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On Preparata's Theory of a Superradiant Phase Transition

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Abstract Expressing the non-relativistic matter-radiation coupling in eigenmodes within dipole approximation the resulting equations of motion are analyzed. Specific stationary oscillating solutions at resonance in two-level approximation are found for which conditions are given leading to a minimum of the total energy that is lower than the energy of the non-interacting groundstate. The main result describing superradiance without population inversion is compared with Preparata's formulation.

1 Introduction

In a recent fascinating little book Giuliano Preparata has reviewed his work on a new theory of matter in which the vacuum fluctuations of the electromagnetic field couple to an internal resonance of the matter system such that this coupling gives rise, in certain cases, to a non-perturbative, "superradiant", groundstate. [1] This program is quite ambitious since Preparata hopes to explain in this way the known collective phenomena of condensed matter such as the Mössbauer effect, superconductivity, superfluidity, ferromagnetism, the particularities of water and more. Proceeding by analogy, replacing the electromagnetic by the pion field he also offers an explanation of the shell model of nuclei and other strong-coupling effects.

Since if true, Preparata's claims have far-reaching consequences for the understanding of the physics of condensed matter, it is of importance to subject this theory to an independent examination. This is the purpose of the present paper in which Preparata's elegant field-theoretic and path-integral methods are replaced by more conventional means. I will follow Preparata's exposition described in Chapters 1 to 3 of his book as well as in earlier lecture notes [2] in its essential steps while using an independent formulation and notation.

"Superradiance", the phase transition of the radiation field coupled to a matter system, has a long and controversial history going back to the seminal paper by Dicke of 1954 where this term was coined and where the "Dicke Hamiltonian" was introduced which consists in a two-level system coupled to the single electromagnetic mode resonating with the system. [3] The mathematical problems related with this model have been investigated in depth by Günter Scharf [4] and by Klaus Hepp and Elliott Lieb. [5] The last two authors have determined in particular the groundstate of the Dicke Hamiltonian which is also the problem addressed in the present paper.

But before being able to enter the subject we must clear away a serious roadblock which has the form of a "no-go theorem". [6] In Ref. 6 the well-known fact is derived that it is always possible to *locally* gauge away the vector potential $\mathbf{A}$. This follows from the identity (see, e.g. Eq. (31.3) of Ref 7)

$$e^{ie\phi(\mathbf{r},t)/\hbar c}[\mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{r},t)]e^{-ie\phi(\mathbf{r},t)/\hbar c} = \mathbf{p} + \frac{e}{c}\big(\mathbf{A}(\mathbf{r},t) - \nabla\phi(\mathbf{r},t)\big) \ , \tag{1.1}$$

valid for any gauge field ϕ, by choosing, for any path through $\mathbf{r}$, $\phi(\mathbf{r},t) = \int^{\mathbf{r}} \mathbf{A}(\mathbf{r}',t)\cdot d\mathbf{r}'$.

For the matter system the dipole approximation then implies that one may have $\mathbf{A} = 0$ at a given atom but certainly not in a whole "coherence domain" (Ref. 1, Sec. 3.1) of the size of the wavelength λ of the resonant radiation which is supposed to contain a large number N of atoms. For the physics of the radiation, however, enforcing $\mathbf{A} = 0$ in one space-time point is of no relevance since the radiation energy density $\frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2)$ is non-zero even at this point, as follows from the relations

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} \ ; \mathbf{B} = \nabla \times \mathbf{A} \tag{1.2}$$

and from the fact that the resonant radiation has a non-zero frequency $\Omega = \frac{2\pi c}{\lambda}$. Here and in what follows the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0 \ , \tag{1.3}$$

valid in the absence of sources, is used.

2 The coupled matter-radiation system

We define the matter system by the non-relativistic one-particle Hamiltonian

$$H = \frac{1}{2m}\big(\mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{r})\big)^2 + W(\mathbf{r}) \tag{2.1}$$

where the radiation-field operator $\mathbf{A}$ is written in Schrödinger representation and the field-free part H_0 determines orthonormal states $|\phi_n\rangle$ through the Schrödinger equation

$$H_0\phi_n(\mathbf{r}) = \varepsilon_n\phi_n(\mathbf{r}) \ . \tag{2.2}$$

Here $\mathbf{r}$ may also stand for other degrees of freedom and n may be a composite index. In what follows we assume the matter system to consist of one-electron atoms defined by the potential W which in a crystal may have the periodicity of the lattice.

In a second-quantized form, defined by the matter-field operator

$$\psi(\mathbf{r}) = \sum_n c_n\phi_n(\mathbf{r}) \tag{2.3}$$

where the c_n satisfy anti-commutation relations, the matter system is described by the Hamiltonians

$$\begin{aligned}
\mathcal{H}_0 &= \langle\psi|H_0|\psi\rangle = \sum_n \varepsilon_n c_n^+ c_n \ ,\\
\mathcal{H}_1 &= \langle\psi|\frac{e}{2mc}(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p})|\psi\rangle = \frac{e}{c}\sum_{nn'}\int_V d^3r\mathbf{A}(\mathbf{r})\cdot\mathbf{j}_{nn'}(\mathbf{r})c_n^+c_{n'} \ ,\\
\mathcal{H}_2 &= \langle\psi|\frac{e^2}{2mc^2}\mathbf{A}^2|\psi\rangle = \frac{r_e}{2}\sum_{nn'}\langle\phi_n|\mathbf{A}^2|\phi_{n'}\rangle c_n^+c_{n'} \ .
\end{aligned} \tag{2.4}$$

Here V is taken as the coherence volume, $r_e \equiv \frac{e^2}{mc^2}$ is the classical electron radius and

$$\mathbf{j}_{nn'}(\mathbf{r}) = \frac{\hbar}{2mi}(\phi_n^*\nabla\phi_{n'} - \phi_{n'}\nabla\phi_n^*) \tag{2.5}$$

the matrix element of the current density.

To Eq. (2.4) must be added the radiation Hamiltonian

$$\mathcal{H}_r = \frac{1}{8\pi}\int_V d^3r(\mathbf{E}^2+\mathbf{B}^2) \tag{2.6}$$

where quantization is defined by the development of the vector-potential into eigenmodes of the volume V,

$$\mathbf{A}(\mathbf{r}) = \sum_k \sqrt{\frac{4\pi\hbar c^2}{\omega_{\mathbf{k}}V}}Q_k\mathbf{e}_k e^{i\mathbf{k}\cdot\mathbf{r}} \ . \tag{2.7}$$

Here $\omega_{\mathbf{k}} = c|\mathbf{k}|$ and

$$Q_k \equiv \frac{1}{\sqrt{2}}(a_k + a_{-k}^+) = Q_{-k}^+ \tag{2.8}$$

where a_k^+ and a_k are the creation and annihilation operators, respectively, and k is a composite index defined by $\pm k = (\pm\mathbf{k}, \pm\lambda)$ where $\lambda = \pm$ is the polarization index defined such that the polarization vectors $\mathbf{e}_k = \mathbf{e}_k^*$ satisfy

$$\mathbf{e}_k = \mathbf{e}_{-k} \ ; \mathbf{k}\cdot\mathbf{e}_k = 0 \ ; \mathbf{e}_{\mathbf{k}\lambda}\cdot\mathbf{e}_{\mathbf{k}\lambda'} = \delta_{\lambda\lambda'} \ . \tag{2.9}$$

Inserting Eq. (2.7) into (2.6) and making use of (1.2), (2.9) one finds

$$\mathcal{H}_r = \frac{\hbar}{2}\sum_k\{\frac{1}{\omega_{\mathbf{k}}}\dot{Q}_k\dot{Q}_{-k} + \omega_{\mathbf{k}}Q_kQ_{-k}\} \tag{2.10}$$

where the first and second terms come from $\mathbf{E}^2$ and $\mathbf{B}^2$, respectively.

The Hamiltonian $\mathcal{H}_2$ in Eq. (2.4) deserves special examination because it may be understood as a renormalization of $\mathcal{H}_r$ (Ref. 1, Sec. 3.4). Making use of Eq. (2.7),

$$\langle\phi_n|\mathbf{A}^2|\phi_{n'}\rangle = \frac{4\pi\hbar c^2}{V}\sum_{kk'}\frac{\mathbf{e}_k\cdot\mathbf{e}_{k'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}Q_kQ_{k'}\langle\phi_n|e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}|\phi_{n'}\rangle\ . \tag{2.11}$$

Here the matrix element of the matter system may be treated in dipole approximation which means that we may set $\mathbf{k}+\mathbf{k}' \simeq 0$. Inserting (2.11) in $\mathcal{H}_2$ one then finds with $\sum_n c_n^+ c_n = N$, which is the number of electrons in the volume V,

$$\mathcal{H}_2 \simeq 2\pi\hbar c^2 r_e\frac{N}{V}\sum_k\frac{1}{\omega_{\mathbf{k}}}Q_kQ_{-k}\ . \tag{2.12}$$

Hence

$$\mathcal{H}_r + \mathcal{H}_2 \simeq \frac{\hbar}{2}\sum_k\frac{1}{\omega_{\mathbf{k}}}\{\dot{Q}_k\dot{Q}_{-k} + c^2[\mathbf{k}^2+\kappa^2]Q_kQ_{-k}\} \tag{2.13}$$

where

$$\kappa \equiv \sqrt{4\pi r_e\frac{N}{V}} \tag{2.14}$$

is the analogue of the reciprocal London penetration depth in the theory of superconductivity (see, e.g. Eq. (25.21) of Ref. 7). κ or, equivalently the mass $\frac{\hbar\kappa}{c}$, implies that Eq. (2.13) gives rise to an equation of motion having the form of the Klein-Gordon equation (see Eq. (3.2) below for $\Lambda_{10} = 0$).

Taking for the average distance between the electrons (atoms) $d = (V/N)^{1/3} \sim 10^{-7}cm$ and for the wavelength $\lambda = 2\pi/|\mathbf{k}| \sim 10^{-4}cm$ so that in a coherence volume V there are $N \sim 10^9$ atoms one finds $\kappa/|\mathbf{k}| \sim 1$. Note that for free but extended particles with no internal degrees of freedom ("one-level atoms") it is $\mathcal{H}_1$ which may be renormalized away by a Bloch-Nordsieck transformation leading to a photon-pair theory. [8] These different treatments of the radiation field are possible because in dipole approximation there is no gauge invariance. [9]

The dynamics is described by equations of motion

$$\dot{O} = \frac{i}{\hbar}[\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_r, O]\ . \tag{2.15}$$

for "observables" O. In the case of the radiation field it is necessary first to define canonical momenta. One easily verifies that the relations

$$P_k \equiv \frac{1}{\omega_{\mathbf{k}}}\dot{Q}_{-k}\ ; i[P_k, Q_{k'}] = \delta_{kk'} \tag{2.16}$$

are compatible with Eq.(2.15) for $O = Q_{-k}$.

In Eq. (2.15) we also need $\mathcal{H}_1$ which may be deduced from Eqs. (2.4) by inserting (2.5), (2.7). One finds

$$\mathcal{H}_1 = -i\hbar \sum_{nn'} \sum_{k} \Lambda_{nn'}(k) Q_{-k} c_n^+ c_{n'} \tag{2.17}$$

where the coupling function

$$\Lambda_{nn'}(k) \equiv ec\sqrt{\frac{4\pi}{\hbar\omega_{\mathbf{k}} V}} \mathbf{e}_k \cdot \mathbf{v}_{nn'}(\mathbf{k}) \ , \tag{2.18}$$

is introduced. Here the dimensionless matrix element

$$\mathbf{v}_{nn'}(\mathbf{k}) \equiv \frac{i}{c} \int_V d^3r \mathbf{j}_{nn'}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \simeq \mathbf{v}_{nn'}(0) \tag{2.19}$$

does not depend on $\mathbf{k}$ in dipole approximation.

For the radiation field the relevant equation of motion (2.15) now is

$$\frac{1}{\omega_{\mathbf{k}}} \ddot{Q}_k = \dot{P}_{-k} = -\left(1 + \frac{\kappa^2}{\mathbf{k}^2}\right) \omega_{\mathbf{k}} Q_k + i \sum_{nn'} \Lambda_{nn'}(k) c_n^+ c_{n'} \ . \tag{2.20}$$

Here use was made of the approximate expression (2.13) and of the definition (2.18). For the matter field the equation of motion (2.15) is just the Schrödinger equation

$$i\dot{c}_n = \frac{\varepsilon_n}{\hbar} c_n - i \sum_{n'} \sum_{k} \Lambda_{nn'}(k) Q_{-k} c_{n'} \tag{2.21}$$

where the contribution from $\mathcal{H}_2$ was neglected since, according to Eq. (2.12), $-[\mathcal{H}_2, c_n] \simeq \mathcal{H}_2 c_n / N$ which is negligible for large N.

3 The dynamics in two-level approximation

The physically important condition is the existence of a resonance of the radiation with a specific transition frequency Ω of the matter system,

$$\omega_{\mathbf{k}} = \Omega \ . \tag{3.1}$$

As a consequence, only two atomic levels which we label $n = 0, 1$ and only the radiation modes satisfying (3.1) are assumed to be relevant. The energy levels defined in Eq. (2.2) may then be chosen such that $\varepsilon_0 = 0$ and $\varepsilon_1 = \hbar\Omega$. We further assume that there are no transport currents, $\mathbf{j}_{nn} = 0$, i.e. the ϕ_n are real. Then $i\mathbf{j}_{01} = -i\mathbf{j}_{10}$ and, according to (2.19) $\mathbf{v} \equiv \mathbf{v}_{10} = -\mathbf{v}_{01}$ is a constant real vector. The equations of motion (2.20) and (2.21) now become

$$\frac{1}{\omega_{\mathbf{k}}}\ddot{Q}_k + \omega_{\mathbf{k}}(1 + \frac{\kappa^2}{\mathbf{k}^2})Q_k = i\Lambda_{10}(k)(c_1^+ c_0 - c_0^+ c_1) \tag{3.2}$$

and

$$\dot{c}_0 = \Lambda_{10}(k)Q_{-k}c_1 \; ; \dot{c}_1 = -i\Omega c_1 - \Lambda_{10}(k)Q_{-k}c_0 \; . \tag{3.3}$$

In view of the fact that $\sum_n c_n^+ c_n = N$ is a very large number it is natural to rescale the dynamical variables which, in addition, may be considered to be classical. The resonance condition (3.1) then implies that the dynamics separates into fast oscillations by Ω and a slow motion which happens in the long time $\tau = \Omega t$. This separation is exhibited in the following "interaction representation" of the variables defined in Eqs. (2.3) and (2.7):

$$Q_k(t) = \sqrt{\frac{N}{2}}\left(\alpha_k(\tau)e^{-i\Omega t} + \alpha_{-k}^*(\tau)e^{+i\Omega t}\right) \tag{3.4}$$

and

$$c_0(t) = \sqrt{N}\gamma_0(\tau) \; ; c_1(t) = \sqrt{N}\gamma_1(\tau)e^{-i\Omega t} \tag{3.5}$$

where now

$$\gamma_0^*\gamma_0 + \gamma_1^*\gamma_1 = 1 \; . \tag{3.6}$$

Separation of the slow motion is obtained by equating all the coefficients of the fast motions $e^{\pm\Omega t}$ in Eqs. (3.1) and (3.2) to zero, a procedure called "rotating-wave approximation" in the literature (Ref. 1, Sec. 3.1). In the notation $\dot{\alpha}_k \equiv d\alpha_k/d\tau$ and $\dot{\gamma}_n \equiv d\gamma_n/d\tau$ the slow equations of motion then are

$$\frac{i}{2}\ddot{\alpha}_k + \dot{\alpha}_k + \frac{i}{2}\mu\alpha_k = \Lambda_k\gamma_0^*\gamma_1 \; , \tag{3.7}$$

valid for $\omega_{\mathbf{k}} = \Omega$ and

$$\dot{\gamma}_0 = {\sum_k}' \Lambda_k\alpha_k^*\gamma_1 \; ; \dot{\gamma}_1 = -{\sum_k}' \Lambda_k\alpha_k\gamma_0 \; . \tag{3.8}$$

Here we have introduced the dimensionless quantities $\mu \equiv \kappa^2/\mathbf{k}^2$ and

$$\Lambda_k \equiv \frac{1}{\Omega}\sqrt{\frac{N}{2}}\Lambda_{10}(k)\big|_{\omega_{\mathbf{k}}=\Omega} = \sqrt{\frac{\mu mc^2}{2\hbar\Omega}}\mathbf{e}\cdot\mathbf{v} \tag{3.9}$$

and the summation convention

$${\sum_k}' \equiv \frac{V}{(2\pi)^3}\sum_{\lambda=\pm}\int d^3k\Delta k\delta(|\mathbf{k}| - \frac{\Omega}{c}) \tag{3.10}$$

where $\Delta k \sim 2\pi V^{-1/3}$ is a spectral width. In view of (2.14), (3.9) it is then evident that the rescaled variables interact strongly with an effective coupling constant $e\sqrt{N}$.

For these equations of motion a first integral may be obtained in the following way: Multiply (3.7) by $\sum_k' \alpha_k^*$ and the first Eq. (3.8) by γ_0^*, equate the left-hand sides and add to the resulting equation its complex conjugate. The result is the time-derivative of the following equation:

$$\sum_k{}' \{|\gamma_0|^2 - |\alpha_k|^2 - \frac{i}{2}(\alpha_k^* \dot{\alpha}_k - \alpha_k \dot{\alpha}_k^*)\} = K_0 \ . \tag{3.11}$$

Similarly we obtain with the second Eq. (3.8)

$$\sum_k{}' \{|\gamma_1|^2 + |\alpha_k|^2 + \frac{i}{2}(\alpha_k^* \dot{\alpha}_k - \alpha_k \dot{\alpha}_k^*)\} = K_1 \tag{3.12}$$

where K_0 and K_1 are real constants. These integrals are not independent, however since, according to (3.6)

$$K_0 + K_1 = \sum_k{}' 1 = \frac{V \Delta k \Omega^2}{\pi^2 c^2} = 1 \ , \tag{3.13}$$

but

$$K \equiv K_1 - K_0 = \sum_k{}' \{|\gamma_1|^2 - |\gamma_0|^2 + 2|\alpha_k|^2 + i(\alpha_k^* \dot{\alpha}_k - \alpha_k \dot{\alpha}_k^*)\} \tag{3.14}$$

is an independent constant of the motion.

Inserting the "interaction representation" (3.4), (3.5) into the Hamiltonians (2.4), (2.13) and (2.17) one obtains the corresponding contributions to the total energy by averaging over the fast motion. Thus

$$E_0 = \hbar N \Omega |\gamma_1|^2 \tag{3.15}$$

and

$$E_r + E_2 = \frac{\hbar}{2} N \Omega \sum_k{}' \{|\dot{\alpha}_k|^2 + i(\alpha_k^* \dot{\alpha}_k - \alpha_k \dot{\alpha}_k^*) + (2+\mu)|\alpha_k|^2\} \ . \tag{3.16}$$

Here an apparent simplification is to substitute the constant of the motion (3.14) for the second and part of the third terms. However, this mixes in the matter variables in a quite unsymmetrical fashion. But even more serious is the fact that it is practically impossible to know the value of K with any precision as is evident from the discussion following Eq. (4.11) below.

As for $\mathcal{H}_1$, elimination of the radiation degrees of freedom with the aid of Eqs. (3.8) leads to

$$E_1 = -i\hbar N \Omega (\dot{\gamma}_0 \gamma_0^* + \dot{\gamma}_1 \gamma_1^*) = \hbar N \Omega \Im (\dot{\gamma}_0 \gamma_0^* + \dot{\gamma}_1 \gamma_1^*) \tag{3.17}$$

where in the last step (3.6) was used.

In the "perturbative groundstate" defined by $\gamma_0 = 0$, $\gamma_1 = 0$ and $\alpha_k = 0$ (all k), the total energy is $E_{min}^{(o)} = 0$. On the other hand, superradiance implies the existence of a groundstate with $E_{min} < E_{min}^{(o)}$. In order to decide this question the dynamics (3.7), (3.8) has to be analysed in more detail. Eq. (3.8) shows that the only radiation mode coupling to the matter system is

$$\beta(\tau) \equiv \frac{1}{g}\sum_k{}' \Lambda_k \alpha_k(\tau) \tag{3.18}$$

where g is the new coupling constant defined in Eq. (3.21) below. The projected Eq. (3.7) is

$$\frac{i}{2}\ddot{\beta} + \dot{\beta} + \frac{i}{2}\mu\beta = 2g\gamma_0^*\gamma_1 \tag{3.19}$$

while Eqs. (3.8) become

$$\dot{\gamma}_0 = g\beta^*\gamma_1 \;; \dot{\gamma}_1 = -g\beta\gamma_0 \; . \tag{3.20}$$

Here

$$g^2 \equiv \frac{1}{2}\sum_k{}' \Lambda_k^2 \; , \tag{3.21}$$

so that g is real. Note that (3.19), (3.20) agree with Eqs. (3.9) of Ref. 1 if γ_0, γ_1, β, μ and g are identified with Preparata's χ_2, $i\chi_1$, $\sqrt{2}A$, 2μ and $g/\sqrt{2}$, respectively.

These slow modes β, γ_0 and γ_1 obey a constant of the motion in addition to the normalization (3.6). In complete analogy to the derivation of Eqs. (3.11), (3.12) one here finds

$$2|\gamma_0|^2 - |\beta|^2 - \frac{i}{2}(\beta^*\dot{\beta} - \beta\dot{\beta}^*) = 2\Delta_0 \tag{3.22}$$

and

$$2|\gamma_1|^2 + |\beta|^2 + \frac{i}{2}(\beta^*\dot{\beta} - \beta\dot{\beta}^*) = 2\Delta_1 \tag{3.23}$$

where Δ_0 and Δ_1 are again real constants which, because of (3.6) satisfy $\Delta_0 + \Delta_1 = 1$. The independent constant of the motion therefore is

$$\Delta \equiv \Delta_1 - \Delta_0 = |\gamma_1|^2 - |\gamma_0|^2 + |\beta|^2 + \frac{i}{2}(\beta^*\dot{\beta} - \beta\dot{\beta}^*) \; . \tag{3.24}$$

4 Conditions for superradiance

A useful equation which is homogeneous in β is obtained by taking the time derivative of (3.19) and substituting $\dot{\gamma}_0$ and $\dot{\gamma}_1$ from (3.20),

$$i\dddot{\beta} + 2\ddot{\beta} + i\mu\dot{\beta} = -4g^2\beta\cos\theta \; . \tag{4.1}$$

Here we have introduced the parametrization $|\gamma_0| = \cos\frac{\theta}{2}$ and $|\gamma_1| = \sin\frac{\theta}{2}$ so that

$$|\gamma_0|^2 - |\gamma_1|^2 = \cos\theta \; . \tag{4.2}$$

$\theta > 0$ then measures the degree of excitation of the matter system and $\theta > \frac{\pi}{2}$ signifies a population inversion.

We first consider the short-time behavior, $\tau = \Omega t < 1$, of Eq. (4.1). Assuming as initial state photon vacuum but allowing for the moment some atomic excitation, $\theta > 0$, the initial time evolution given by Eq. (4.1) has the form $\beta \propto e^{-ip\tau}$ where p is determined by the characteristic equation

$$f(p) \equiv p(p^2 + 2p - \mu) = 4g^2 \cos\theta \ . \tag{4.3}$$

Since $f(p)$ has extrema at $f_\pm(\mu) \equiv f(p_\pm(\mu))$ where $p_\pm$ are the roots of $f'(p) = 0$, Eq. (4.3) has only real solutions provided that the right-hand side of (4.3) lies in the interval

$$f_+(\mu) < 4g^2 \cos\theta < f_-(\mu) \ . \tag{4.4}$$

The initial, aperiodic, evolution must therefore lie outside of the interval (4.4). Numerically, $f_-(0) = \frac{32}{27} = 1.185$, $f_+(0) = 0$ and $f_-(1) = 2.63$, $f_+(1) = -0.113$. Thus $4g^2 \cos\theta < f_+(\mu) \leq 0$ requires a population inversion $\theta > \frac{\pi}{2}$ while for $4g^2 \cos\theta > f_-(\mu) \geq \frac{32}{27}$ the system evolves even for $\theta = 0$, provided that

$$g^2 > \frac{8}{27} \ . \tag{4.5}$$

In these outer domains $f(p)$ has one real and two complex conjugate solutions, $p = r \pm is$, for which $\beta \propto e^{(-ir\pm s\tau)}$. Hence the system possesses a run-away instability which is a necessary condition for reaching a superradiant groundstate starting from the perturbative groundstate $\theta = 0$.

We next investigate the oscillating stationary states that this run-away solution may eventually reach by writing

$$\beta = Be^{i(\varphi - \nu\tau)} \ ; \gamma_0 = \cos\frac{\theta}{2} e^{-i\nu_0\tau} \ ; \gamma_1 = \sin\frac{\theta}{2} e^{i(\chi - \nu_1\tau)} \tag{4.6}$$

with real positive amplitude B. This gives for the constant of the motion (3.24)

$$\Delta = B^2(1 + \nu) - \cos\theta \tag{4.7}$$

while (4.1) becomes (4.3) with $p = \nu$,

$$f(\nu) = 4g^2 \cos\theta \tag{4.8}$$

where, however, we are interested in real values of ν. Insertion of (4.6) into Eqs. (3.20) yields the remaining equations,

$$\nu_0 = gB \tan\frac{\theta}{2} \ ; \nu_1 = gB \cot\frac{\theta}{2} \tag{4.9}$$

and

$$\nu + \nu_0 - \nu_1 = 0 \; ; \varphi - \chi = \pm\frac{\pi}{2} \; . \tag{4.10}$$

Combination of (4.9) and the first Eq. (4.10) gives

$$\nu = 2gB\cot\theta \; ; \nu = \pm|\nu| \; ; \theta = \pm|\theta| \; . \tag{4.11}$$

In principle, Eqs. (4.7), (4.8) and (4.11) determine the amplitude B and the atomic excitation θ as functions of the coupling constant g, the "mass" $\sqrt{\mu}$ and the constant of the motion Δ. Since, physically, the system starts to evolve from the perturbative groundstate one is tempted to conclude from Eq. (3.24) that $\Delta = -1$. This, however, is questionable since the classical motion emerges from the initial quantum motion so that the initial value of Δ is unsharp.

The crucial question now is, what does the solution (4.6)-(4.11) imply for the energies (3.15)-(3.17). The result for (3.15) and (3.17) is simple since it is expressed entirely in terms of the matter variables:

$$E_{mat} \equiv E_0 + E_1 = \frac{\hbar}{2}N\Omega\{1 - \cos\theta - 2gB\sin\theta\} \; . \tag{4.12}$$

On the other hand, the motion of the radiation variables α_k is best described by the analogue of Eq. (3.19). Taking the time derivative of (3.7) and inserting Eqs. (3.8) one obtains

$$i\dddot{\alpha}_k + 2\ddot{\alpha}_k + i\mu\dot{\alpha}_k = -2g\Lambda_k\beta\cos\theta \; . \tag{4.13}$$

Again we are interested in an oscillatory stationay state as in (4.6),

$$\alpha_k = A_k e^{-i\nu_k\tau} \tag{4.14}$$

with real ν_k. Now, since Eq. (3.18) is supposed to be valid for all times τ, β and α_k must oscillate with the same frequency, i.e. according to (4.6), $\nu_k = \nu$ (all k). Inserting (4.14) with this restriction in (4.13) the result is

$$f(\nu)A_k = 2g\Lambda_k Be^{i\varphi}\cos\theta \; . \tag{4.15}$$

Comparison with (4.8) then gives

$$A_k = \frac{\Lambda_k}{2g}Be^{i\varphi} \tag{4.16}$$

which, inserted together with (4.14) and $\nu_k = \nu$ in (3.16), yields

$$E_{rad} \equiv E_r + E_2 = \frac{\hbar}{4}N\Omega\{(\nu+1)^2 + 1 + \mu\}B^2 \tag{4.17}$$

where use was made of (3.21). Since $E_{rad} \geq 0$ the existence of a superradiant groundstate demands that $E_{mat} < 0$. Note that in Ref. 1 the constant of the motion $Q = \frac{\Delta+1}{2}$ is inserted in Eq. (3.15) for the total energy.

The convenient parameter to analyse the total energy (4.12), (4.17) turns out to be the frequency ν. This means that the atomic excitation $\cos\theta$ has to be expressed with the aid of (4.8) as

$$\cos\theta(\nu) = \frac{f(\nu)}{4g^2} \; ; \nu = \pm|\nu| \tag{4.18}$$

while the amplitude B is obtained by substituting this relation in (4.11),

$$B(\nu) = \frac{\nu}{2g} \tan\theta(\nu) \tag{4.19}$$

and the constant of the motion (4.7) becomes

$$\Delta = \frac{\nu^2(1+\nu)}{4g^2} \tan^2\theta(\nu) - \cos\theta(\nu) \; . \tag{4.20}$$

But since the initial value of Δ is expected to be unsharp we are not using this equation, leaving ν as a free parameter. (Note that another possibility would be to use (4.20) in extremal form, $\delta\Delta = 0$.)

With (4.18), (4.19) the total energy may now be expressed as

$$\varepsilon \equiv \frac{2}{\hbar N\Omega}(E_{mat} + E_{rad}) = [1 - \cos\theta(\nu)]\{1 - \frac{\psi(\nu)}{\varphi^2(\nu)} 2g^2[1 + \cos\theta(\nu)]\} \tag{4.21}$$

where $\varphi(\nu) \equiv \nu^2 + 2\nu - \mu = f(\nu)/\nu$ and $\psi(\nu) \equiv \nu^2 + 2\nu - 2 - 3\mu = \varphi(\nu) - 2(1+\mu)$. Eq. (4.21) shows that a necessary condition for $\varepsilon < 0$ to hold is $\psi(\nu) > 0$. This is the case for $\nu > \nu_{3+}$ and for $\nu < \nu_{3-}$ where $\nu_{3\pm}$ are the zeros of $\psi(\nu)$,

$$\nu_{n\pm} \equiv -1 \pm \sqrt{n(1+\mu)} \tag{4.22}$$

and the zeros of $\varphi(\nu)$ are $\nu_{1\pm}$. The condition for a negative total energy as obtained from Eq. (4.21) reads

$$\frac{\varphi^2(\nu)}{\psi(\nu)} < 2g^2 + \frac{\nu}{2}\varphi(\nu) \; ; \psi(\nu) > 0 \; . \tag{4.23}$$

Now it is not difficult to show that in the outer regions $\nu > \nu_{3+}$ and $\nu < \nu_{3-}$ the function $\varphi^2(\nu)/\psi(\nu)$ is parabola-like with minima at $\nu = \nu_{5\pm}$ and that $\varphi(\nu_{5\pm}) = 4(1+\mu)$ and $\psi(\nu_{5\pm}) = 2(1+\mu)$. Hence

$$\min\Big(\frac{\varphi^2(\nu)}{\psi(\nu)}\Big) = \frac{\varphi^2(\nu_{5\pm})}{\psi(\nu_{5\pm})} = 8(1+\mu) \; ; \nu > \nu_{3+} \textit{ or } \nu < \nu_{3-} \; . \tag{4.24}$$

We may take $\nu = \nu_{5\pm}$ as representative points rather than determine the minimum value of g^2 from (4.23) or that of ε from (4.21). Then, using definition (4.22), Eqs. (4.18) and (4.23) become, respectively,

$$\cos\theta(\nu_{5\pm}) = \frac{1+\mu}{g^2}(-1 \pm \sqrt{5(1+\mu)}) \tag{4.25}$$

and

$$g^2 > 5(1+\mu)\left[1 \mp \sqrt{\frac{1+\mu}{5}}\right] . \tag{4.26}$$

Here the upper value of the bound is larger than the one in Eq. (4.5) for $\mu < 3.88$. But it is still much smaller than the lower value which corresponds to a population inversion of the atomic system, $\cos\theta < 0$. This shows that population inversion is unfavorable for superradiance, in striking difference to the laser.

A somewhat simpler but less explicit form of the total energy (4.21) is obtained by using as parameter the atomic excitation θ. Substituting in Eq. (4.8) $g\cos\theta$ with the help of (4.11) one arrives at an equation which is only quadratic in ν,

$$(\nu+1)^2 = 2g\frac{\sin\theta}{B} + 1 + \mu \tag{4.27}$$

where, however, B is a complicated function of θ. Four of the eight solutions $\nu = \pm|\nu|$ of Eq. (4.27) are given in Preparata's Eq. (3.25c) where 2α is our θ and ϵ our sign of $\theta = \pm|\theta|$. From Eqs. (4.12), (4.17) one now deduces the following expression for the total energy:

$$\varepsilon = 1 - \cos\theta - gB\sin\theta + (1+\mu)B^2 . \tag{4.28}$$

This expression corresponds to Preparata's $2H$ given by his Eqs. (3.23), (3.30), provided that the appropriate signs ϵ are chosen.

Considering now B as a free parameter the minimum of ε is situated at

$$B = \frac{g\sin\theta}{2(1+\mu)} \tag{4.29}$$

and has the value

$$\varepsilon_{min} = 1 - \cos\theta - \frac{g^2\sin^2\theta}{4(1+\mu)} . \tag{4.30}$$

Hence a necessary condition for superradiance is

$$g^2 > \frac{4(1+\mu)}{1+\cos\theta} \tag{4.31}$$

which again shows that population inversion, $\theta > \pi/2$, is unfavorable for superradiance.

Surprisingly, if the value (4.29) is inserted into Eq. (4.27) one recovers $\nu = \nu_{5\pm}$ which, more precisely, are the four values $\nu = \pm|\nu_{5\pm}|$. This means that the renormalized frequency $\Omega_{ren} = |1+\nu|\Omega$ has the values $\sqrt{5(1+\mu)} + 2n$ where $n = 0, \pm 1$ and, for $n = -1$, is smaller than Ω if $\mu < 4/5$. Note that Preparata's relation (3.33) agrees with this expression for $n = 0$, provided that care is taken in choosing the appropriate signs ϵ, and hence, obeys $\Omega_{ren} > \Omega$ ($\omega > \omega_0$ in his notation).

5 Conclusion

The results of the last section confirm Preparata's conclusion that, at least in two-level approximation, a superradiant groundstate below the perturbative groundstate exists for sufficiently strong coupling $g^2 \propto Ne^2$, i.e. if the number N of atoms contained in the coherence volume $V \sim \lambda^3$ is large enough. He is also right in emphasizing that, in distinction to the laser, superradiance does not require a population inversion of the atomic system but occurs preferentially at low excitation.

An assertion which is more delicate to assess is Preparata's claim that superradiance is self-contained, i.e. that there is total reflection at the boundary of the matter system. If true this property would preclude any detection of superradiant photons outside the system. This is the most crucial point because such a detection, I believe, would represent the only possibility of an experimental verification.

Now, the system will radiate at a frequency $\Delta\Omega \equiv N\Omega(\varepsilon/2) + \Omega_{ren}$, provided that $\Delta\Omega > 0$. Thus the condition which allows a collective phenomenon in condensed matter to be explained in terms of superradiance is even more severe than the existence of a negative groundstate energy, $\varepsilon < 0$, namely $\Delta\Omega < 0$. However, if the number N of resonators in the coherence volume satisfies $N \gg 1$, the two conditions merge, provided that ε is of order one.

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