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General Decomposition Theory of Spin Connections, Topological Structure of Gauss-Bonnet-Chern Density and the Morse Theory[1]

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Abstract. By means of methods of the geometric algebra the general decomposition of the spin connections on the sphere bundle of a compact n-dimensional Riemannian manifold has been studied in detail. Using this decomposition theory it is shown that the Gauss-Bonnet-Chern density of the Euler-Poincaré characteristic can be expressed as a δ-function of a smooth vector field $\delta(\vec{\phi})$. The topological structure of the Gauss-Bonnet-Chern density is detailed. Furthermore the Morse theory formula of the Euler-Poincaré characteristic has been obtained via the topological structure.

1. Introduction

The Gauss-Bonnet-Chern (GBC) theorem is one of the most significant results in differential geometry. It relates the curvature of the compact and oriented even-dimensional Riemannian

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manifold M with an important topological invariant, the Euler-Poincaré characteristic $\chi(M)$. An elegant intrinsic proof of the theorem was discovered by Chern[1,2], whose instructive idea was to work on the sphere bundle $S(M)$ rather than on M. A summary and some historical comments on the GBC theorem are given by Kobayashi, Nomizu[3], and Spivak[4] respectively.
Recently, a detailed review of Chern's proof of the GBC theorem has been presented in Ref[5].

A great advance in this field was the discovery of a relationship between supersymmetry and the index theorem, which includes the derivation of the GBC theorem via supersymmmetry and path integral techniques as presented by Alvarez-Gaumé et al.[6]. In topological quantum field theory which was initiated by Witten[7], the GBC theorem can be derived by means of Morse theory[8]. On the physics side, the optical Berry phase is a direct result of the Gauss-Bonnet theorem[9] and the black hole entropy emerges as the Euler class through dimensional continuation of the Gauss-Bonnet theorem[10].

Using the special decomposition of a spin connection for the group $SO(n)$ in a previous work[11] by one of the authors (Duan) of this paper, the GBC density (the Euler-Poincaré characteristic $\chi(M)$ density) can be taken as a δ-function $\delta(\vec{\phi})$ of a smooth vector field $\vec{\phi}$ which implies that only the zeros of a smooth vector field $\vec{\phi}$ on the manifold M contribute to $\chi(M)$. This fact is the classical Hopf theorem[12,13]. But we must point out here that the decomposition of the spin connection in the previous paper[11] was carried out only for a special gauge condition.

In this paper we will try to establish a general decomposition theory of the spin connections for an $SO(n)$ gauge theory in terms of the unit vector field $\vec{n}$ by means of the methods of geometric algebra and give a general decomposition formula of the spin connections with a global property. The Chern-Simons $(n-1)$-form on the whole bundle $S(M)$ can be rigorously obtained without imposing a gauge condition. One shows that the GBC density takes the form of the δ-function $\delta(\vec{\phi})$ of a smooth vector field $\vec{\phi}$ and the topological structure of the GBC density can be labeled by the Brouwer degrees and the Hopf indices. Furthermore we show that the expression for $\chi(M)$ in the Morse theory can be represented by the Hopf indices and the Hessian matrices via the topological structure of the Gauss-Bonnet-Chern density.

This paper is arranged as follows. In Sec.2 we will study a general decomposition theory of spin connections in an $SO(n)$ gauge theory on a sphere bundle $S(M)$. In Sec.3 we will derive the Chern-Simons $(n-1)$-form from the Chern formula on the sphere bundle $S(M)$, using the general decomposition expression of the spin connections and express it completely in terms of a unit vector field $\vec{n}$. In Sec.4 we investigate the topological structure of the GBC density. In Sec.5 we derive the expression for $\chi(M)$ in the Morse theory by the topological structure of the GBC density given in Sec.4.

2. The Sphere Bundle and the General Decomposition Theory of the Spin Connections

In this section we will begin with the introduction of some definitions and the basic notions which are necessary for the discussion of the general decomposition problems. The intrinsic proof of the Gauss-Bonnet-Chern theorem for the Euler-Poincaré characteristic $\chi(M)$ of the Riemannian manifold M given by Chern[1,2] makes use of a unit vector field on M with only a finite number of isolated singular points via introducing a sphere bundle[14,15] $S(M)$ over M. We recall also that the topological invariant $\chi(M)$ is identified as the sum of the indices of a smooth vector field on M at its zeros, which was given by a famous Hopf theorem. To give a unified version of the two viewpoints above, let M be the compact and oriented n-dimensional Riemannian manifold and $\vec{\phi}$ be a smooth vector field (a section of vector bundle) on M, as in Refs.[16,17,18,19]. We define a unit vector field $\vec{n}$ on M as

$$n^a = \phi^a/\phi, \qquad \phi \equiv \| \vec{\phi} \|, \qquad \phi^2 = \phi^a\phi^a, \qquad a = 1, 2, ..., n. \tag{1}$$

in which the superscript "a" is the local orthonormal frame index.

From (1) we see that the zeros of $\vec{\phi}$ are just the singularities of $\vec{n}$. This expression naturally gives the constraint

$$n^a n^a = 1, \tag{2}$$

and we get

$$n^a dn^a = 0. \tag{3}$$

In fact, $\vec{n}$ is just a section of the sphere bundle $S(M)$[5]. For the $SO(n)$ gauge theory, i.e., the principal bundle $P(\pi, M, G)$ with the structure group $G = SO(n)$, let x^μ be the local coordinates on the base manifold M. The covariant derivative 1-form $D_\omega n^a$ of the unit vector field n^a is defined by

$$D_\omega n^a = dn^a - \omega^{ab} n^b \tag{4}$$

and the curvature 2-form by

$$F^{ab}(\omega) = d\omega^{ab} - \omega^{ac} \wedge \omega^{cb}, \tag{5}$$

where ω^{ab} is a spin connection 1-form (the $SO(n)$ gauge potential)

$$\omega^{ab} = \omega_\mu^{ab} dx^\mu, \qquad \omega^{ab} = -\omega^{ba}. \tag{6}$$

From (3), (4) and (6) we see that

$$n^a D_\omega n^a = 0. \tag{7}$$

Let the n anti-commuting Dirac matrices γ_a $(a = 1, ..., n)$ in the n-dimensional vector space of the vector field $\vec{\phi}$ be a base of the Clifford algebra[20], satisfying the anticommutation relations

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab} I.$$

In the geometric algebra theory [21,22] the unit vector field $\vec{n}$ on M can be expressed in the following matrix form

$$n = n^a \gamma_a, \tag{8}$$

as a 1-vector. Then the spin connection 1-form and the curvature 2-form are respectively

$$\omega = \frac{1}{2}\omega^{ab} I_{ab} \ , \qquad F(\omega) = \frac{1}{2}F^{ab}(\omega) I_{ab} \ , \tag{9}$$

in which I_{ab} are the generators of the group $SO(n)$, i.e.,

$$I_{ab} = \frac{1}{4}[\gamma_a, \gamma_b], \tag{10}$$

and

$$[I_{ab}, \gamma_c] = \gamma_a \delta_{bc} - \gamma_b \delta_{ac}. \tag{11}$$

Using (8) and (11) it is easy to prove that the covariant derivative 1-form (4) can be rewritten in terms of n and ω:

$$D_\omega n = dn - [\omega, n] \tag{12}$$

and the curvature 2-form:

$$F(\omega) = d\omega - \omega \wedge \omega. \tag{13}$$

Let f and h be two 1-vectors in geometric algebra theory, i.e., $f = f^a \gamma_a, \quad h = h^a \gamma_a$. The scalar product of $\vec{f}$ and $\vec{h}$ can be defined by

$$\vec{f} \cdot \vec{h} = \frac{1}{2}(fh + hf) = f^a h^a. \tag{14}$$

It can be shown that the geometric product of f and h is

$$fh = f^a h^a + (f^a h^b - h^a f^b) I_{ab}. \tag{15}$$

Making use of (15) and from (2), (3), (7), (8) we have

$$nn = n^a n^a = I, \tag{16}$$

$$dnn + ndn = 0, \tag{17}$$

$$D_\omega nn + nD_\omega n = 0, \tag{18}$$

and

$$dnn = (dn^a n^b - n^a dn^b) I_{ab}, \tag{19}$$

$$nD_\omega n = (n^a D_\omega n^b - n^b D_\omega n^a) I_{ab}. \tag{20}$$

Let T_r be a r-vector in the geometric algebra

$$T_r = \frac{1}{r!} T^{a_1 \ldots a_r} \gamma_{a_1} \cdots \gamma_{a_r}.$$

We can prove that the covariant derivative 1-form of T_r is

$$D_\omega T_r = dT_r - [\omega, T_r]. \tag{21}$$

Now we return to the main discussion on the decomposition theory of the spin connections. In gauge theory it is well-known that when the connection 1-form ω undergoes the gauge transformation

$$\omega' = S\omega S^{-1} + dSS^{-1}, \tag{22}$$

the curvature 2-form F should be transformed as

$$F' = SFS^{-1}. \tag{23}$$

These are fundamental requirements for a gauge field theory. In our viewpoint[17], the spin connection 1-form ω^{ab} can be decomposed and may has an inner structure, which has been effectively used to study the magnetic monopole problem in the $SU(2)$ gauge theory[17,18,19], the topological gauge theory of the dislocations and disclinations in condensed matter physics[23], and the geometrization of the Planck constant in terms of the space-time defect in General Relativity[24]. The main feature of the decomposition theory of the connections (the gauge potential)[17] is that the connection ω can be generally decomposed by

$$\omega = A + b, \tag{24}$$

where A and b are respectively required to satisfy the gauge transformation and vector covariant transformation rules, i.e.,

$$A' = SAS^{-1} + dSS^{-1}, \tag{25}$$

$$b' = SbS^{-1}. \tag{26}$$

From (25) and (26) one shows that

$$\omega' = A' + b' = S(A + b)S^{-1} + dSS^{-1}.$$

This means that the decomposition of the connection $\omega = A+b$ with (25) and (26) rigorously satisfies the gauge transformation rules.

Using (16), by (12) we see that

$$D_\omega nn = dnn - \omega + n\omega n. \tag{27}$$

From the above we find that

$$\omega = \frac{1}{2}(dnn + nD_\omega n) + \frac{1}{2}J_n(\omega), \tag{28}$$

where we have defined the useful symbol

$$J_n(\omega) = n\omega n + \omega. \tag{29}$$

The expression (28) with (29) is the general decomposition formula of the spin connections for the $SO(n)$ gauge theory. We call the term $\frac{1}{2}(dnn + nD_\omega n)$ in (28) the fundamental term of the general decomposition of the connection ω with respect to n, and $\frac{1}{2}J_n(\omega)$ the compensative term of the general decomposition of the connection ω respect to n, which makes the whole general decomposition expression of the connection ω to satisfy the gauge transformation rule.

Let a family $\{W, V, U, ...\}$ be an open cover of M and S_{VU} be the transition matrix functions which satisfy the following condition[25]

$$S_{UU} = I, \qquad S_{VU}^{-1} = S_{UV}, \qquad S_{WV}S_{VU}S_{UW} = I \qquad W \cap V \cap U \neq 0.$$

For any two open neighborhoods V and U, if $V \cap U \neq 0$, then

$$n_v = S_{VU} n_u S_{VU}^{-1}, \tag{30}$$

where n_v and n_u are two smooth vector fields on V and U respectively and the spin connections ω_v and ω_u satisfy the relation

$$\omega_v = S_{VU}\omega_u S_{VU}^{-1} + dS_{VU}S_{VU}^{-1}, \tag{31}$$

which is the fundamental condition for the existence of the connection on the principal bundle $P(\pi, M, G)$. In the physics terminology S_{VU} is just the gauge transformation (22) in the gauge field theory. In the following, for abbreviation, we shall use the notation $S_{VU} = S$. From (28), (29) and making use of (30) and (31), one can prove that[26]

$$\frac{1}{2}(dn_v n_v + n_v D_{\omega_v} n_v) + \frac{1}{2}J_{n_v}(\omega_v) =$$

$$S[\frac{1}{2}(dn_u n_u + n_u D_{\omega_u} n_u) + \frac{1}{2}J_{n_u}(\omega_u)]S^{-1} + dSS^{-1}. \tag{32}$$

And using (31) and (32) we see that

$$\omega_v - [\frac{1}{2}(dn_v n_v + n_v D_{\omega_v} n_v) + \frac{1}{2}J_{n_v}(\omega_v)] =$$

$$S[\omega_u - \frac{1}{2}(dn_u n_u + n_u D_{\omega_u} n_u) - \frac{1}{2}J_{n_u}(\omega_u)]S^{-1}.$$

The expression above shows that if the decomposition formula on the open neighborhood U

$$\omega_u = \frac{1}{2}(dn_u n_u + n_u D_{\omega_u} n_u) + \frac{1}{2}J_{n_u}(\omega_u)$$

holds true, then the decomposition formula on the open neighborhood V

$$\omega_v = \frac{1}{2}(dn_v n_v + n_v D_{\omega_v} n_v) + \frac{1}{2}J_{n_v}(\omega_v)$$

must hold true, too. This means that the general decomposition formula (28) has a global property and is independent from the choice of the local coordinates.

Let us show now that the decomposition formula (28) is independent of the choice of the unit vector field $\vec{n}$. To do this let k, in geometric algebra, be a unit vector field different from the unit vector field n. Then the quantity nk, in the terminology of the geometric algebra, is a 2-vector and, using (21), its covariant derivative 1-form is

$$D_\omega(nk) = d(nk) - [\omega, nk]. \tag{33}$$

This expression gives the relation

$$D_\omega nk + nD_\omega k = dnk + ndk - \omega nk + nk\omega.$$

Multiplying both sides of the formula above by n from the left and by k from the right, and using

$$kk = k^a k^a I = I, \tag{34}$$

it is not difficult to get

$$nD_\omega n + D_\omega kk = ndn + dkk - n\omega n + k\omega k. \tag{35}$$

Making use of (17) and (18) and

$$kdk + dkk = 0, \qquad kD_\omega k + D_\omega kk = 0, \tag{36}$$

from the expression (35) we find that

$$\frac{1}{2}(dnn + nD_\omega n) + \frac{1}{2}J_n(\omega) = \frac{1}{2}(dkk + kD_\omega k) + \frac{1}{2}J_k(\omega). \tag{37}$$

This means that the general decomposition formula (28) is indeed independent from the unit vector field n we chose.

To give a more concrete decomposition formula of the connections on the sphere bundle $S(M)$ we suppose in (24) that

$$[A, n] \neq 0, \tag{38}$$

$$[b, n] = 0. \tag{39}$$

Let a be an arbitrary element of the Lie algebra $so(n)$. It is expressed by

$$a = \frac{1}{2}a^{ab}I_{ab} \qquad a' = SaS^{-1}. \tag{40}$$

In the geometric algebra theory a is an antisymmetric tensor. Using the unit vector field n we can always construct a quantity b:

$$b = \frac{1}{2}(nan + a), \tag{41}$$

which obviously satisfies the commutation relation (39). Substituting (41) and (24) into (29) we have

$$J_n(\omega) = n(A + a)n + A + a. \tag{42}$$

Since, as we have mentioned above, a is arbitrary, the argument $(A+a)$ in (42) should also be arbitrary, and by (25) and (40) we see that it satisfies the gauge transformation rule. This means that

$$B = A + a \tag{43}$$

is an arbitrary connection for the $SO(n)$ gauge theory with

$$B' = SBS^{-1} + dSS^{-1}. \tag{44}$$

Making use of (43) and (41), we can rewrite (42) as

$$J_n(\omega) = J_n(B) = nBn + B, \tag{45}$$

or

$$J_n(B) = n[B, n] + 2B. \tag{46}$$

Thus the general decomposition formula (28) becomes

$$\omega = \frac{1}{2}(dnn + nD_\omega n) + \frac{1}{2}J_n(B). \tag{47}$$

We can always choose a as $a = B_0 - A$, where B_0 is a flat connection

$$B_0 = dUU^{-1} \tag{48}$$

in which U is a local matrix in the spinor representation corresponding to the group $SO(n)$. From (48), (45) and (47) we finally get

$$\omega = \frac{1}{2}(dnn + nD_\omega n) + \frac{1}{2}J_n(B_0), \tag{49}$$

where

$$J_n(B_0) = nB_0 n + B_0. \tag{50}$$

Substituting (49) into (13), for the flat connection B_0, we get $F(B_0) = 0$. Using (50) we derive the following decomposition expression for the curvature 2-form:

$$\begin{aligned} F(\omega) = \frac{1}{4}[&-D_0 n \wedge D_0 n + D_\omega n \wedge D_\omega n + ndD_\omega n - dD_\omega nn \\ &+ D_\omega n \wedge (B_0 n + nB_0) - (B_0 n + nB_0) \wedge D_\omega n], \end{aligned} \tag{51}$$

where

$$D_0 n = dn - [B_0, n]. \tag{52}$$

From (52), (51) can be rewritten in the covariant formulation as

$$F(\omega) = \frac{1}{4}[-D_0 n \wedge D_0 n + D_\omega n \wedge D_\omega n + nD_0 D_\omega n - D_0 D_\omega nn]. \tag{53}$$

Using (9), (10), (15) and noticing that $J_n(B_0) = \frac{1}{2}J^{ab}(B_0)I_{ab}$ (throughout this paper), we see from (46), (49) and (53) that the component decomposition formulas of $\omega, F(\omega)$ and $J_n(\omega)$ are respectively given by

$$\omega^{ab} = dn^a n^b - dn^b n^a + n^a D_\omega n^b - n^b D_\omega n^a + \frac{1}{2}J_n^{ab}(B_0), \tag{54}$$

$$F^{ab}(\omega) = -D_0 n^a \wedge D_0 n^b + D_\omega n^a \wedge D_\omega n^b + n^a D_0 D_\omega n^b - n^b D_0 D_\omega n^a \tag{55}$$

and

$$J_n^{ab}(B_0) = 2(B_0^{bc} n^c n^a - B_0^{ac} n^c n^b + B_0^{ab}), \tag{56}$$

where

$$D_0 n^a = dn^a - B_0^{ab} n^b. \tag{57}$$

It is easy to see that the special case of (54) and (55) with $B_0 = 0$ (i.e., $J_n^{ab}(B_0) = 0$) is just the result adopted in Ref.[11]. One notes that the special decomposition expressions in terms of the unit vector field $\vec{n}$ for the spin connection 1-form ω^{ab} and the curvature 2-form $F^{ab}(\omega)$ in Ref.[11] do not satisfy the gauge transformation rules (22) and (23). It may be regarded as a special gauge condition.

3. Chern- Simons Form on the Sphere Bundle $S(M)$

In this section we will discuss the Chern-Simons $(n-1)$-form by means of the decomposition expression (41). For an even dimensional compact and oriented Riemannian manifold M, there exists a unique n-form Λ over M such that

$$\Lambda = \frac{(-1)^{n/2}}{2^n \pi^{n/2}(n/2)!} \epsilon_{a_1 a_2 \ldots a_{n-1} a_n} F^{a_1 a_2}(\omega) \wedge \ldots \wedge F^{a_{n-1} a_n}(\omega), \tag{58}$$

which is a closed n-form. Let π be a natural projection, thus $\pi^{-1}(\mathrm{p})$ is all the part of S(M) lying above the point $p \in M$. This fiber is isometric to the $(n-1)$-dimensional sphere S^{n-1}. The transformation group in the fiber can be taken as the group $SO(n)$. The n-form Λ over M, when pulled back to the sphere bundle $S(M)$[13,5], is exact

$$\pi^* \Lambda = d\Omega \tag{59}$$

Chern[1,5] has proved that the $(n-1)$ form Ω on the sphere bundle $S(M)$ is

$$\Omega = \frac{1}{(2\pi)^{n/2}} \sum_{k=0}^{n/2-1} (-1)^k \frac{2^{-k}}{(n-2k-1)!!k!} \Theta_k, \quad n \geq 4 \tag{60}$$

which is called the Chern formula, with

$$\Theta_k = \epsilon_{a_1 a_2 \ldots a_{n-2k} a_{n-2k+1} a_{n-2k+2} \cdots a_{n-1} a_n} n^{a_1} D_\omega n^{a_2} \wedge \ldots \wedge D_\omega n^{a_{n-2k}} \wedge \ldots$$
$$\wedge F^{a_{n-2k+1} a_{n-2k+2}}(\omega) \wedge \ldots \wedge F^{a_{n-1} a_n}(\omega), \tag{61}$$

where F^{ab} is the curvature 2-form. Using the Bianchi identity

$$DF^{ab}(\omega) = 0, \tag{62}$$

it can be proved that

$$D\Lambda = 0. \tag{63}$$

This means that Λ is independent of the connections and determines a cohomology class belonging to the cohomology group $H^n(M)$ of the manifold M. This is equivalent to saying that the integral $\int_M$ taken over a closed manifold M is a topological invariant, which is called the Euler-Poincaré characteristic $\chi(M)$. π^* maps the cohomology class of M into that of $S(M)$, while $\vec{n}^*$ performs the inverse operation, i.e., $\vec{n}^*\pi^*$ amounts to the identity. The famous GBC theorem can thus be expressed as

$$\chi(M) = \int_M \Lambda = \int_M \vec{n}^*\pi^*\Lambda = \int_M \vec{n}^* d\Omega. \tag{64}$$

Substituting (55) into (61) and noting that the two last terms in (55) do not contribute to Θ_k due to the complete antisymmetry of the tensor $\epsilon_{a_1 \dots, a_n}$ in the expression (61), we find the expansion of Θ_k:

$$\begin{aligned} \Theta_k = \sum_{l=0}^{k} (-1)^k C_k^l (-1)^l \\ \epsilon_{a_1 a_2 \dots a_{n-2k} a_{n-2k+1} a_{n-2k+2} \dots a_{n-2k+2l-1} a_{n-2k+2l} a_{n-2k+2l+1} a_{n-2k+2l+2} \dots a_{n-1} a_n} \\ n^{a_1} D_\omega n^{a_2} \wedge \dots \wedge D_\omega n^{a_{n-2k}} \wedge D_\omega n^{a_{n-2k+1}} \wedge D_\omega n^{a_{n-2k+2}} \wedge \dots \\ \wedge D_\omega n^{a_{n-2k+2l-1}} \wedge D_\omega n^{a_{n-2k+2l}} \wedge D_0 n^{a_{n-2k+2l+1}} \wedge D_0 n^{a_{n-2k+2l+2}} \wedge \dots \\ \wedge D_0 n^{a_{n-1}} \wedge D_0 n^{a_n}, \end{aligned} \tag{65}$$

where

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

The two vectors $D_\omega \vec{n}$ and $D_0 \vec{n}$ are both perpendicular to $\vec{n}$, i.e., $n^a D_\omega n^a = n^a D_0 n^a = 0$. Defining $\vec{k}$ by

$$D_\omega n^a - D_0 n^a = \alpha k^a, \tag{66}$$

where α is the magnitude of the vector $D_\omega \vec{n} - D_0 \vec{n}$, we find that $\vec{k}$ is perpendicular to $\vec{n}$, i.e.,

$$n^a k^a = 0, \tag{67}$$

and it follows that

$$\begin{aligned} k^a dn^a + n^a dk^a = 0, \\ n^a D_\omega k^a + k^a D_\omega n^a = 0, \qquad n^a D_0 k^a + k^a D_0 n^a = 0. \end{aligned} \tag{68}$$

From (66) and (34) one obtains

$$\alpha = (D_\omega n^a - D_0 n^a) k^a. \tag{69}$$

Substituting this into (66) we have

$$D_\omega n^a = D_0 n^a + k^a k^b (D_\omega n^b - D_0 n^b).$$

Using (68) the expression above becomes

$$D_\omega n^a = D_0 n^a - k^a n^b (D_\omega k^b - D_0 k^b). \tag{70}$$

Putting (70) into (65), we get

$$\Theta_k = \sum_{l=0}^{k}(-1)^k(-1)^l C_k^l \epsilon_{a_1a_2a_3\ldots a_n}(n^{a_1}D_0 n^{a_2} \wedge D_0^{a_3} \ldots \wedge D_0 n^{a_n}$$

$$- (n + 2l - 2k - 1)n^{a_1}k^{a_2}n^{b_2}(D_\omega k^{b_2} - D_0 k^{b_2}) \wedge D_0 n^{a_3} \wedge \ldots \wedge D_0 n^{a_n}). \quad (71)$$

From the formulas

$$(-1)^m C_{n-1}^m = \sum_{j=0}^{m} C_n^j, \qquad m \neq 0$$

and

$$\sum_{j=0}^{m}(-1)^{m+1} j C_k^m = -\sum_{j=0}^{m}(-1)^m j C_k^m = 0, \qquad k \neq 1,$$

it is not difficult to show that

$$\sum_{l=0}^{k}(-1)^k(-1)^l C_k^l = (-1)^k[\sum_{l=0}^{k-1}(-1)^l C_k^l + (-1)^k] = 0, \quad k \neq 0$$

and

$$-\sum_{l=0}^{k}(-1)^k(-1)^l(n + 2l - 2k - 1)C_k^l = \begin{cases} -2, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases}$$

Substituting these into (71) we obtain

$$\Theta_k = \begin{cases} \epsilon_{a_1a_2\ldots a_n}(n^{a_1}D_0 n^{a_2} \wedge \ldots \wedge D_0 n^{a_n} & \\ -(n-1)n^{a_1}k^{a_2}n^{b_2}(D_\omega k^{b_2} - D_0 k^{b_2}) \wedge \ldots \wedge D_0 n^{a_n}), & \text{if } k = 0, \\ -2\epsilon_{a_1a_2\ldots a_n} n^{a_1}k^{a_2}n^{b_2}(D_\omega k^{b_2} - D_0 k^{b_2}) \wedge \ldots \wedge D_0 n^{a_n}, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases} \quad (72)$$

Using the expression (72), from (60) we find for Ω the compact form

$$\Omega = \frac{1}{(n-1)!A(S^{n-1})}\epsilon_{a_1a_2\ldots a_n} n^{a_1}D_0 n^{a_2} \wedge \ldots \wedge D_0 n^{a_n}, \quad (73)$$

where $A(S^{n-1})$ is the area of S^{n-1}:

$$A(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Using the geometric algebra methods, for the flat connection $B_0 = dUU^{-1}$ where we take the simple matrix U:

$$U = nl, \qquad U^{-1} = ln, \quad (74)$$

which is a local spinor matrix representation for the group $SO(n)$ (see[27]), n being the unit vector field defined as before, and l another unit vector field perpendicular to n:

$$ln + nl = 0. \quad (75)$$

It is not difficult to verify that the matrix U given by (74) transforms l into $-l$ and n into $-n$:

$$UlU^{-1} = -l, \qquad UnU^{-1} = -n. \tag{76}$$

A more detailed discussion of the construction of the spinor matrix representation is given in the Appendix. Substituting (74) into (45) and using (75) we find that the flat connection is of the form

$$B_0 = dn\, n + n\, dl\, l\, n. \tag{77}$$

By (77) and (52) we have

$$D_0 n = -dn + [dl\, l, n]. \tag{78}$$

Making use of (15) we can be express $dl\, l$ explicitly as

$$dl\, l = (dl^a l^b - l^a dl^b) I_{ab}.$$

From (78) and the formula above, using (8), (11) and $dn^a l^a + n^a dl^a = 0$, one obtains

$$D_0 n^a = -dn^a + 2l^a l^b dn^b. \tag{79}$$

Putting (79) into (73), we can express Ω as

$$\Omega = \frac{1}{(n-1)!A(S^{n-1})}\epsilon_{a_1 a_2 \ldots a_n}(-n^{a_1} dn^{a_2} \wedge \ldots \wedge n^{a_n}$$

$$+2(n-1)n^{a_1} dn^{a_2} \wedge \ldots \wedge dn^{a_{n-1}} \wedge l^{a_n} l^b dn^b).$$

Using the the formula given in the Appendix of Ref [10]),

$$\epsilon_{A_1 a_2 \ldots a_n} n^{a_1} dn^{a_2} \wedge \ldots \wedge dn^{a_{n-1}} \wedge dn^b =$$

$$\frac{1}{n-1}(\delta^b_{a_n} - n^b n^{a_n})\epsilon_{a_1 a_2 \ldots a_n} n^{a_1} dn^{a_2} \wedge \ldots \wedge dn^{a_n},$$

we find that Ω on M can be reduced to

$$\Omega = \frac{1}{(n-1)!A(S^{n-1})}\epsilon_{a_1 a_2 \ldots a_n} n^{a_1} dn^{a_2} \wedge \ldots \wedge dn^{a_n}. \tag{80}$$

The expression (80) above is the Chern-Simons $(n-1)$-form expressed cleanly in terms of the unit vector field n^a on the whole sphere bundle $S(M)$. This is a generalization of the result of Ref [5], which was only valid on a fibre of $S(M)$.

4. The Topological Structure of the GBC Density and the GBC Theorem

In the present section, from the Chern-Simons $(n-1)$-form in its expression (80), we will show that the singularities of a n-form $d\Omega$ can be directly expressed by $\delta(\Phi)$. In order to do this, let us note that from (1) we have

$$dn^a = \frac{d\phi^a}{\phi} + \phi^a d(\frac{1}{\phi}).$$

By substituting this into (80), Ω becomes

$$\Omega = \frac{1}{(n-1)!A(S^{n-1})}\epsilon_{a_1 a_2 \ldots a_n}\frac{\phi^{a_1}}{\phi^n}d\phi^{a_2}\wedge \ldots \wedge d\phi^{a_n}. \tag{81}$$

Using

$$\frac{\phi^a}{\phi^n} = -\frac{1}{n-2}\frac{\partial}{\partial\phi^a}(\frac{1}{\phi^{n-2}}),$$

one can write the pull-back of the exterior derivative of (81) to M as

$$\vec{\phi}^* d\Omega = -\frac{1}{A(S^{n-1})(n-1)!(n-2)}\epsilon_{a_1 a_2 \ldots a_n}\frac{\partial}{\partial\phi^a}\frac{\partial}{\partial\phi^{a_1}}(\frac{1}{\phi^{n-2}})\times$$

$$\partial_{\mu_1}\phi^a\partial_{\mu_2}\phi^{a_2}\ldots\partial_{\mu_n}\phi^{a_n}\frac{\epsilon^{\mu_1\mu_2\ldots\mu_n}}{\sqrt{g}}\sqrt{g}d^n x, \tag{82}$$

where $g = \det(g_{\mu\nu})$, $g_{\mu\nu}$ being the metric tensor on M. Using the Jacobian $D(\phi/x)$ defined by

$$\epsilon^{aa_2\ldots a_n}D(\phi/x) = \epsilon^{\mu_1\mu_2\ldots\mu_n}\partial_{\mu_1}\phi^a\partial_{\mu_2}\phi^{a_2}\ldots\partial_{\mu_n}\phi^{a_n}$$

and the formula

$$\epsilon_{a_1 a_2\ldots a_n}\epsilon^{aa_2\ldots a_n} = (n-1)!\delta^a_{a_1},$$

one can rewrite (82) as

$$\vec{n}^* d\Omega = -\frac{1}{A(S^{n-1})(n-2)}\frac{\partial^2}{\partial\phi^a\partial\phi^a}(\frac{1}{\phi^{n-2}})D(\phi/x)d^n x.$$

By means of the general Green function formula in the vector space for ϕ:

$$\Delta_\phi(\frac{1}{\phi^{n-2}}) = -\frac{4\pi^{n/2}}{\Gamma(n/2-1)}\delta(\vec{\phi}) \qquad n \geq 3,$$

where

$$\Delta_\phi = \frac{\partial^2}{\partial\phi^a\partial\phi^a}$$

is the n-dimensional Laplacian operator in the vector space. Thus $\vec{n}^* d\Omega$ becomes

$$\vec{n}^* d\Omega = \delta(\vec{\phi})D(\phi/x)d^n x. \tag{83}$$

Therefore we may define a Gauss-Bonnet-Chern density ρ on M as

$$\rho = \frac{1}{(n-1)!A(S^{n-1})}\epsilon_{a_1 a_2\ldots a_n}\epsilon^{\mu_1\mu_2\ldots\mu_n}\partial_{\mu_1}n^{a_1}\partial_{\mu_2}n^{a_2}\ldots\partial_{\mu_n}n^{a_n}$$

$$= \delta(\vec{\phi})D(\phi/x). \tag{84}$$

We see that $\rho \neq 0$ and $\vec{n}^* d\Omega \neq 0$ only if $\vec{\phi} = 0$. The two expressions (83) and (84) above are of great importance: they yield, in our case, an evident result of the Hopf theorem.

Suppose that $\phi^a(x)$ $(a = 1..., n)$ possess N isolated zeros and let the ith zero be $\vec{x} = \vec{z}_i$. According to the δ-function theory [28], $\delta(\vec{\phi})$ can be expressed by

$$\delta(\vec{\phi}) = \sum_{i=1}^{N} \frac{\beta_i \delta(\vec{x} - \vec{z})}{|D(\phi/x)|_{\vec{x}=\vec{z}_i}}, \tag{85}$$

and one then obtains

$$\delta(\phi) D(\phi/x) = \sum_{i=1}^{N} \beta_i \eta_i \delta(\vec{x} - \vec{z}_i), \tag{86}$$

where β_i is a positive integer (the Hopf index of the ith zero) and η_i, the Brouwer degree[12]:

$$\eta_i = \mathrm{sgn} D(\phi/x)|_{\vec{x}=\vec{z}_i} = \pm 1.$$

The meaning of the Hopf index β_i is that while $\vec{x}$ covers the region neighbouring the zero $\vec{z}_i$ once, the vector field function $\vec{\phi}$ covers the corresponding region β_i times. From the above and from (86), the GBC density on M has the following topological structure:

$$\rho = \delta(\vec{\phi}) D(\phi/x) = \sum_{i=1}^{N} \beta_i \eta_i \delta(\vec{x} - \vec{z}_i), \tag{87}$$

which shows that the local structure of ρ is labeled by the Brouwer degrees and the Hopf indices. The integration of ρ on M is just the Euler-Poincaré characteristic

$$\chi(M) = \int_M \rho d^n x = \int_M \vec{n}^* d\Omega = \sum_{i=1}^{N} \beta_i \eta_i. \tag{88}$$

The result (88) says that the sum of the indices of the zeros of the vector field $\vec{\phi}$ – or of the singularities of the unit vector field $\vec{n}$ – is the Euler-Poincaré characteristic $\chi(M)$. Therefore the topological structure of the GBC density yields the expected Hopf theorem. On the other hand, we must point out the important and simple fact that from (87) and (88) we have

$$\chi(M) = \int_M \delta(\vec{\phi}) D(\phi/x) d^n x = \deg\phi \int_V \delta(\vec{\phi}) d^n \phi = \deg\phi, \tag{89}$$

and

$$\deg\phi = \sum_{i=1}^{N} \beta_i \eta_i, \tag{90}$$

where V is the elementary volume in the vector space of $\vec{\phi}$ and $\deg\phi$ is the degree of the mapping ϕ. The mapping degree $\deg\phi$ represents the global topological property of the Euler-Poincaré characteristic.

5. From the Gauss-Bonnet-Chern Theorem to the Morse Theory

In the following, we will study the Euler-Poincaré characteristic $\chi(M)$ in the Morse theory making use of the topological structure given in Sec.4. To reach our goal, we first briefly review the elements of the Morse theory[16,28]. Then we use (64), (83) and (85) in order to give the formula for $\chi(M)$ in the Morse theory.

Let f be a Morse function on the manifold M. For a point $p \in M$, if the function f satisfies

$$df\mid_p = \partial_\mu f dx^\mu \mid_p = 0, \tag{91}$$

then the point p is a critical point of the function f. The Hessian at p, $H_p f$, is a quadratic form on $T_p M$, the tangent space to M at p. In local coordinates $\{x^\mu\}$ centered at p, the matrix of $H_p f$ relative to the base ∂_μ at p is

$$\{H_p f\}_{\mu\nu} = \partial_\mu \partial_\nu f, \tag{92}$$

which is called the Hessian matrix. If, at p,

$$\det H_p f \neq 0, \tag{93}$$

then the point p is called a non-degenerate critical point of the function f. The index of p is the number of negative eigenvalues of the Hessian matrix $\det H_p f$ and it will be denoted by $\lambda_i(f)$.

Now let us suppose that a smooth vector field $\vec{\phi}$ on M is the gradient field [28] of the Morse function f on the manifold M, i.e.,

$$\phi^a = e^{a\mu} \partial_\mu f, \tag{94}$$

where $e^{a\mu}$ are the vierbeins $(g^{\mu\nu} = e^{a\mu} e^{a\nu})$. The relation (94) means that the critical points of the Morse function f are just the zero points of the vector field $\vec{\phi}$.

Using (94) we get, by differentiation,

$$\partial_\mu \phi^a \mid_p = e^{a\nu} \partial_\mu \partial_\nu f \mid_p . \tag{95}$$

Making use of the formula

$$\epsilon_{a_1 \ldots a_n} e^{a_1 \mu_1} \ldots e^{a_n \mu_n} = \epsilon^{\mu_1 \ldots \mu_n} \frac{1}{\sqrt{g}}, \tag{96}$$

from (95) one obtains

$$D(\phi/x)\mid_p = \frac{1}{\sqrt{g}} \det H_p f \mid_p . \tag{97}$$

Using (85) and (97), from the expression (83) we find that

$$\vec{n}^* d\Omega = \sum_{i=1}^{N} \beta_i \delta(\vec{x} - \vec{p}_i) \frac{\det H_p f}{\mid \det H_p f \mid} \mid_{p_i} d^n x. \tag{98}$$

Substituting (98) into the Gauss-Bonnet-Chern theorem (64), we get the formula

$$\chi(M) = \sum_{i=1}^{N} \beta_i \frac{\det H_{p_i} f}{\mid \det H_{p_i} f \mid}, \tag{99}$$

where N is the number of critical points of the Morse function f. This means that the Euler-Poincaré characteristic $\chi(M)$ is related, not only to the Hessian matrix, but also to the Hopf index.

In the Morse theory it is well-known that the Morse function f at the neighbourhood of any critical point p_i can take the following form:

$$f = f(p_i) - (x^1)^2 - ... - (x^{\lambda_i})^2 + ... + (x^n)^2, \tag{100}$$

where $\lambda_i(f) = 0, 1...n$. Since the vierbein $e^{a\mu}$ is a single-vector function on the manifold M, it is easy to show from (94) and (100) that ϕ^a also is single-valued. This means that the Hopf indices $\beta_i = 1$ (i=1,2,...,N). Substituting (100) into (99), we obtain

$$\chi(M) = \sum_{i=1}^{N}(-1)^{\lambda_i(f)}. \tag{101}$$

This is a famous result in the Morse theory .

We have thus seen that the topological structure of the Gauss-Bonnet-Chern density induces namely the Euler-Poincaré characteristic $\chi(M)$ in the Morse theory.

The conclusion of this paper is therefore that the general decomposition theory of connections in a gauge field theory is an important and powerful tool for studying topological problems, not only in theoretical physics, but also in differential geometry.

Appendix

Theorem[27]: Let $\vec{n}$ and $\vec{l}$ be two unit vector fields in a n-dimensional vector space with $\vec{n} \cdot \vec{l} \neq -1$, and $\vec{h}$ be a unit vector field in the $\vec{n} - \vec{l}$ plane which bisects the angle θ between $\vec{n}$ and $\vec{l}$. Let n, l and h be three 1-vectors in the geometric algebra. Then there exist two elementary transformations which bring l into n via a similarity transformation, i.e.,

$$n = L_i l L_i^{-1}, \quad i = 1, 2,$$

where

$$L_1 = h \quad L_1^{-1} = L_1,$$
$$L_2 = hl = nh \quad L_2^{-1} = lh = hn,$$

where h has been defined by

$$h = \frac{n + l}{2(cos\theta/2)},$$

and θ is

$$cos\theta = n^a l^a \neq -1.$$

L_i $(i = 1, 2)$ are regarded as the spinor matrix representation of the orthogonal transformations, L_1 representing the axial involution with respect to h, and L_2 representing the plane rotation in the $\vec{n} - \vec{l}$ plane.

Now, let us further consider the case where the unit vector field $\vec{l}$ is perpendicular to the unit vector field $\vec{n}$. Then

$$L_2 = \frac{1}{\sqrt{2}}(1 + nl), \quad L_2^{-1} = \frac{1}{\sqrt{2}}(1 + ln).$$

Using L_2 we can construct a simple spinor matrix representation of the orthogonal transformation U as

$$U = L_2 L_2 = nl, \quad U^{-1} = ln.$$

Then

$$UlU^{-1} = -l, \quad UnU^{-1} = -n$$

This shows that U brings l into $-l$, n into $-n$, respectively.

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