Classical and Quantum Equivalence Principle in Terms of the Path Group

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Abstract. A natural mapping of paths in a curved space onto the paths in the corresponding (tangent) flat space may be used to reduce the curved-space-time path integral to the flat-space-time path integral. The dynamics of the particle in a curved space-time is expressed then in terms of an integral over paths in the flat (Minkowski) space-time. This may be called quantum equivalence principle. Contrary to the known DeWitt's definition of a curved-space path integral, the present definition leads to the covariant equation of motion without a scalar curvature term. The reduction of a curved-space path integral to the flat-space path integral may be expressed in terms of a representation of the path group. With the help of this representation all the results may be generalized to the case of an arbitrary external field.

1   Introduction

The motion of a free classical point particle is described by a direct line in the Minkowski space-time. The evolution of a quantum point particle may be described by a path integral in the Minkowski space. According to the Einstein's equivalence principle, a point classical particle in a curved space-time moves along a geodesic line. The motion of a quantum particle may be described by an integral over paths in the curved space-time, however the definition of such an integral is ambiguous. The first definition given by B.DeWitt [1] and some other definitions lead to the equations of motions differing by the coefficient in the

term proportional to the scalar curvature R.

It will be pointed out in the present paper that the evolution of both classical and quantum particles in a curved space-time may be naturally described in terms of the Minkowski space-time. This description is based on the natural but non-holonomic mapping of curves in the curved space-time onto the curves in the Minkowski space-time. This results in the description of a classical particle moving in the curved space-time, by a direct line in the Minkowski space. The description of a quantum particle is given by the integral over paths in the Minkowski space, but with the corresponding operator in the integrand. This path-depending operator may be shown to be a representation of the path group.

2   Mapping of curves and the equivalence principle

It is well known that there is no natural point-to-point mapping of a curved space onto the Minkowski space. However a natural correspondence exists between the curves in the Minkowski space and the curves in the curved space provided the starting point of these curves and (local) reference frame in this point are fixed. To describe this correspondence, let us identify the Minkowski space $\mathcal{M}$ with a tangent space $\mathcal{M}_x$ to the curved (pseudo-Riemannian) space $\mathcal{X}$ in the starting point x of the curves, and the reference frame of the Minkowski space with some orthonormal local frame $n = \{n_\alpha \in \mathcal{M}_x : \alpha = 0, 1, 2, 3\}$ in the specified point x. 1. Let $[x] = \{x(\tau') \in \mathcal{X} : 0 \leq \tau' \leq \tau\}$ be a curve in $\mathcal{X}$ starting in x and $n(\tau')$ a result of parallel transport of n along the curve $[x]$. Then the curve $[\xi] = \{\xi(\tau') \in \mathcal{M} : 0 \leq \tau' \leq \tau\}$ in $\mathcal{M}$ (with $\xi(0)$ coinciding with the origin of $\mathcal{M}$) may be defined unambiguously by the condition that its tangent vector $\dot{\xi}(\tau')$, being expanded in respect to the reference frame of the Minkowski space, has the same coefficients as the tangent vector $\dot{x}(\tau')$ expanded in respect to the local frame $n(\tau')$.

As a result, the natural correspondence of curves in $\mathcal{M}$ (starting in the origin) with the curves in $\mathcal{X}$ (starting in x) is established, provided the local frame n in the point x is given. Instead of the curve in $\mathcal{M}$ with the fixed starting point, $[\xi]$ may be understood as a class of curves differing by general translation, $[\xi] \sim [\xi']$, if $\xi'(\tau') = \xi(\tau') + a$.

Naturalness of the mapping defined in this way may be formulated as follows: If the observer located in the space-time point x has to describe the path $[x]$ (in the curved space-time $\mathcal{X}$) as a line in his "flat map" (with the geometry of the Minkowski space $\mathcal{M}$), he should choose the curve $[\xi]$. Such a "flat modelling" is defined for an arbitrary curve $[x]$ in $\mathcal{X}$.

Geodesic lines $[x]$ will be associated then with direct lines $[\xi]$. This gives a simple formulation of the Einstein's equivalence principle [2]: local "flat models" for trajectories of a point particle are direct lines.

2. By the described procedure, a class of curves $[\xi]$ in $\mathcal{M}$ and an orthonormal local frame $n \in \mathcal{N}$ ($\mathcal{N}$ being a fiber bundle of all orthonormal local frames) determine one more local frame $n[\xi] \in \mathcal{N}$ by the rule: if $n = n(0)$, then $n[\xi] = n(\tau)$ in the notations introduced earlier.

Therefore, each class of curves $[\xi]$ determines a mapping of $\mathcal{N}$ onto itself.

This mapping may be described in a more formal way [3] with the help of the concept of the basis vector fields in the fiber bundle of local frames. To introduce this concept, it is convenient to consider the fiber bundle $\mathcal{B}$ of all local frames over $\mathcal{X}$. The coordinates x^μ of the points in $\mathcal{X}$ and the components b_α^μ of the vectors of the local frame b in this point may serve as coordinates of the manifold $\mathcal{B}$. Then the basis vector fields B_α are following:

$$B_\alpha = b_\alpha^\mu \frac{\partial}{\partial x^\mu} + b_\alpha^\mu b_\beta^\nu \Gamma_{\mu\nu}^\lambda(x) \frac{\partial}{\partial b_\beta^\lambda}. \tag{2.1}$$

The vector fields B_α are horizontal in the fiber bundle $\mathcal{B}$ and their restrictions on the fiber bundle $\mathcal{N}$ of orthonormal local frames are horizontal in $\mathcal{N}$. We shall need these restrictions rather than the complete fields B_α. For simplicity, we shall denote them by the same letters.

The horizontal vector fields allow one to define, as an ordered exponential of an integral, the following operator acting in the space of functions on $\mathcal{N}$:

$$V[\xi] = P e^{\int B_\alpha d\xi^\alpha} = \lim_{N \to \infty} e^{B_\alpha \Delta \xi_N^\alpha} \dots e^{B_\alpha \Delta \xi_1^\alpha}. \tag{2.2}$$

The set of operators $V[\xi]$ forms a representation of the path group [4] (generalizing translations).

In terms of these operators, the mapping $n \to n[\xi]$ may be defined as follows:

$$(V[\xi]\Psi)(n) = \Psi(n[\xi]). \tag{2.3}$$

3 Quantum equivalence principle

Evolution of a quantum particle is described by the propagator which may be expressed in the form of a path integral. The path integral in a curved space-time may be reduced to the path integral in the Minkowski space-time but with the operator $V[\xi]$ in the integrand.

1. To do this, let us describe states of the particle by functions $\Psi(n)$ on the fiber bundle of (orthonormal) local frames instead of usual functions $\psi(x)$ on the space-time [3]. Both functions are connected in a very simple way for the scalar particle:

$$\Psi(n) = \psi(x), \quad x = \pi(n)$$

where the canonical projection $\pi : \mathcal{N} \to \mathcal{X}$ associates the point x with the local frame n in this point. This definition may be naturally generalized onto the case of a spinning particle:

$$\Psi(n\lambda) = D(\lambda^{-1})\,\Psi(n), \quad \psi(x) = \Psi(\sigma(n)), \quad x = \pi(n).$$

Here σ is an arbitrary section of the fiber bundle $\mathcal{N}$, and D a representation of the Lorentz group describing spin of the particle.

2.The evolution of a quantum particle may be described by a propagator $U(x, x')$, but we shall use the corresponding operator U (for which the propagator is a kernel). In the Minkowski space-time this operator may be expressed in the form of a path integral

$$U = \int_0^\infty d\tau\, \epsilon^{-im^2\tau} U_\tau, \quad U_\tau = \int d[\xi]\, \epsilon^{(-i/4)\int_0^\tau d\tau\, \dot{\xi}^\alpha \dot{\xi}_\alpha}\, V[\xi] \tag{3.1}$$

where $V[\xi]$ is an operator of displacement along the path $[\xi]$:

$$(V[\xi]\psi)(\xi) = \psi(\xi - \Delta\xi), \quad \Delta\xi = \xi(\tau) - \xi(0).$$

The propagator in a curved space-time may be defined [3] by the same formulas (3.1) but with the expression (2.2) for the operator $V[\xi]$. The dynamics of the particle in a curved space-time is expressed then in terms of the integral over paths in the flat space-time (Minkowski space). This may be called *quantum equivalence principle*. The resulting definition of the curved-space path integral differs from the known DeWitt's definition [1] in that it leads to the covariant equation of motion with no term proportional to the scalar curvature. An essentially equivalent though apparently different definition of the path integral in a curved space has been given in [5].

The operator $V[\xi]$ which maps flat-space paths onto the curved-space paths may be shown to form a representation of the path group [4]. With the help of this representation all the results may be generalized on the case of an arbitrary external field (gauge, gravitational or gauge plus gravitational fields).

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References

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