brief.

1 Introduction

Modern relativistic field theory deals with fields of different nature, the simplest of which is a real scalar field φ describing neutral zero spin particles of the mass m. The action and the corresponding Euler-Lagrange equation for such a field in Minkowski space-time, respectively, have the form: [1]

$$S = \frac{1}{2} \int \left(\eta^{\mu\nu} \partial_\mu \varphi(x) \partial_\nu \varphi(x) - m^2 \varphi^2(x) \right) d^4x \quad ; \tag{1.1}$$

[1] We use units $\hbar = c = 1$.

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\varphi(x) = 0 \tag{1.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the metric tensor in Minkowski space-time.

Everyone studying general relativity encounters an ambiguity when trying to consider the action for the scalar field in curved space-time. From one hand, following the standard prescription of textbooks (see,e.g., [1]), when going from Minkowski space-time to curved Riemannian with the metric $g_{\mu\nu}$ one should only replace ordinary derivatives ∂_μ by covariant derivatives ∇_μ and the volume d^4x by the covariant volume $\sqrt{-g}d^4x$ (this is a so-called rule "a comma transforms to the semicolon" [1]):

$$\partial_\mu \longrightarrow \nabla_\mu \quad ; \tag{1.3}$$

$$d^4x \longrightarrow \sqrt{-g}d^4x. \tag{1.4}$$

Then in curved space-time (1.1) and (1.2) respectively have the form:

$$S_{\min} = \frac{1}{2}\int\left(g^{\mu\nu}\nabla_\mu\varphi(x)\nabla_\nu\varphi(x) - m^2\varphi^2(x)\right)\sqrt{-g}d^4x \quad ; \tag{1.5}$$

$$\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2\right)\varphi(x) = 0. \tag{1.6}$$

The interaction of scalar field φ with gravity deduced in this way is called the *minimal coupling* for the reason that the field φ "feels" gravity through covariant derivatives only. We have provided the corresponding action with the subscript "min".

From the other hand, one may observe that Eq.(1.6) in the massless case $m = 0$ is not invariant under conformal scale transformations of the metric: $g_{\mu\nu} \to \exp[2\lambda(x)]g_{\mu\nu}$ ($\lambda(x)$ is an arbitrary smooth function of spacetime point) provided that the field φ transforms as $\varphi \to \exp[-\lambda(x)]$. The physical meaning of the conformal invariance is that the massless field does not possess an internal scale of length (for a massive field the scale is its Compton length $\lambda_C = m^{-1}$, provided that the gravitational coupling constant is fixed). This fact indicates that for compatibility with the massless case Eq.(1.6) should be improved.

To account the conformal invariance in the massless case one must take the action

$$S_{\mathrm{conf}} = \frac{1}{2}\int\left(g^{\mu\nu}\nabla_\mu\varphi(x)\nabla_\nu\varphi(x) - m^2\varphi^2(x) - \frac{1}{6}R(x)\varphi^2(x)\right)\sqrt{-g}d^4x \tag{1.7}$$

where $R(x)$ is the scalar curvature of space-time. This action gives the following equation of motion for the field $\varphi(x)$:

$$\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2 + \frac{1}{6}R\right)\varphi(x) = 0. \tag{1.8}$$

The interaction of a scalar field with gravity obtained in this way is called the *conformal coupling*. We have provided the corresponding action with the subscript "conf".

Usually both cases (1.5),(1.6) and (1.7),(1.8) are written in a uniform way:

$$S = \frac{1}{2}\int \left[g^{\mu\nu}\nabla_\mu\varphi(x)\nabla_\nu\varphi(x) - (m^2 + \xi R)\varphi^2(x)\right]\sqrt{-g}d^4x; \tag{1.9}$$

$$\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2 + \xi R\right)\varphi(x) = 0. \tag{1.10}$$

where ξ is a numerical constant taking the value $\xi = 0$ for the minimal coupling and $\xi = 1/6$ for the conformal. But the representation (1.9),(1.10) means more than the above two cases because, from the mathematical point of view, ξ may take any real value. Thus, instead of the above two types of couplings, we are led to a one parameter family of interactions of scalar field with gravity. However, as we shall show below, this is not so from the physical point of view because Eq.(1.10) for different ξ leads to drastically different physical consequenses. What value of ξ is correct? Does general relativity prescribe a definite value to ξ, or may one choose it according to taste? This problem has a long history and, in our opinion, has a definite solution: the correct value prescribed by general relativity is $\xi = 1/6$. This result has been established in the paper of Chernikov and Tagirov [2]. Unfortunately this fact is not widely known to physical community and there is a misunderstanding in modern physical literature according to which the choice of the value of ξ is a matter of taste. Moreover in some textbooks (see for example [3]) one can find a claim that the conformal coupling ($\xi = 1/6$) is wrong and the correct one is the minimal coupling ($\xi = 0$). Recently an investigation in this direction was undertaken by Sonego and Faraoni [4]. They analyzed the behavior of the retarded Green's function of scalar field in curved background and have shown that the correct behaviour in the limit of coinciding points can be obtained for the conformal coupling ($\xi = 1/6$) only.

The aim of this paper is to show that the rule "a comma transforms to the semicolon" when one is going from special to general relativity is not so simple, and it may easily lead one to wrong results without careful accounting of basic principles of general relativity. We also shall try to prove more or less rigorously that basic principles of field theory and general relativity inevitably lead to the unique value of ξ, namely $\xi = 1/6$. All the other values of ξ for the free scalar field turn out to be unphysical. However, as we show in the paper, the situation can be improved by taking into consideration its self-interaction. We hope that our consideration will help to remove the above misunderstanding among physicists.

The paper is organized as follows. In sec.2 we analyze in detail the problem of "anomalous R-forces" and prove that no such forces arise for the conformally coupled scalar field. We demonstrate the drastical difference in quasiclassical behavior between conformal and minimal couplings and show that the minimal coupling leads to the tachyonic behavior whereas the conformal coupling have the correct quasiclassical limit. In sec.3 by calculating exactly the difference between the Feynman propagator of scalar field in Minkowski space-time and the corresponding one in Riemannian space-time in Riemannian normal coordinates we show that it tends to zero only in the case of conformal coupling ($\xi = 1/6$). In the case of non-conformal couplung $\xi \neq 1/6$ there exists a finite remainder contradicting physical reality. In sec.4 we study the effect of spontaneous symmetry breaking for the minimally coupled scalar field and show that via this effect the minimally coupled scalar field can be cured from tachyonic behaviour. Sec.5 contains the discussion of some crucial consequences of the obtained results.

2 The Problem of Anomalous R-forces in Minimal and Conformal Couplings

In the literature (see, e.g.,[3]) one can find claims that Eq. (1.8) violates the strong equivalence principle and leads to the appearance of anomalous R-term forces between two "scalar charged" particles. In this section we show that the situation is quite opposite and such claims are caused by incorrect applications of basic principles of general relativity. Let us clarify our point of view by analyzing in detail the problem suggested in [3] indicating exactly where the authors of [3] are wrong.

Consider the conformally coupled massless scalar field in curved space-time with a point-like source. The equation of motion for such a field is

$$\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{6}R\right)\varphi = \mu_1\delta(x,x'). \tag{2.1}$$

To extract anomalous R-forces from this equation the authors of [3] suggest the following argumentation. Let us write Eq.(2.1) in a locally Lorentz coordinate system where the point-like source is at rest in the form

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{6}R\right)\varphi = \mu_1\delta(\boldsymbol{r}). \tag{2.2}$$

Let us comment that in writing down Eq.(2.2) the following replacements were done:

$$\sqrt{-g}\to 1;\quad g^{\mu\nu}\to\eta^{\mu\nu};\quad \nabla_\mu\to\partial_\mu;\quad \delta(x,x')\to\delta(\boldsymbol{r});\quad R\to R. \tag{2.3}$$

The correctness of these replacements is discussed at the end of this section.

Using (2.3) one immediately obtains a Yukawa's potential solution to Eq.(2.2):

$$\varphi = -\frac{\mu_1}{r}\exp\left(-\frac{r}{a\sqrt{6}}\right), \tag{2.4}$$

where $a = R^{-1/2}$ which leads to the anomalous R-force.

It is easy to see, however, that, if we find the exact solution of Eq. (2.1), we do not obtain the Yukawa solution. Let us demonstrate this in the most simple case of conformally flat Friedmann quasi-Euclidean space-time with the metric:

$$\begin{aligned} ds^2 &= a^2(\eta)(d\eta^2 - d\boldsymbol{l}^2);\\ d\boldsymbol{l}^2 &= dx^2+dy^2+dz^2, \end{aligned} \tag{2.5}$$

where η is the conformal time. Eq. (2.1) for the frame where a point-like source is at rest has the form:

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{6}R\right)\varphi = \mu_1\frac{\delta(\boldsymbol{r})}{\sqrt{g^{(3)}}}. \tag{2.6}$$

Here $\delta(\boldsymbol{r})$ is the ordinary δ-function but we must have $\sqrt{g^{(3)}}$ where $g^{(3)}$ is the determinant of the 3-metric, i.e., $\sqrt{g^{(3)}} = a^3$ in the metric (2.5). Eq.(2.6) in the metric (2.5) takes the form

$$\frac{1}{a^2}\frac{\partial^2\varphi}{\partial\eta^2} + \frac{2}{a^3}a'\frac{\partial\varphi}{\partial\eta} - \frac{1}{a^2}\Delta\varphi + \frac{R}{6}\varphi = \mu_1\frac{\delta(\boldsymbol{r})}{a^3}. \tag{2.7}$$

where the prime denotes differentiation in η.

Let us look for a solution of (2.7) in the form:

$$\varphi \longrightarrow \tilde{\varphi}; \qquad \varphi = a^{-1}\tilde{\varphi}. \tag{2.8}$$

Substituting this into (2.7), ater some calculations, we get the following equation:

$$\frac{1}{a^3}[\tilde{\varphi}'' - \Delta\tilde{\varphi}] = \frac{1}{a^3}\mu_1\delta(\boldsymbol{r}), \tag{2.9}$$

and the static solution to this equation is the usual Coulomb potential but with the conformal factor

$$\tilde{\varphi} = -\frac{\mu_1}{r} \qquad \text{and} \qquad \varphi = -\frac{\mu_1}{ar}. \tag{2.10}$$

On the contrary, for the minimal coupling instead of Eq. (2.9) we have

$$\left(\tilde{\varphi}'' - \frac{a''}{a}\tilde{\varphi} - \Delta\tilde{\varphi}\right) = \mu_1\delta(\boldsymbol{r}). \tag{2.11}$$

The situation is the same as if in flat space-time we had a mass term due to $\frac{a''}{a} \neq 0$. This is the reason why, contrary to [3], we have no usual massless behavior.

For a dust-like universe we have $R \neq 0$ and $a(\eta) = a_0\eta^2$, so that the term $\frac{a''}{a} > 0$, and for $\eta \approx$ const it has properties of $m^2a^2 < 0$, i.e., of "tachyonic mass".

Now let us return to the question: what is the error when the authors of [3] naively write Eq.(2.1) in a locally Lorentz coordinate system in the form (2.2) by making the replacements (2.3)? The answer to this question has a great methodological meaning.

The operator $g^{\mu\nu}\nabla_\mu\nabla_\nu$ in (2.1) acting on the scalar field has one Christoffel symbol only, so it seems that "at the point" one can write $\eta^{\mu\nu}\partial_\mu\partial_\nu$ as in Minkowsky space-time. But in order to find a solution to the differential equation at a given point of the manifold one must study properties of this solution in the neigborhood of the point taking into account the boundary conditions properly. If $\{\xi^\mu\}$ are locally Lorentz coordinates and $\{x^\mu\}$ are arbitrary coordinates, and $\{X^\mu\}$ are the coordinates of P_0 (the origin of the locally Lorentz frame), then

$$\xi^\alpha(x) = a^\alpha + b^a_\mu(x^\mu - X^\mu) + \frac{1}{2}b^\alpha_\lambda\Gamma^\lambda_{\mu\nu}(X)(x^\mu - X^\mu)(x^\nu - X^\nu), \tag{2.12}$$

with $a^\alpha = \xi^\alpha(x)\Big|_{x=X}$; $\quad b^\alpha_\lambda = \frac{\partial\xi^\alpha}{\partial x^\lambda}\Big|_{x=X}$.

Also

$$\Gamma^\alpha_{\beta\gamma,\mu}(\xi^\mu)\Big|_{\xi^\mu_0} = -\frac{1}{3}\left(R^\alpha_{\beta\gamma\mu}(\xi^\mu) + R^\alpha_{\gamma\beta\mu}(\xi^\mu)\right)\Big|_{\xi^\mu_0}.$$

Then, for $\xi^\mu \neq \xi^\mu(P_0) = \xi^\mu_0$, quantities $\Gamma^\alpha_{\beta\mu}(\xi^\mu)$ are not zero and the simple replacements (2.3) are correct at the single point only, namely at the origin of the frame. This means that, in order to obtain a solution to Eq.(2.1) with a source term in the $\{\xi^\mu\}$ coordinates valid for a neighborhood of the origin of the frame and taking into account the boundary conditions properly one must solve instead of Eq. (2.2) a very complicated equation in these coordinates. The problem is not so simple in general but if this is done carefully (as we do it in the next section) one must obtain the correct result founded in this section for the case of conformally flat manifold.

Now let us discuss for this case the quasiclassical behavior when φ has the form

$$\varphi = \frac{1}{a}\rho\exp\left(i\frac{S}{\hbar}\right). \tag{2.13}$$

Then for the conformal coupling we find the following equations valid up to the order $\hbar^{-2}$

$$\frac{\partial S}{\partial\eta}\frac{\partial S}{\partial\eta} - \sum_{i=1}^{3}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_i} = 0 \qquad \text{for} \qquad m = 0; \tag{2.14}$$

$$\frac{1}{a^2}\left\{\frac{\partial S}{\partial\eta}\frac{\partial S}{\partial\eta} - \sum_{i=1}^{3}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_i}\right\} = m^2 \qquad \text{for} \qquad m \neq 0, \tag{2.15}$$

which is just

$$g^{00}\frac{\partial S}{\partial\eta}\frac{\partial S}{\partial\eta} - g^{ij}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_j} = m^2.$$

For the minimal coupling we obtain

$$\frac{1}{a^2}\left\{\frac{\partial S}{\partial\eta}\frac{\partial S}{\partial\eta} - \sum_{i=1}^{3}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_i}\right\} = -\frac{a''}{a^3} = -\frac{R}{6}. \tag{2.16}$$

So, it is easy to see that geodesics for these particles cannot live on light cones. Even worse, they can be spacelike.

Before ending this section we find instructive to discuss how to solve Eq.(1.8) for $m = 0$ in the coordinate system used by astronomers. To do this, besides the sinchronous reference frame

$$ds^2 = c^2dt^2 - a^2(t)d\boldsymbol{l}^2 = c^2dt^2 - a^2(t)[dr^2 + r^2(\sin^2\theta' d\varphi^2 + d\theta'^2)] \quad (2.17)$$

connected with the conformal one by $cdt = ad\eta$, we introduce another coordinate system where the space distance is given by $D = a(t)r$, so, that

$$dD = adr + rda \quad (2.18)$$

and

$$\begin{aligned} ds^2 &= (1 - D^2\frac{\dot{a}^2}{c^2a^2})c^2dt^2 + 2D\frac{\dot{a}}{ca}dDcdt - \\ & dD^2 - D^2(\sin^2\theta d\varphi^2 + d\theta^2). \end{aligned} \quad (2.19)$$

The advantage of the reference frame associated to these coordinates is that for the observer on the Earth, when $\varepsilon = D\frac{\dot{a}}{ca}$ is small, one has a good approximation to the Minkowski metric, and only for large enough D one has curved spacetime. Really, we can use the parameter ε as a small parameter. Let us write equation (1.8) in these coordinates. From:

$$\begin{cases} g_{00} = 1 - \left(\frac{D\dot{a}}{ca}\right)^2 \\ g_{01} = 2D\left(\frac{\dot{a}}{ca}\right); g_{ii} = -1; g_{\theta\theta} = -D^2, \\ g_{\varphi\varphi} = -D^2\sin^2\theta' \end{cases} \quad (2.20)$$

and using the relations

$$\begin{cases} g^{0r}g_{r0} + g^{00}g_{00} = 1 \\ g^{r0}g_{00} + g^{rr}g_{r0} = 0 \\ 2g^{r0}g_{0r} + g^{rr}g_{rr} = 1 \end{cases} , \quad (2.21)$$

we obtain

$$g^{0r} = \frac{2D\frac{\dot{a}}{ca}}{1 + 7\left(D\frac{\dot{a}}{ca}\right)^2}. \quad (2.22)$$

Then, in the first approximation in ε we have:

$$g^{0r} = 2D\frac{\dot{a}}{ca} = g_{0r} \quad (2.23)$$

Putting this into eq (1.8) for $m = 0$, noting that terms depending on ε^2 after differentiation still will contain ε, we can put them away when $\varepsilon \to 0$. The only term which cannot be put away is the one containing derivatives in D of g^{0r}. So, taking for $\sqrt{-g}$ the Minkowski value, we obtain the equation:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right]\varphi + 2\frac{\dot{a}}{ca}\frac{\partial}{\partial t}\varphi + \frac{R}{6}\varphi = 0 \tag{2.24}$$

So, the nondiagonal components g_{0r} lead to an extra term in the approximation $\varepsilon \to 0$.

But in order to solve this equation we must take

$$\varphi = \frac{1}{a}\widetilde{\varphi} \tag{2.25}$$

which, as in the previous case of conformal time, immediately leads to the cancellation of the $\frac{R}{6}$ term, so that we end with:

$$\frac{1}{c^2 a}\ddot{\widetilde{\varphi}} - \frac{1}{a}\Delta\widetilde{\varphi} = 0. \tag{2.26}$$

Can we put away the term $2\frac{\dot{a}}{c^2 a}\frac{\partial}{\partial t}\varphi$ but retain the $\frac{R}{6}$ term? From Einstein's equations, we can see that for usual cosmological models we must put away the term $\frac{R}{6}\varphi$ if we take $\frac{\dot{a}}{a}$, the Hubble's constant, equal to zero. At the modern epoch of evolution of the universe we can do this and use the Minkowski metric as a very good approximation near the Earth.

But for R large enough in the early epochs one cannot do this. But, then, the nondiagonal terms $g^{0r} \neq 0$ lead to the impossibility of having a unique time and to define space distance unambiguously. It follows that we cannot write the usual $\delta(\boldsymbol{r})$-function for the charge distribution and eq (2.2) does not have unambiguous sense.

At last to finish the story about "Yukawa forces" for the conformal massless case we consider the de Sitter universe. As is well known, the de Sitter metric can be written, depending on coordinates used, in nonstationary or stationary forms. In "curvature" coordinates the interval can be written as

$$\begin{aligned} ds^2 &= \left(1 - \frac{r^2}{a_0^2}\right)dt^2 - \left(1 - \frac{r^2}{a_0^2}\right)^{-1} dr^2 - r^2(\sin^2\theta d\varphi^2 + d\theta^2) = \\ & \left(1 - \frac{r^2}{a_0^2}\right)dt^2 - \left(1 - \frac{r^2}{a_0^2}\right)^{-1} dr^2 - r^2 d\sigma^2 \end{aligned} \tag{2.27}$$

In "orispheric" coordinates it can be written as (2.5) with $a(\eta) = \frac{a_0}{\eta}$.

In the synchronous reference frame one has due to $a(\eta)d\eta = d\tau$ that $a_0 \ell n\eta = \tau$ leading to $a(\tau) = \exp -\frac{1}{a_0}\tau$, and for $k = 0$

$$ds^2 = c^2 d\tau^2 - \mathrm{e}^{-\frac{2\tau}{a_0}}(d\chi^2 + \chi^2(\sin^2\theta d\varphi^2 + d\theta^2)) \tag{2.28}$$

Here we use notations τ, χ instead of t, r used for the stationary case.

The connection between coordinates τ, χ, t, r is given by the following formula. First from t, r one goes to

$$\tau = t + \int \frac{f(r)dr}{1 - \frac{r^2}{a_0^2}}$$
$$R = t + \int \frac{dr}{(1 - \frac{r^2}{a_0^2})f(r)} \tag{2.29}$$

and $f(r) = \dfrac{r}{a_0}$. Then

$$R - \tau = a_0 \ell n r \tag{2.30}$$

and

$$r = \exp\left(\frac{R - \tau}{a_0}\right) \tag{2.31}$$

The interval (2.27) is written as

$$ds^2 = d\tau^2 - \frac{r^2}{a_0^2}dR^2 - r^2 d\sigma^2 = d\tau^2 - e^{-\frac{2\tau}{a_0}}(d\chi^2 + \chi^2 d\sigma^2) \tag{2.32}$$

if $\chi \equiv \exp\left(\dfrac{R}{a_0}\right)$.

But then it is easy to write our general solution of the Klein-Gordon equation for the massless conformal coupling case as:

$$\varphi = -\frac{1}{a(\tau)}\frac{\mu_1}{\chi} = \mu_1 \exp\left(\frac{1}{a_0}\tau\right)\exp\left(-\frac{R}{a_0}\right) = -\mu_1 \exp\left(-\frac{R - \tau}{a_0}\right) = -\frac{\mu_1}{r}, \tag{2.33}$$

so it is just the usual Coulomb potential and we consider this to be the end of "Yukawa-like" forces for the massless conformal coupling case!

3 Feynman Propagator in Curved Space-Time and the Equivalence Principle

In flat Minkowski space-time the Feynman propagator for the equation

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\varphi(x) = \delta(x, x') \tag{3.1}$$

has the form [5]: [2]

$$G_F^M(x, x') = \frac{1}{(4\pi)^2}\int_0^\infty \frac{1}{s^2}\exp\left\{-i\left[m^2 s + \frac{\sigma^M(x, x')}{2s}\right]\right\} ds \tag{3.2}$$

[2]We have provided all important quantities referred to Minkowski space-time with the superscript M.

where $\sigma^M(x,x')$ is the half of the square of the interval between the points x and x':

$$\sigma^M(x,x') = \frac{1}{2}\eta_{\mu\nu}(x^\mu - x^{\mu'})(x^\nu - x^{\nu'}) \tag{3.3}$$

$G_F^M(x,x')$ is a complex function and can be decomposed in real and imaginary parts:

$$G_F^M(x,x') = \bar{G}^M(x,x') + \frac{i}{2}G^{(1)M}(x,x') \tag{3.4}$$

where $G^{(1)M}(x,x')$ is the Hadamard function in Minkowski space-time.

Now let us turn to the curved space-time. Here instead of (3.1) one should solve the following equation:

$$\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2 + \xi R(x)\right)\varphi(x) = \delta(x,x'). \tag{3.5}$$

This problem is not so simple in arbitrary curved space-time, so that the solution in general can be found in the WKB approximation [5, 6, 7]. The corresponding Feynman propagator $G_F(x,x')$ in curved space-time has the form [7]:

$$G_F(x,x') = \frac{\Delta^{1/2}}{(4\pi)^2}\int_0^\infty \frac{ds}{s^2}\exp\left\{-i\left[m^2 s + \frac{\sigma(x,x')}{2s}\right]\right\}\sum_{n=0}^\infty a_n(x,x')(is)^n \tag{3.6}$$

where $\sigma(x,x')$ is the so-called geodesic interval equal to the half of the square of the geodesic connecting the points x and x' in curved space-time:

$$\sigma(x,x') = l^2/2; \quad l = \int_0^\lambda d\lambda'\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda'}\frac{dx^\nu}{d\lambda'}\right)^{1/2}, \tag{3.7}$$

$\Delta^{1/2}(x,x') = -\det\left(\frac{\partial^2\sigma(x,x')}{\partial x^\mu \partial x^{\nu'}}\right)[g(x)g(x')]^{-1/2}$, and $a_n(x,x')$ are the coefficients of the WKB decomposition satisfying recurrent equations. [3]

As in the flat space $G_F(x,x')$ can be split in real and imaginary parts

$$G_F(x,x') = \bar{G}(x,x') + \frac{i}{2}G^{(1)}(x,x'). \tag{3.8}$$

Now let us apply the equivalence principle to Eq. (3.6) which states that in a local inertial frame of reference all the gravitational effects disappear. This means that in such a frame there must be no difference between (3.2) and (3.6).

Let us introduce in the neighborhood of the point x' a system of Riemannian normal coordinates y^μ for the point x with the origin at x'. In such coordinates the decomposition

[3] For simplicity we do not write down these equations. Details can be found in [5, 6, 7].

of the metric tensor $g_{\mu\nu}$ has the form [6]:

$$\begin{aligned} g_{\mu\nu}(x) &= \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta - \frac{1}{6}(\nabla_\gamma R_{\mu\alpha\nu\beta})y^\alpha y^\beta y^\gamma \\ &+ \left[\frac{1}{20}\nabla_\gamma\nabla_\delta R_{\mu\alpha\nu\beta} + \frac{2}{45}R_{\alpha\mu\beta\lambda}R^\lambda_{\gamma\nu\beta}\right] y^\alpha y^\beta y^\gamma y^\delta + \cdots \end{aligned} \tag{3.9}$$

where $\eta_{\mu\nu}$ is the metric tensor of Minkowski space, and all the coefficients are calculated in the origin of the frame ($y = 0$).

To what extent can one use these coordinates as locally Minkowskian? Let in the neighborhood of x' one has

$$R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} \sim \rho^{-4} \tag{3.10}$$

where ρ is a characteristic curvature radius. Then, as is seen from (3.9), our frame will be locally Lorenzian up to the distances of the order of ρ from the point x'. All the gravitational effects in such a frame will be contained in terms of the second order and higher.

Let us use the DeWitt decomposition of the Feynman propagator in curved space-time [5] taken in the Riemannian normal coordinates (3.9) and compare it with the analogous decomposition in Minkowski space. Then, for the difference $G_{\mathrm{F}}(x,x') - G^M_{\mathrm{F}}(x,x')$ we have:

$$\begin{aligned} \bar{G}(x,x') - \bar{G}^M(x,x') &= -\frac{1}{8\pi}\theta\left(-\sigma^M(x,x')\right)\left\{\frac{1}{2}\left[\left(\frac{1}{6}-\xi\right)R + \frac{1}{2}\left(\frac{1}{6}-\xi\right)R_{;\alpha}y^\alpha\right]\right\} \\ &+ O_1(y^2/\rho^2)\cdot\delta\left(\sigma^M(x,x')\right) - O_2(y^2/\rho^2)\cdot\theta\left(-\sigma^M(x,x')\right) \\ &+ O_3(y^2/\rho^2); \end{aligned} \tag{3.11}$$

$$\begin{aligned} G^{(1)}(x,x') - G^{(1)M}(x,x') &= -\frac{1}{2\pi^2}\left(\gamma - \frac{1}{2}\ln 2 + \frac{1}{2}\ln|2m^2\sigma^M(x,x')|\right) \\ &\times \left\{\frac{1}{2}\left[\left(\frac{1}{6}-\xi\right)R - \frac{1}{2}\left(\frac{1}{6}-\xi\right)R_{;\alpha}y^\alpha\right]\right\} \\ &+ \frac{1}{2\pi^2}\left\{\left[\frac{a_2(x,x')}{4m^2} + \frac{a_3(x,x')}{4m^4} + \frac{a_4(x,x')}{2m^6} + \ldots\right] + \ldots\right\} \\ &+ O_4(y^2/\rho^2)\cdot\frac{1}{4\pi\sigma^M(x,x')} + O_5(y^2/\rho^2)\cdot\frac{1}{2\pi^2} \\ &\times \left(\gamma - \frac{1}{2}\ln 2 + \frac{1}{2}\ln|2m^2\sigma^M(x,x')|\right) + O_6(y^2/\rho^2) \end{aligned} \tag{3.12}$$

where terms $O_i(y^2/\rho^2)$, $i = 1,2,3,4,5,6$ contain corrections of the order of y^2/ρ^2 and higher. We see that for $\xi \neq \frac{1}{6}$ these differences have non-gravitational origin because all gravitationally induced corrections are gathered in terms $O_i(y^2/\rho^2)$. For $\xi = 1/6$ we have no difference between $G_{\mathrm{F}}(x,x')$ and $G^M_{\mathrm{F}}(x,x')$ up to the order of y^2/ρ^2.

The second term in (3.12) has also a pure gravitational origin. Really, introducing the Compton length of a particle, $\lambda_{\mathrm{C}} = m^{-1}$, one can represent this term in the form

$$\frac{1}{2\pi^2}\left\{m^2\left[\lambda_{\mathrm{C}}^4 a_2(x,x') + \lambda_{\mathrm{C}}^6 a_3(x,x') + \lambda_{\mathrm{C}}^8 a_4(x,x')\right] + \ldots\right\}.$$

Of the order of magnitude the products $\lambda_C^{2n} a_n(x,x')$ are

$$\lambda_C^{2n} a_n(x,x') \sim (\lambda_C/\rho)^{2n},$$

so that this term is responsible for particle creation by gravitational field.

Therefore we see that for $\xi = \frac{1}{6}$ the difference $G_F(x,x') - G_F^M(x,x')$ in Riemannian normal coordinates is completely induced by gravitational field that is in accordance with the equivalence principle, whereas for $\xi \neq \frac{1}{6}$ it contains terms violating the equivalence principle. From this we conclude that general relativity *does prescribe* the definite value $\xi = \frac{1}{6}$ for interaction of gravity with scalar field. The "minimal coupling" for the free scalar field turns out to be unphysical. In the next section we discuss how this situation can be improved.

4 Spontaneous Symmetry Breaking of the Minimally Coupled Scalar Field in Curved Space-Time

Now let us investigate what happens if we shall try to account a self-interaction of minimally coupled scalar field. For simplicity consider the massless field with the action

$$S = \int \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi - \frac{\lambda}{3!} (\varphi^* \varphi)^2 \right] d^4x. \tag{4.1}$$

The corresponding Euler-Lagrange equation is

$$\nabla_\rho \nabla^\rho \varphi + \frac{\lambda}{3} \varphi^* \varphi^2 = 0 \tag{4.2}$$

Consider the most simplest homogeneous Friedmann metric

$$ds^2 = a^2(\eta)(d\eta^2 - d\boldsymbol{l}^2) \tag{4.3}$$

where η is the "conformal time": $d\eta = a(t)dt$, and the spatial part $d\boldsymbol{l}^2$ of the metric has the form $(\boldsymbol{r}, \theta, \varphi$ are spherical coordinates):

$$d\boldsymbol{l}^2 = \frac{d\boldsymbol{r}^2}{1 - k\boldsymbol{r}^2} - \boldsymbol{r}^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \tag{4.4}$$

where $k = +1, 0, -1$ for closed, Euclidean, and open space respectively.

Let us denote by $|0>$ the Heisenberg vacuum state defined at $t = 0$. From the spatial homogeneity of the metric it follows that, if the vacuum expectation value $<0|\varphi|0>$ is nonzero, then it can depend on t only (hereafter we shall use the conformal time η):

$$<0|\varphi(\eta, \boldsymbol{x})|0> = <0|\varphi(\eta, 0)|0> \equiv g(\eta) \tag{4.5}$$

Its nonzero value means spontaneous symmetry breaking. Let us show that such a situation really takes place.

Averaging Eq.(4.2) in the state $|0>$ and setting (in the tree approximation)

$$\left\langle 0|\varphi^*\varphi^2|0\right\rangle \approx \langle 0|\varphi^*|0\rangle \langle 0|\varphi|0\rangle^2 = g^3, \tag{4.6}$$

we can write it in the form

$$g'' + \frac{2a'}{a}g' + \frac{\lambda a^2}{3}g^3 = 0. \tag{4.7}$$

Making in this equation the replacement

$$g(\eta) = \sqrt{\frac{3}{\lambda}\frac{f(\eta)}{a(\eta)}} \quad , \tag{4.8}$$

we obtain for $f(\eta)$ the equation of Duffing's type:

$$f'' - \frac{a''}{a}f + f^3 = 0. \tag{4.9}$$

If $\frac{a''}{a} > 0$ then this term plays the role of negative mass. It is well known that for $\frac{a''}{a} > 0$ the trivial solution $f = 0$ to equations of such type is unstable. To find exact solutions to Eq. (4.9) is not a simple problem. Nevertheless it is possible to investigate the behavior of solutions to this equation in physically relevant situations.

For this goal consider the effective potential $V(f)$ corresponding to (4.9)

$$V(f) = \frac{1}{2}f^2\left(\frac{1}{2}f^2 - \frac{a''}{a}\right). \tag{4.10}$$

This potential has minima at $f = f_0$ where

$$f_0 = \pm\sqrt{\frac{a''}{a}} \quad ; \qquad V(f_0) = -\frac{1}{4}\left(\frac{a''}{a}\right)^2, \tag{4.11}$$

so that $f = f_0$ turns out to be energetically preferable compared to $f = 0$. But the problem is that f_0 depends on η. Nevertheless it is reasonable to take this value at the initial moment of time to define initial conditions.

Consider now the case $\frac{a''}{a}$ =const (for example, it corresponds to the Miln universe for $k = -1$, but our consideration does not depend on k). Let us pass from the field φ to the fields φ_1^0 and φ_2 with nonzero vacuum expectation values:

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \quad ; \qquad \varphi_1^0 = \varphi_1 - \sqrt{\frac{6}{\lambda}\frac{f_0}{a}}. \tag{4.12}$$

Substituting (4.12) into (4.1) we find (c- numbers and 4-divergences are dropped out):

$$\begin{aligned} S &= \frac{1}{2}\int d^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\varphi_1^0\partial_\nu\varphi_1^0 - \frac{3f_0^2}{a^2}\varphi_1^{0^2} + g^{\mu\nu}\partial_\mu\varphi_2\partial_\nu\varphi_2 - \frac{f_0^2}{a^2}\varphi_2^2\right.\\ &- \left.\sqrt{\frac{2\lambda}{3}}\frac{f_0}{a}\varphi_1^0(\varphi_1^{0^2} + \varphi_2^2) - \frac{\lambda}{12}(\varphi_1^{0^2} + \varphi_2^2)^2\right]. \end{aligned} \tag{4.13}$$

Therefore, due to the spontaneous symmetry breaking instead of the massless field there arise two real fields with masses

$$m_1^2 = \frac{3f_0^2}{a^2}, \quad m_2^2 = \frac{f_0^2}{a^2}. \tag{4.14}$$

For the de Sitter universe where $a(\eta) = a_0/\eta$ and $k = 0$ one obtains the constant masses:

$$m_1^2 = \frac{6}{a_0^2}; \qquad m_2^2 = \frac{2}{a_0^2}. \tag{4.15}$$

From different considerations the similar result was obtained within the framework of inflationary model in [8].

It is easy to see that these masses have a purely geometrical origin because

$$\frac{f_0^2}{a^2} = \frac{1}{6}R - \frac{k}{a^2}. \tag{4.16}$$

Equations of motion following from (4.13) drastically differ from the free minimal coupling case. Their main feature is that now they have correct quasiclassical behavior. Really, taking quasiclassical approximation for φ_1^0 and φ_2 in the form

$$\varphi_1^0 = \frac{1}{a}\rho \exp\left(i\frac{S_1}{\hbar}\right), \quad \varphi_2 = \frac{1}{a}\rho \exp\left(i\frac{S_2}{\hbar}\right) \tag{4.17}$$

one obtains for S_1 and S_2 respectively

$$g^{00}\frac{\partial S_1}{\partial \eta}\frac{\partial S_1}{\partial \eta} - g^{ij}\frac{\partial S_1}{\partial x^i}\frac{\partial S_1}{\partial x^j} = m_1^2 - \frac{R}{6} + \frac{k}{a^2} = 2\left(\frac{R}{6} - \frac{k}{a^2}\right) = 2\frac{a''}{a^3} \geq 0, \tag{4.18}$$

$$g^{00}\frac{\partial S_2}{\partial \eta}\frac{\partial S_2}{\partial \eta} - g^{ij}\frac{\partial S_2}{\partial x^i}\frac{\partial S_2}{\partial x^j} = m_2^2 - \frac{R}{6} + \frac{k}{a^2} = 0, \tag{4.19}$$

so that φ_1^0 describes the massive particle and φ_2 corresponds to the Goldstone massless meson.

If the initial mass of the field is nonzero and the curvature is large enough then one obtains similar results with

$$f_0 = \pm\sqrt{\frac{a''}{a} - m^2a^2} \tag{4.20}$$

for the epoch when $\frac{a''}{a} > m^2a^2$.

As is easy to see the above consideration is correct also for $\frac{a''}{a} \neq$const at any given moment of time $\eta = \eta_0$ or within the interval $\eta_0 \leq \eta \leq \eta_1$ where $\frac{a''}{a}$ is slowly varied.

Therefore the self–interaction can "cure" the minimally coupled scalar field from pathological behavior via the mechanism of spontaneous symmetry breaking. Here we have some sort of "tachyon conspiracy": Nature prohibits observation of classical tachyons.

5 Concluding Remarks

From our consideration it is seen that for the massless case there are serious reasons to take the conformal coupling for a scalar field in curved space-time. For the massive field, if the scalar curvature is large enough, the conformal coupling is also preferable if one does not want to deal with the "tachyonic type" behavior. Nevertheless, if one takes into account the self-interaction for the minimal coupling, one can have the spontaneous symmetry breaking effect leading to the change of the physical mass of the scalar field, so that there will be no "tachyonic" behavior as it is in the usual case for the Goldstone model.

It is well known that the minimally coupled massive scalar field is important in the inflationary scenario where it plays the role of the "inflaton". Its unusual properties are also used to obtain galaxy formations in the Friedmann universe due to the growth of small primordial fluctuations in the de Sitter stage. That is why we think that our remarks are important for understanding the inflation scenario.

Despite that the conformal coupling is preferable there are cases when one must deal with some kind of the minimally coupled scalar field. These are cases of gravitons and vector massive mesons. As is known the equation of motion of the graviton is not conformally invariant [7], so that for $R \neq 0$ they are not really massless particles. For the longitudinal component of massive vector bosons in curved space-time one also has the minimally coupled equation. Therefore both gravitons and massive vector bosons in curved space-time with R large enough must have features described in this paper, and we think that nonlinear terms (the self-interaction) must be taken into account.

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