Zeitschrift: Helvetica Physica Acta

Band: 69 (1996)

Heft: 3

Artikel: A classification of embedding class 2 vacua

Autor: Bergh, Norbert van den

DOI: https://doi.org/10.5169/seals-116948

Helv Phys Acta
Vol. 69 (1996)

0018-0238/96/030325-04$1.50+0.20/0

A Classification of Embedding Class 2 Vacua

By Norbert Van den Bergh

Faculty of Applied Sciences, University of Gent,
B9000-Gent, Belgium

Abstract. A new approach is presented to the problem of embedding a vacuum space-time (M, g) in a 6-dimensional (pseudo-)Euclidean manifold.

1 Introduction

As vacuum solutions necessarily have an embedding class > 1 and as a number of important vacuum space-times (Schwarzschild, the pp-waves ...) are known to be of embedding class 2, it is a natural question to ask for a list of all embedding class 2 vacuum solutions. It is known [5] that the two second fundamental forms $\mathbf{\Omega}^{(1)}$ and $\mathbf{\Omega}^{(2)}$ of a vacuum metric necessarily *commute*. This allows one to write down all possible pairs of second fundamental forms ($\mathbf{\Omega}^{(1)}$, $\mathbf{\Omega}^{(2)}$), after which a calculation of the curvature (via the Gauss equations) and expressing the fact that $R_{ab} = 0$, leaves one with only the Codazzi equations to be solved (the Ricci equations are accounted for automatically as the torsion vector t_a is necessarily a gradient):

$$\Omega_{ab;c} - \Omega_{ac;b} = \epsilon_2(t_c\Lambda_{ab} - t_b\Lambda_{ac}) \tag{1.1}$$

and

$$\Lambda_{ab;c} - \Lambda_{ac;b} = \epsilon_1(t_b\Omega_{ac} - t_c\Omega_{ab}) \tag{1.2}$$

where $\Omega = \mathbf{\Omega}^{(1)}$ and $\Lambda = \mathbf{\Omega}^{(2)}$.

A complication is that Ω and Λ are not uniquely determined by the embedding. This is due to the possibility of performing at each point a *hyperrotation* of the 2 'external' basisvectors orthogonal to space-time. As a first step one should therefore construct a classification, invariant under Lorentz-transformations as well as hyperrotations, of commuting and symmetric tensors.

The resulting canonical set is quite formidable [4], but after introducing the condition $R_{ab} = o$, it reduces to a list of 18 canonical types: 6 for Petrov type D, 5 for type N, 2 for type II and 5 for type I (the absence of Petrov type III is in agreement with [5]).

In the following discussion of the different Petrov types, the *Gauss tetrad* will refer to the tetrad in which the second fundamental forms assume the canonical form associated with the specified generalized Segre type [4] of the (Ω, Λ) pairs.

2 Petrov type D

Whereas for the Segre types $[(1111)_T, (11)_T(11)_S]$, $[(211)_N, (11)_S2]$ the Codazzi equations admit no solutions, the canonical expressions for the second fundamental forms for the type $[(111)_T1,\ (11)_T11]$ allow one to deduce from the Gauss equation and the Bianchi identity $D\psi_2 = \rho\psi_2$ that $\rho = 0$: solutions belong then to Kundt's class, for which (in the standard Kinnersley tetrad [2]) the metrics are given by

$$\omega^1 = \frac{1}{2\xi}dx - \frac{i\xi}{2}dy + \frac{2ia\xi r}{x^2+a^2}dw \ ,\ \omega^3 = dw \ ,\ \omega^4 = dr + \frac{lr^2}{2a(x^2+a^2}dw - \frac{2xr}{x^2+a^2}dx$$

with $\xi = [2amx + l(a^2 - x^2)/2a(x^2+a^2)]^{1/2}$. The general (Kinnersley IVa) case is easily excluded and one only has to consider the limit $a \to 0$ with $l = -2Ca$ (the so called Ehlers-Kundt or Kinnersley IVb metrics), for which one can show that the Codazzi equations form an integrable system.

In the remaining 3 Segre types $[(111)_T1, (11)_S11]$, $[(111)_S1, (11)_S11]$ and $[(11)_T(11)_S, Z\overline{Z}(11)_S]$ one can construct a canonical Ω and Λ expression for each, after which the Codazzi equations imply $\kappa = \lambda = \sigma = \nu = 0$ (such that ψ_2 is the only non-zero Weyl coefficient) and

$$\tau = \pi = \alpha + \overline{\beta}\ ,\ \overline{\rho} = \rho\ ,\ \overline{\mu} = \mu \tag{2.1}$$

with ρ and $\mu \neq 0$. Solutions therefore belong to the Robinson-Trautman family, but are further restricted by the fact that the conditions 2.1 hold in a tetrad where only ψ_2 is non-zero: this excludes the C-metric and we are left with the Schwarzschild metric and its generalizations. These metrics are known [3] to be of embedding class 2 and one can indeed show that they can be embedded in three algebraically inequivalent ways.

3 Petrov type N

In each of the 5 possible Gauss tetrads for type N, ψ_4 is the only non-vanishing Weyl coefficient, such that the null congruence $\mathbf{e}_4$ is geodesic and shearfree: $\kappa = \sigma = 0$. Analyzing the

Codazzi equations shows that only two cases may lead to embedding class 2 solutions which are *not* pp-waves.

In the first case ($[(111)_T1,(21)_N1]'$) solutions, if they exist, will necessarily belong to the Cahen-Spelkens family [1],

$$\omega^1 = d\zeta \ , \ \omega^3 = dw \ , \ \omega^4 = dv + Wd\zeta + \overline{W}d\overline{\zeta} + Hdw \tag{3.1}$$

with $W = -2v/\zeta + \overline{\zeta}$, $H = -v^2/(\zeta+\overline{\zeta})^2 + H^0(w,\zeta,\overline{\zeta})$.

Using the fact that the original Gauss tetrad is connected to the tetrad 3.1 by a sequence of null rotations, spatial rotations and boosts, one can write down an explicit expression for the second fundamental forms corresponding to 3.1.

Using the simplifications on the spin coefficients resulting from 3.1,

$$\begin{aligned} \kappa &= \sigma = \mu = \lambda = \rho = \epsilon = 0 \\ \beta &= \alpha = \frac{1}{2}\tau = -\frac{1}{2}\pi \ , \ \overline{\gamma} = \gamma \end{aligned} \tag{3.2}$$

one deduces from the Codazzi equations and the Bianchi identities a restriction on the Cahen-Spelkens curvature,

$$\overline{\delta}\log\psi_4 = -\pi(1-2(\psi_4/\overline{\psi_4})^{1/2}), \tag{3.3}$$

an inspection of which shows that the function H^0 is given by

$$H^0(x,y,w) = ax\log(x^2+y^2) + bxy + cx^2 + dx + e \tag{3.4}$$

with a, b, c, d and e functions of w only. Herewith it is easy to verify that the remaining Codazzi equations form an integrable system. The only Cahen-Spelkens metric of embedding class 2 is therefore given by 3.1, 3.4.

In the remaining case ($[(11)_S2,112]'$) one obtains from the Codazzi equations the following simplifications on the spin coefficients,

$$\kappa = \sigma = \tau = \pi = \epsilon = \overline{\rho} - \rho = \overline{\mu} - \mu \tag{3.5}$$

after which the Newman-Penrose equations imply that $\lambda = 0$. Solutions belong then to the Robinson-Trautman type N family, given by

$$\omega^1 = xP^{-1}d\zeta \ , \ \omega^3 = dw \ , \ \omega^4 = dx + Hdw \tag{3.6}$$

As in this tetrad ψ_4 is the only non-zero Weyl coefficient and $\tau = 0$, $\rho \neq 0$, it will differ from the Gauss tetrad by at most spatial rotations and boosts. This enables one to write down again an explicit expression for the second fundamental forms and, after inserting in the Codazzi equations the restrictions on the spin coefficients corresponding to 3.6, it follows that these equations are incompatible with the type N condition.

The only type N vacua of embedding class 2 are therefore given by the pp-waves and by the special Cahen-Spelkens metrics of the previous paragraph.

4 Petrov types I and II

For Petrov types I and II the only results obtained so far can be summarized as follows:

The Codazzi equations admit *no solutions* for the canonical types $[(11)_T(11)_S,\ 112]$ and $[(11)_T(11)_S,\ [(11)_T(11)_S]$.

For Petrov type II only one canonical type survives ($[(11)_S2,\ 112]''$). The second fundamental forms are then given by

$$\begin{aligned}
\boldsymbol{\Omega} &= \pm\,\boldsymbol{\omega}^1 \otimes \boldsymbol{\omega}^1 + R_3(-2\boldsymbol{\omega}^1 \otimes \boldsymbol{\omega}^2 - 2\boldsymbol{\omega}^2 \otimes \boldsymbol{\omega}^1 \\
&\quad + \boldsymbol{\omega}^3 \otimes \boldsymbol{\omega}^3 + \boldsymbol{\omega}^4 \otimes \boldsymbol{\omega}^4) \\
\boldsymbol{\Lambda} &= M\boldsymbol{\omega}^1 \otimes \boldsymbol{\omega}^1 + S_3\boldsymbol{\omega}^3 \otimes \boldsymbol{\omega}^3 + S_4\boldsymbol{\omega}^4 \otimes \boldsymbol{\omega}^4
\end{aligned} \tag{4.1}$$

and, although the Codazzi equations lead to considerable simplifications on the Newman-Penrose spin coefficients ($\kappa = \sigma = \lambda = \rho = \mu = \epsilon = 0$ and $\overline{\gamma} = \gamma$, $\tau = -\overline{\pi}$), the integrability of these equations still remains an open question.

References

[1] M. Cahen and J. Spelkens, *Bull. Acad. Roy. Belgique Cl. Sci.* **53** 817 (1967)

[2] W. Kinnersley, *J. Math. Phys.* **10** 1195 (1969)

[3] D. Kramer, H. Stephani, M.A.H. MacCallum and E. Herlt, *Exact solutions of Einstein's field equations* (V.E.B. Berlin , 1980)

[4] N. Van den Bergh, *Lorentz- and hyperrotation-invariant classification of symmetric tensors and the embedding class 2 problem*, *Class. Quantum Grav.* (to be published)

[5] M.Sh. Yakupov, *Grav. i Teor. Otnos.* Univ. Kazan **9** 109 (1973)