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Determination of the chiral pion-pion scattering parameters: a proposal

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Abstract. An explicitly crossing-symmetric decomposition of the pion-pion scattering amplitudes into low- and high-energy components is established. The high-energy components are entirely determined by absorptive parts at high energies. They impose constraints on the behavior of the low-energy amplitudes. The use of these constraints for the determination of the parameters in the one- and two-loop amplitudes is proposed.

1 Introduction and statement of results

Chiral perturbation theory of the meson sector is an effective field theory providing a successful description of low-energy strong interaction processes in terms of expansions in powers of external momenta and quark masses [1]. These expansions are derived from an effective Lagrangian which is itself a series of powers of the pion field, its derivatives and the quark mass matrix. It contains effective coupling constants whose number increases dramatically as one proceeds from the leading terms to higher order corrections [2]. The problem of determining the values of these coupling constants arises. As one is dealing with an effective low-energy theory, part of the coupling constants is meant to encode the low-energy manifestations of high-energy phenomena, therefore it must be possible to relate some of the coupling constants to the characteristics of such phenomena. In the present context, high energies are of course very modest, starting around 500 MeV. Order of magnitude estimates can be obtained by evaluating contributions of the high-energy states (resonances), by means of a Lagrangian describing these states and their coupling to the pion field [3]. This procedure amounts to saturating high-energy cross-sections by resonance contributions in a narrow width approximation. Other, potentially more accurate, methods are based on the use of dispersion relations which connect low- and high-energy processes [4, 5].

In this paper I follow the dispersion relation path and consider a special process, pion-pion scattering. I address the problem of determining the parameters appearing in the chiral amplitudes of this process. These parameters are related in a known way to the coupling constants of the chiral Lagrangian. More precisely, I am asking two questions:

1. Is it possible to decompose a pion-pion amplitude into a high- and a low-energy component? The high-energy component should be determined by high-energy absorptive parts and it should be possible to obtain the low-energy component from the chiral absorptive part.

2. Is it possible to determine unambiguously the chiral parameters and, consequently, the chiral coupling constants, with the aid of the high-energy components?

As crossing symmetry is a basic property of pion-pion scattering, I require the decomposition into low- and high-energy components to be explicitly crossing symmetric. If one works with dispersion relations, the main difficulty of question (1) comes from this last requirement. In fact, ordinary dispersion relations are not convenient tools and a technique developed thirty years ago turns out to be more appropriate [6]. It is based on the following considerations. The isospin I s-channel amplitude T^I is a function of the three Mandelstam variables s, t and u ($I = 0, 1, 2$) [7, 8]. Crossing symmetry dictates the transformation of the T^I under permutations of s, t and u. One defines three amplitudes G_i, linearly related to the T^I, which are totally symmetric functions of s, t, u ($i = 0, 1, 2$). Conversely the total symmetry of the G_i implies crossing symmetry for the T^I. When expressed in terms of appropriate new variables the G_i obey dispersion relations which do not spoil their symmetry properties. The low- and high-energy components L_i and H_i of G_i are obtained by splitting its dispersion integral into low- and high-energy parts. The L_i and H_i are totally symmetric and define a crossing symmetric decomposition of the T^I. Therefore the answer to question (1) is affirmative.

A strategy for fixing the chiral parameters is to adjust them in such a way that the chiral amplitudes T^I_χ are good approximations of the true amplitudes T^I at points where the chiral expansion has to be valid. I adopt and implement this strategy by requiring that truncated Taylor expansions of T^I and T^I_χ coincide at a conveniently chosen point where both amplitudes are regular. Points in the Mandelstam triangle $s < 4M_\pi^2$, $t < 4M_\pi^2$, $u < 4M_\pi^2$ are good candidates and I shall work with Taylor expansions around the symmetry point $s = t = u = 4M_\pi^2/3$. The outcome will be a set of equations relating the parameters of the higher-order terms of the chiral amplitudes to high-energy pion-pion scattering. Consequently the answer to question (2) is also affirmative to some extent.

As a by-product of my investigations I obtain upper bounds for the Taylor coefficients of the high-energy components at the symmetry point. They seem to be compatible with good convergence properties of the chiral expansion.

A similar method for the determination of pion-pion parameters has been developed in [4]. The main difference between this method and my proposal lies in a treatment of crossing symmetry which does not depend on the order of the chiral expansion.

The principal aim of this paper is to establish that the idea of obtaining restrictions on the chiral coupling constants from high-energy processes can be implemented precisely and unambiguously in the special case of pion-pion scattering. My technique cannot be extended to other processes in a straightforward way. Here, I am mainly interested in questions of principle and the practical application of my constraints is another task. Due to the poor shape of our information on high-energy pion-pion scattering, it is doubtful that they really can improve the results already obtained [4].

The paper is organized as follows. Section 2 contains an outline of the derivation of dispersion relations for the totally symmetric amplitudes G_i. Section 3 is technical: the analyticity properties allowing Taylor expansions in two variables around the symmetry point are established. The results are stated in two propositions. Constraints for the chiral coupling constants are derived in Section 4. Explicit equations up to the sixth order of the chiral expansion are written down. Technical details are presented in two Appendices.

2 Dispersion relations for totally symmetric amplitudes

High energy components of the pion-pion amplitudes T^I will be defined with the help of three totally symmetric functions $G_i(s,t,u)$:

$$\begin{aligned} G_0(s,t,u) &= \frac{1}{3}\left(T^0(s,t,u)+2T^2(s,t,u)\right), \\ G_1(s,t,u) &= \frac{T^1(s,t,u)}{t-u}+\frac{T^1(t,u,s)}{u-s}+\frac{T^1(u,s,t)}{s-t}, \\ G_2(s,t,u) &= \frac{1}{s-t}\left(\frac{T^1(s,t,u)}{t-u}-\frac{T^1(t,s,u)}{s-u}\right) \\ &\quad +\frac{1}{t-u}\left(\frac{T^1(t,u,s)}{u-s}-\frac{T^1(u,t,s)}{t-s}\right)+\frac{1}{u-s}\left(\frac{T^1(u,s,t)}{s-t}-\frac{T^1(s,u,t)}{u-t}\right). \end{aligned} \tag{2.1}$$

The Mandelstam variables will be expressed in units of M_π^2, M_π = pion mass ($s+t+u=4$). No poles are produced by the denominators because $T^1(s,t,u)$ is antisymmetric in t and u. The functions G_0, G_1 and G_2 have been introduced by Roskies [7]: G_0 is simply the $\pi^0-\pi^0$ amplitude. Crossing symmetry is encoded in the total symmetry of the G_i. The individual amplitudes T^I are reconstructed in the following way.

$$\begin{aligned}
T^0(s,t,u) &= \frac{5}{3}G_0(s,t,u) + \frac{2}{9}(3s-4)G_1(s,t,u) - \frac{2}{27}[3s^2+6tu-16]G_2(s,t,u), \\
T^1(s,t,u) &= (t-u)\left[\frac{1}{3}G_1(s,t,u) + \frac{1}{9}(3s-4)G_2(s,t,u)\right], \\
T^2(s,t,u) &= \frac{2}{3}G_0(s,t,u) - \frac{1}{9}(3s-4)G_1(s,t,u) + \frac{1}{27}[3s^2+6tu-16]G_2(s,t,u).
\end{aligned} \tag{2.2}$$

The symmetry of G_i implies that it can be expressed as a function of two independent variables which are totally symmetric and homogeneous in s, t and u, for instance the variables x and y defined by

$$x = -\frac{1}{16}(st+tu+us), \qquad y = \frac{1}{64}stu. \tag{2.3}$$

No singularities are induced by the change of variables $(s,t,u) \to (x,y)$: each singularity of G_i as a function of x and y is the image of a singularity in the (s,t,u)-space. Analyticity properties of $G_i(x,y)$ have been established [6, 8] by looking at its restrictions to complex straight lines

$$y = a(x-x_0) + y_0, \qquad a, x_0, y_0 \in \mathbb{C}. \tag{2.4}$$

As a function of x, at a fixed value of the slope a and for a given point (x_0, y_0), the restriction

$$F_i(x; a, x_0, y_0) \doteq G_i(x, a(x-x_0)+y_0) \tag{2.5}$$

has simple analyticity properties. As long as a and y_0 belong to an x_0-dependent neighborhood $V(x_0)$ of the origin, F_i is regular in the x-plane with a cut $C(a, x_0, y_0)$. This cut is the image in the x-plane of the physical cut $\{s,t,u \,|\, 4 \le s < \infty\}$. The Froissart bound for the asymptotic behavior of the pion-pion amplitudes implies a once subtracted dispersion relation for F_0 and F_1 and an unsubtracted relation for F_2:

$$\begin{aligned}
&F_i(x; a, x_0, y_0) - (1-\delta_{i2})F_i(x_1, a, x_0, y_0) \\
&= \frac{1}{\pi}\int_{C(a,x_0,y_0)} \mathrm{d}\xi \left[\frac{1}{\xi - x} - (1-\delta_{i2})\frac{1}{\xi - x_1}\right] \operatorname{Disc} F_i(\xi, a, x_0, y_0),
\end{aligned} \tag{2.6}$$

where Disc F_i is the discontinuity of F_i across the cut C. The relation (2.6) holds whenever $(a, y_0) \in V(x_0)$; it is an exact consequence of the general principles of quantum field theory.

My high-energy components of the pion-pion amplitudes will be obtained from a decomposition of the right-hand side integral in (2.6) into low- and high-energy parts. To this end, it is convenient to parameterize the cut C by means of the energy squared s and rewrite the right-hand side of (2.6) as an integral over s. The variable ξ in (2.6) becomes a function of s:

$$\xi(s; a, x_0, y_0) = \frac{1}{16(s+4a)}\left[s^2(s-4) + 64(ax_0 - y_0)\right]. \tag{2.7}$$

The discontinuity of F_i is related to the absorptive parts $A^I(s,t)$ of the pion-pion amplitudes evaluated at a (complex) value $\tau(s;a,x_0,y_0)$ of the squared momentum transfer t:

$$\tau(s;a,x_0,y_0) = -\frac{1}{2}\left\{(s-4) - \left[(s-4)^2 - \frac{16}{s+4a}(as(s-4) - 16(ax_0 - y_0))\right]^{\frac{1}{2}}\right\}. \quad (2.8)$$

With these prerequisites the change of variable $\xi \to s$ transforms (2.6) into

$$G_i(x,y) = (1-\delta_{i2})G_i(x_1,y_1) + \frac{1}{16\pi}\int_4^\infty \mathrm{d}s \frac{1}{s+4a}\left[\frac{1}{\xi - x} - (1-\delta_{i2})\frac{1}{\xi - x_1}\right] B_i(s,\tau). \quad (2.9)$$

The relation (2.6) has been written in terms of G_i, the points (x,y) and (x_1,y_1) belonging to the straight line (2,4). In the integral, ξ and τ denote the functions defined in (2.7) and (2.8). The function B_i is proportional to Disc F_i:

$$B_i(s,\tau) = (s-\tau)(2s-4+\tau)\mathrm{Disc}\, F_i(\xi;a,x_0,y_0). \quad (2.10)$$

It is obtained from the absorptive parts A^I:

$$\begin{aligned} B_0(s,t) &= \frac{1}{3}(s-t)(2s-4+t)\left(A^0(s,t) + A^2(s,t)\right), \\ B_1(s,t) &= \frac{1}{6}(3s-4)\left(2A^0(s,t) - 5A^2(s,t)\right) \\ &\quad + \left[\frac{(s-t)(2s-4+t)}{(2t-4+s)} - \frac{1}{2}(2t-4+s)\right] A^1(s,t), \\ B_2(s,t) &= -\frac{1}{2}\left(2A^0(s,t) - 5A^2(s,t)\right) + \frac{3}{2}\frac{3s-4}{2t-4+s}A^1(s,t). \end{aligned} \quad (2.11)$$

The construction of the domain $V(x_0)$ specifying the validity of relation (2.9) can now be explained. It is based on the known exact properties of absorptive parts [9]. At fixed s $(4 \le s < \infty)$ $A^I(s,t)$ is an analytic function of t which is regular in an ellipse $E(s)$ with foci at $t = 0$ and $t = -(s-4)$ and right extremity $r(s)$ given by

$$r(s) = \begin{cases} \dfrac{16s}{s-4} & \text{for} \quad 4 < s < 16, \\ \dfrac{256}{s} & \text{for} \quad 16 \le s \le 32, \\ \dfrac{4s}{s-16} & \text{for} \quad 32 \le s < \infty. \end{cases} \quad (2.12)$$

The integrand in (2.9) is defined if $\tau(s;a,x_0,y_0)$ always stays within the ellipse $E(s)$. This is precisely the condition defining the domain $V(x_0)$ which I shall use:

$$V(x_0) = \{a, y_0 \mid \tau(s;a,x_0,y_0) \in E(s), \quad 4 \le s < \infty\}. \quad (2.13)$$

Figure 1 illustrates the limitations resulting from the condition

$$(a, y_0) \in V(x_0) \tag{2.14}$$

for real values of the parameters a, x_0 and y_0. Figure 2 displays the permitted values of the slope a when $x_0 = -50$ and $y_0 = 1/27$. If (2.14) is fulfilled, the relation (2.9) not only holds true but the absorptive parts appearing in B_i are given by their convergent partial wave expansions. In this sense the integral in (2.9) only involves physical quantities.

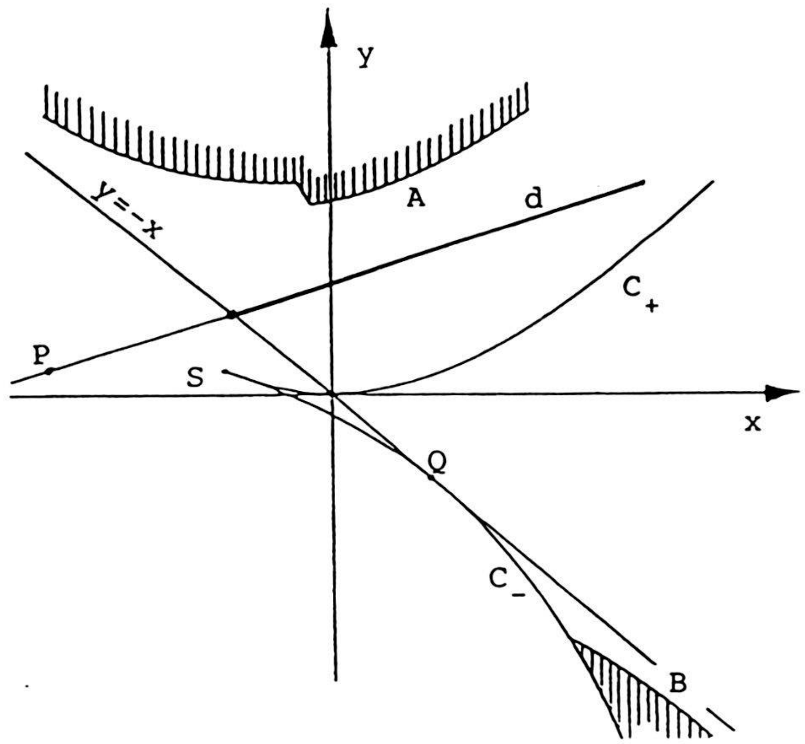

Figure 1: Qualitative picture of the real (x, y)-plane. The real (s, t, u)-space is mapped onto a domain bounded by the curves C_+ and C_- (C_- = image of the line $s = t$, $t > 4/3$, C_+ = image of the line $u = t$, $t > 4/3$). The point s corresponds to the symmetry point $s = t = u = 4/3$. The line $y = -x$ is the image of $s = 4$. The curves B and A are the images of the extremities of the semi-major and semi-minor axes of the ellipses $E(s)$ ($s > 4$). For a real slope a and a real point $P(x_0, y_0)$, the restriction F_i to the line d ($y = a(x - x_0) + y_0$) has a real cut starting on the line $y = -x$ if the point Q is below d. It starts on C_- if Q is above d. The dispersion relation (2.6) is valid if d avoids the shaded region.

3 Defining high-energy components

From now on, I keep x_0 and y_0 fixed, and confine myself to the family of straight lines (2.4) passing through the point (x_0, y_0). Furthermore, the subtraction point (x_1, y_1) is identified

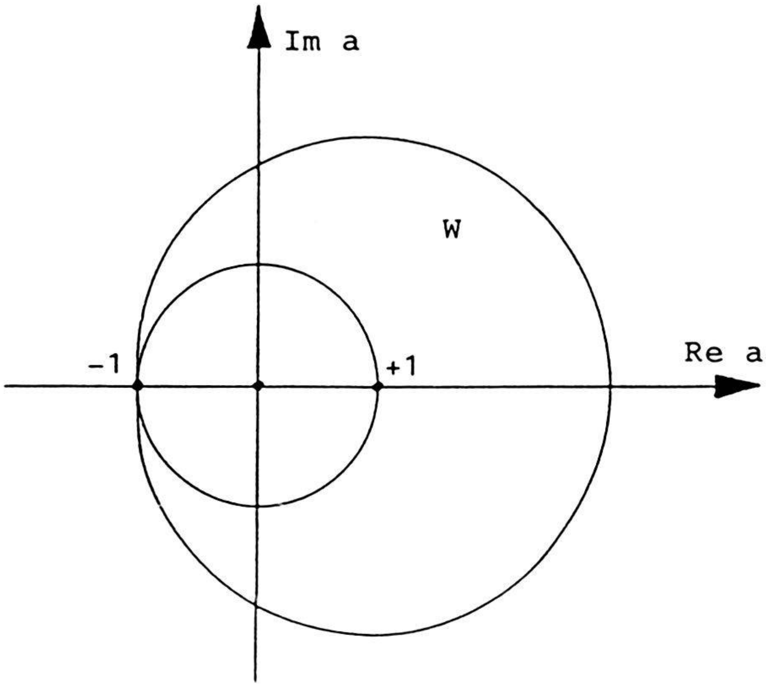

Figure 2: Domain W of the permitted values of the slope a defined via (2.14) for $x_0 = -50$ and $y_0 = 1/27$. The dispersion relation (2.6) is valid if a is within this domain; it contains the circle $|a| = 1$.

with (x_0, y_0) and the dispersion relation (2.9) is used as a representation of the function $G_i(x, y)$ in a domain of the (x, y)-space determined by condition (2.14) with $a = (y - y_0)/(x - x_0)$. I choose (x_0, y_0) in such a way that this representation holds in a neighborhood of the point $x = x_s = -1/3$, $y = y_s = 1/27$, which is the image of the symmetry point $s = t = u = 4/3$. The Taylor expansion of G_i around this point can then be obtained from the representation and the parameters of the chiral pion-pion amplitudes are constrained by equating Taylor coefficients of the chiral G_i with the coefficients derived from (2.9). This explicitly crossing symmetric procedure will be explained in detail in Section 4.

The main aim of the present Section is the extraction of high-energy components from equation (2.9), but it is first necessary to ensure that this equation really provides a representation of G_i in a neighborhood of the symmetry point.

Proposition 1 *If x_0 is real, $-72 < x_0 < 3x_s/2$, $y_0 = y_s$, the function $G_i(x, y)$ given by equation (2.9) is regular for $(x, y) \in M$ where M is the cartesian product $D_x \times D_y$ of two disks D_x and D_y in the x- and y-plane respectively, centered on x_s and y_s.*

The integral $J(x, a)$ in (2.9) is primarily a function of x and a. Since x_0 and y_0 are fixed, condition (2.14) defines a domain W for a. The integral $J(x, a)$ is defined and regular if a belongs to W and x is in $\mathbb{C} \backslash C(a)$, $C(a)$ being an abbreviation for $C(a, x_0, y_0)$ (the point (x_0, y_0) being fixed, explicit reference to the x_0- and y_0-dependence will also be dropped in ξ and τ). Information on the location of the cut $C(a)$ is needed in order to proceed. If the slope a is real, inspection of Fig. 1 shows that

$$\xi(s,a) \geq \xi(4,a_0) \tag{3.1}$$

if $s \geq 4$ and $-1 < a \leq a_0$. This means that for such values of a, the cut $C(a)$, which is on the real x-axis, is entirely on the right of the point $x = \xi(4,a_0) = (a_0 x_0 - y_0)/(1+a_0)$.

An inequality similar to (3.1) holds for complex slopes and in a more general context.

Lemma 1 *If* $|a| < a_0$, $a_0 < \Lambda^2/4$, $\Lambda^2 \geq 4$, $y_0 > 0$ *and* $x_0 + y_0 < 0$, *the inequality*

$$\mathrm{Re}\ \xi(s,a) > \xi(\Lambda^2, a_0) \tag{3.2}$$

holds for $\Lambda^2 \leq s < \infty$.

This lemma follows from a straightforward computation.

Setting $\Lambda^2 = 4$ in (3.2) one sees that the whole complex cut $C(a)$ is on the right of the line $\mathrm{Re}\ x = \xi(4,a_0)$.

Lemma 2 *The integral* $J(x,a)$ *is defined and regular for* $\mathrm{Re}\ x < \xi(4,a_0)$ *and* $|a| < a_0$ *if* $x_0 < 3x_s/2$ *and* $a_0 < 1$. *The circle* $|a| = a_0$ *has to be inside the domain* W.

To prove this lemma I choose a_0 in such a way that $\xi(4,a_0) = x_s/2 = -1/6$. With $y_0 = y_s$ this gives

$$a_0 = -\frac{x_s + 2y_s}{x_s - 2x_0}. \tag{3.3}$$

As, by assumption, $x_0 < 3x_s/2$, the condition $a_0 < 1$ is satisfied. One verifies that the circle $|a| = a_0$ is contained in W if $-72 < x_0 < 3x_s/2$. Consequently, Lemma 2 shows that $J(x,a)$ is regular in the product $D_x \times D_a$ of two disks, D_x with center $x = x_s$ and radius $\rho_x = -x_s/2$, and D_a with center $a = 0$ and radius a_0 given in (3.3). This result implies Proposition 1. Indeed, the representation (2.9) can be rewritten as

$$G_i(x,y) = G_i(x_0,y_0) + J\left(x, \frac{y-y_0}{x-x_0}\right). \tag{3.4}$$

If $x \in D_x$, $y \in D_y$, the radius of D_y being $\rho_y = a_0((3x_s/2) - x_0)$, the slope $a = (y-y_s)/(x-x_0)$ verifies the inequality

$$|a| = \frac{|y-y_s|}{|x-x_0|} < a_0, \tag{3.5}$$

because $|y-y_s| < \rho_y$ and $|x-x_0| > (3x_s/2) - x_0$ when $|x-x_s| < \rho_x$. Therefore $a \in D_a$ and the right-hand side of (3.4) is defined and regular. □

The proof of Proposition 1 contains arbitrary choices leading to special, non-optimal, values of the radii ρ_x and ρ_y. This does not matter because the role of Proposition 1 is simply to ensure analyticity in a product of two disks which, in turn, guarantees the convergence of the Taylor expansion of G_i in both variables x and y.

The validity of (3.4) in a neighborhood of the symmetry point being established, a decomposition of G_i into a low- and high-energy component valid in that neighborhood can be defined simply by splitting $J(x,a)$ into an integral from 4 to Λ^2 and an integral from Λ^2 to infinity. The high-energy componentH_i of G_i is defined by

$$H_i(x,y) = \frac{1}{16\pi}\int_{\Lambda^2}^{\infty} \mathrm{d}s \frac{1}{(s+4a)}\left[\frac{1}{\xi - x} - (1-\delta_{i2})\frac{1}{\xi - x_0}\right] B_i(s,\tau) \tag{3.6}$$

where $a = (y-y_0)/(x-x_0)$. The following proposition, an analogue of Proposition 1, holds for H_i.

Proposition 2 *If*

$$-72 < x_0 < -\frac{1}{64}\Lambda^4(\Lambda^2-4) + \left(\frac{1}{4}\Lambda^2 + 1\right) x_s + y_s \tag{3.7}$$

the function H_i defined in (3.6) is regular in the union M_H of a family of cartesian products of two disks:

$$M_H = \bigcup_{\rho_x} [D_x(\rho_x) \times D_y(\rho_y)]\,. \tag{3.8}$$

The disks D_x and D_y are centered at $x = x_s$ and $y = y_s$ and their radii ρ_x and ρ_y are related by

$$\rho_y = \frac{1}{64}\frac{x_s - \rho_x - x_0}{x_s + \rho_x - x_0}\left[\Lambda^4(\Lambda^2-4) - 16\Lambda^2(x_s+\rho_x) - 64 y_s\right], \tag{3.9}$$

with

$$0 < \rho_x < \frac{\Lambda^2}{16}(\Lambda^2-4) - x_s - \frac{4y_s}{\Lambda^2}. \tag{3.10}$$

The proof of Proposition 2 is a paraphrase of the proof of Proposition 1. If $J_H(x,a)$ denotes the integral in (3.6), this function is defined if a belongs to a domain W_H obtained from (2.13) by restricting s to values larger than Λ^2. As a function of x $J_H(x,a)$ has a cut $C_H(a)$:

$$C_H(a) = \left\{x \mid x = \xi(s,a), \quad s \geq \Lambda^2\right\}. \tag{3.11}$$

Lemma 1 now indicates that the cut $C_H(a)$ is entirely on the right of the line Re $x = \xi(\Lambda^2, a_0)$ if $|a| < a_0 < (\Lambda^2/4)$, $\xi(\Lambda^2, a_0)$ being given by (2.7). This quantity is the abscissa of the intersection of the line $y = a_0(x-x_0) + y_s$ with the image

$$64y = \Lambda^2\left(\Lambda^2(\Lambda^2-4) - 16x\right) \tag{3.12}$$

of the line $s = \Lambda^2$ in the real (x,y)-plane.

By analogy with Lemma 2, it now appears that the integral J_H is defined and regular for Re $x < \xi(\Lambda^2, a_0)$ and $|a| < a_0$, provided that the circle $|a| = a_0$ is within W_H. If x_0 verifies (3.7) this last condition is satisfied for $a_0 < 1$.

The regularity of $H_i(x,a)$ for $x \in D_x$ is ensured if the radius ρ_x is such that

$$\rho_x < \xi(\Lambda^2, a_0) - x_s. \tag{3.13}$$

At a given ρ_x this fixes the maximal slope a_0:

$$a_0 = \frac{\Lambda^4(\Lambda^2 - 4) - 16\Lambda^2(x_s + \rho_x) - 64y_s}{64(x_s + \rho_x - x_0)}. \tag{3.14}$$

As a_0 must be positive, ρ_x has the upper bound (3.10). Furthermore, a_0 has to be smaller than 1 in order to secure regularity with respect to a in D_a, the disk $|a| < a_0$, a_0 given by (3.14). If ρ_x is allowed to vanish, this imposes the upper limit in (3.7). As in the last step of the proof of Proposition 1, the regularity of $J_H(x,a)$ for $(x,a) \in D_x \times D_a$ now implies the regularity of $H_i(x,y)$ in $D_x(\rho_x) \times D_y(\rho_y)$, the radius of D_y being $\rho_y = a_0(x_s - \rho_x - x_0)$. The expression (3.14) for a_0 leads to the relation (3.9) between ρ_x and ρ_y. □

As the radii of convergence of the Taylor expansion of H_i will matter, Proposition 2 goes into greater detail than Proposition 1 although it is not aimed at being optimal.

Whereas (3.6) defines crossing-symmetric high-energy components of the T^I via (2.2), a drawback of these components is that they depend on the choice of the subtraction point (x_0, y_0). This comes from the explicit appearance of x_0 in the integral of (3.6) (if $i \neq 2$) and the (x_0, y_0)-dependence of ξ and τ (cf. (2.7) and (2.8)). In fact, after identification of (x_0, y_0) and (x_1, y_1) in (2.9), the right-hand side has to be independent of (x_0, y_0) and this leads to constraints on the absorptive parts already noticed in [8]. If the integral is split into a low- and a high-energy part, there is a coupling between low- and high-energy absorptive parts, which I shall not discuss.

4 High-energy constraints on the one- and two-loop chiral pion-pion parameters

The outcome of the previous Sections is a representation of the symmetric amplitudes $G_i(x,y)$ in a neighborhood of the symmetry point. It provides a decomposition into low- and high-energy contributions,

$$G_i(x,y) = L_i(x,y) + H_i(x,y), \tag{4.1}$$

where

$$L_i(x,y) = (1 - \delta_{i2})G_i(x_0, y_0) + \frac{1}{\pi}\int_4^{\Lambda^2} ds \frac{1}{s + 4a}\left[\frac{1}{\xi - x} - (1 - \delta_{i2})\frac{1}{\xi - x_0}\right] B_i(s, \tau) \tag{4.2}$$

is the low-energy component and H_i the high-energy component defined in (3.6). Proposition 1 applies to L_i: this function is known to be regular in the domain M of Proposition 1.

The high-energy component H_i is certainly regular in the larger domain M_H defined in (3.8). These analyticity properties imply that the Taylor expansion of $G_i(x,y)$ around (x_s, y_s) in the two complex variables x and y can be extracted from the representations (3.6) and (4.2). It converges in a domain containing M whereas the expansion of the high-energy component converges in a larger domain containing M_H.

In order to derive well defined constraints on the parameters appearing in the chiral amplitudes T_χ^I from (4.1), I make two assumptions:

(i) The symmetric amplitudes G_i^χ obtained from the $2n$-th order chiral amplitudes approximate the true symmetric amplitudes G_i in a neighborhood of the symmetry point up to higher order corrections.

(ii) The discontinuities Disc G_i^χ of the $2n$-th chiral symmetric amplitudes approximate Disc G_i in a bounded interval above threshold up to higher order corrections.

This means that the representation (4.1) can be rewritten in the following way if Λ^2 is conveniently chosen and if (x,y) is close to (x_s, y_s):

$$G_i^\chi(x,y) = L_i^\chi(x,y) + H_i(x,y) + \text{ higher order terms.} \tag{4.3}$$

The low-energy component L_i^χ is obtained from (4.2) where B_i is replaced by B_i^χ.

The precise value of Λ^2 plays no role in what follows. A special value I have in mind is $\Lambda^2 = 16$ corresponding to an energy of 560 MeV.

Each chiral amplitude T_χ^I is a sum of a polynomial in s, t and u and non-polynomial terms exhibiting the cuts necessarily present in any scattering amplitude. According to (2.1) the symmetric amplitudes G_i^χ have the same structure. The polynomial part of G_i^χ is $O(p^{2n_i})$ where n_i is determined by n and depends on i. Although the G_i^χ do not have the same asymptotic behavior as the G_i, they share the regularity properties we have established. The coefficients appearing in the polynomials and in the non-polynomial terms are determined by the chiral coupling constants.

By construction G_i^χ and L_i^χ have the same discontinuity across the cut $C(a)$ as long as $4 \le s \le \Lambda^2$. This implies that the difference $(G_i^\chi - L_i^\chi)$ can be written as

$$G_i^\chi(x,y) - L_i^\chi(x,y) = -(1-\delta_{i2})G_i(x_0,y_0) + P_i(x,y) + H_i^\chi(x,y) \tag{4.4}$$

where P_i is a low-energy component and H_i^χ is the high-energy component of G_i^χ. As P_i has no discontinuity across the low-energy part of the cut $C(a)$, it is regular in a domain which is larger than M. Up to the sixth order of the chiral expansion P_i is in fact a polynomial of degree $2n_i$. The following discussion applies to that situation, i.e. I assume that $n \le 3$ from now on. I show in Appendix A how P_i and H_i^χ are constructed.

Combining (4.4) and (4.3) gives

$$P_i(x,y) - (1-\delta_{i2})G_i(x_0,y_0) = H_i(x,y) - H_i^\chi(x,y) + O\left(\lambda^{2(n_i+1)}\right). \tag{4.5}$$

The strengths of the successive terms of the chiral expansion are measured by means of a parameter λ: a convenient choice is $\lambda^2 = M_\pi^2/(16\pi F_\pi^2)$. The relation (4.5) has to hold in a neighborhood of the symmetry point. For consistency H_i and H_i^χ are replaced by their $2n_i$-order truncated Taylor expansions Q_i and Q_i^χ:

$$P_i(x,y) + Q_i^\chi(x,y) = (1-\delta_{i2})G_i(x_0,y_0) + Q_i(x,y) + O\left(\lambda^{2(n_i+1)}\right). \tag{4.6}$$

The left-hand side is entirely determined by the $2n$-th order chiral amplitudes whereas the right-hand side involves the pion-pion absorptive parts above Λ^2 and the value of G_i at the subtraction point if $i = 0, 1$. Equating the coefficients of the left- and right-hand side polynomials gives a series of constraints on the parameters of the chiral amplitudes. The $i = 0$ and $i = 1$ constraints coming from the constant terms in (4.6) have a special status because of the presence of $G_i(x_0, y_0)$, the unknown value of G_i at the subtraction point. The remaining constraints relate the chiral parameters to high-energy pion-pion scattering.

The regularity of H_i in the family of products M_H implies upper bounds for the coefficients $C_{n,m}$ of its Taylor expansion

$$H_i(x,y) = \sum_{n,m} C_{n,m}^i (x-x_s)^n (y-y_s)^m. \tag{4.7}$$

If ρ_x and ρ_y are such that $D_x(\rho_x) \times D_y(\rho_y)$ is inside M_H, $|H_i|$ is finite on the boundary of this product of two disks and

$$\left|C_{n,m}^i\right| < \frac{K_i}{(\rho_x)^n(\rho_y)^m}. \tag{4.8}$$

One checks that $\rho_x = \rho_y = 9$ fulfills the above requirements if $x_0 = -50$, $y_0 = y_s$ and $\Lambda^2 = 16$ (notice that these values are compatible with (3.7)). This leads to the simple but severe bound $|C_{n,m}^i| < K_i/9^{(n+m)}$. A more refined bound is derived in Appendix B. The same bounds hold for the Taylor coefficients of H_i^χ. Inequality (4.8) is an important result: it shows that the coefficients of the high-order polynomials Q_i fall off exponentially. In view of (4.6) this indicates that a rapid decrease of the size of the high-order terms in the chiral expansion is conceivable.

Finally I examine the nature of the conditions that equation (4.6) imposes on the one- and two-loop chiral amplitudes [10, 11]. These amplitudes are obtained in a standard way from a single function $A^\chi(s,t,u)$ which is the sum of a second-, fourth- and sixth-order term

$$A^\chi(s,t,u) = \lambda^2 A_2(s,t,u) + \lambda^4 A_4(s,t,u) + \lambda^6 A_6(s,t,u). \tag{4.9}$$

The polynomial parts of these terms have the form

$$\begin{aligned} A_2^{\rm pol}(s,t,u) &= a_{2,0} + a_{2,1}s \\ A_4^{\rm pol}(s,t,u) &= a_{4,0} + a_{4,1}s + a_{4,2}s^2 + a_{4,3}tu \\ A_6^{\rm pol}(s,t,u) &= a_{6,0} + a_{6,1}s + a_{6,2}s^2 + a_{6,3}tu + a_{6,4}s^3 + a_{6,5}stu \end{aligned} \tag{4.10}$$

The non-polynomial parts are sums of products of polynomials with analytic functions of a single variable s, t or u exhibiting the cut $[4,\infty)$. At fixed $a_{2,0}$ and $a_{2,1}$, the parameters appearing in these non-polynomial terms are linear in the $a_{4,\alpha}$, $\alpha = 0,1,2,3$. The $2n$-th order term $P_{i,2n}$ of the polynomial P_i defined in (4.4) is either a constant or a polynomial of first degree.

$$\begin{array}{lll} P_{0,2} = \alpha_{0,2}, & P_{1,2} = \alpha_{1,2}, & P_{2,2} = 0, \\ P_{0,4} = \alpha_{0,4} + \beta_{0,4}(x - x_s), & P_{1,4} = \alpha_{1,4}, & P_{2,4} = \alpha_{2,4}, \\ P_{0,6} = \alpha_{0,6} + \beta_{0,6}(x - x_s) + \gamma_{0,6}(y - y_s), & P_{1,6} = \alpha_{1,6} + \beta_{1,6}(x - x_s), & P_{2,6} = \alpha_{2,6}. \end{array} \tag{4.11}$$

The $\alpha_{i,2n}$, $\beta_{i,2n}$, $\gamma_{i,2n}$ are linear combinations of the $a_{2n,m}$.

The $Q_i^{\chi(2n)}$ have the same form as the $P_i^{(2n)}$: they are obtained from (4.11) by replacing $\alpha_{i,2n}, \ldots$ by new coefficients $\alpha^H_{i,2n}, \ldots$ which are linear in the $a_{4,\alpha}$ at fixed $a_{2,0}$ and $a_{2,1}$.

From (4.11) and (4.6) one obtains two constraints at leading, second order, four constraints for the fourth-order one-loop amplitudes and six constraints for the one- and two-loop sixth-order amplitudes. If the conditions involving the values of G_0 and G_1 at the symmetry point are disregarded, two second-order constraints remain:

$$\begin{aligned} \lambda^4 \left(\alpha_{2,4} + \alpha^H_{2,4}\right) &= H_2(x_s, y_s) + O(\lambda^6), \\ \lambda^4 \left(\beta_{0,4} + \beta^H_{0,4}\right) &= (\partial H_0/\partial x)(x_s, y_s). \end{aligned} \tag{4.12}$$

At sixth order we obtain four conditions:

$$\begin{aligned} \lambda^4 \left(\alpha_{2,4} + \alpha^H_{2,4}\right) + \lambda^6 \left(\alpha_{2,6} + \alpha^H_{2,6}\right) &= H_2(x_s, y_s) + O(\lambda^8), \\ \lambda^4 \left(\beta_{0,4} + \beta^H_{0,4}\right) + \lambda^6 \left(\beta_{0,6} + \beta^H_{0,6}\right) &= (\partial H_0/\partial x)(x_s, y_s) + O(\lambda^8), \\ \lambda^4 \beta^H_{1,4} + \lambda^6 \left(\beta_{1,6} + \beta^H_{1,6}\right) &= (\partial H_1/\partial x)(x_s, y_s) + O(\lambda^8), \\ \lambda^4 \gamma^H_{0,4} + \lambda^6 \left(\gamma_{0,6} + \gamma^H_{0,6}\right) &= (\partial H_0/\partial y)(x_s, y_s) + O(\lambda^8). \end{aligned} \tag{4.13}$$

The fact the $\beta_{1,4} = \gamma_{0,4} = 0$ has been taken into account.

In the right-hand sides of these equations we have integrals over absorptive parts $A^I(s, \tau(s))$ evaluated at $\tau(s) \approx -(64/27)/8s(s-4)^2)$, $s > \Lambda^2$, very close to the forward direction, if $x_0 = -50$, $y_0 = y_s$.

Similar constraints have been derived in [5] and [10].

The equations (4.12-13) have not been analyzed in detail until now: this is beyond the scope of the present work. This means that I end up with a proposal whose practicality remains to be explored.

Appendix A Constructing the polynomial P_i: a sample calculation

The polynomial part of A^χ produces a polynomial part of G_i^χ which appears unchanged in the polynomial P_i defined in (4.4). The main point is to find out the contribution to P_i coming from the non-polynomial terms of A^χ. Up to sixth order, these terms have a simple structure, some of them having the form

$$\tilde{A}(s,t,u) = R(s)f(s), \tag{A.1}$$

where R is a polynomial and f an analytic function with a right-hand cut $[4,\infty)$. As an illustration I compute the terms of P_0 and L_0^χ coming from $\tilde{A}$. This produces the following term of G_0^χ:

$$\tilde{G}_0^\chi(s,t,u) = R(s)f(s) + R(t)f(t) + R(u)f(u). \tag{A.2}$$

The functions f of the one- and two-loop amplitudes obey once-subtracted dispersion relations. This allows a decomposition of f into a low- and a high-energy component:

$$\begin{aligned} f(s) &= f_{\mathrm{L}}(s) + f_{\mathrm{H}}(s), && \text{(A.3)}\\ f_{\mathrm{L}}(s) &= f(0) + \frac{s}{\pi}\int_4^{\Lambda^2} \frac{\mathrm{d}\sigma}{\sigma}\,\frac{\mathrm{Im}\ f(\sigma)}{\sigma - s}, && \text{(A.4)}\\ f_{\mathrm{H}}(s) &= \frac{s}{\pi}\int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\sigma}{\sigma}\,\frac{\mathrm{Im}\ f(\sigma)}{\sigma - s}. && \text{(A.5)} \end{aligned}$$

The high-energy term $\tilde{H}_0^\chi$ is simply obtained by replacing f by f_H in (A.2). Equation (4.4) becomes

$$\tilde{P}_0(x,y) = R(s)f_{\mathrm{L}}(s) + R(t)f_{\mathrm{L}}(t) + R(u)f_{\mathrm{L}}(u) - L_0^\chi(x,y) + G_0(x_0,y_0). \tag{A.6}$$

Inserting the representation (A.4) and introducing Disc $\tilde{G}_0^\chi = R(s)\,\mathrm{Im}\ f(s)$ into (4.2) gives an explicit expression for the polynomial $\tilde{P}_0$:

$$\tilde{P}_0 = C + f(0)\,[R(s)+R(t)+R(u)] - \frac{1}{\pi}\int_4^{\Lambda^2} \frac{\mathrm{d}\sigma}{\sigma}\,[S(s,\sigma)+S(t,\sigma)+S(u,\sigma)]\,\mathrm{Im}\, f(\sigma), \tag{A.7}$$

where $S(s,\sigma)$ is a polynomial in two variables:

$$S(s,\sigma) = \frac{sR(s) - \sigma R(\sigma)}{s-\sigma} \tag{A.8}$$

and the constant C is given by

$$C = -\frac{1}{\pi}\int_4^{\Lambda^2} \mathrm{d}\sigma \left[\frac{1}{\sigma - s_0} + \frac{1}{\sigma - t_0} + \frac{1}{\sigma - u_0}\right] R(\sigma)\,\mathrm{Im}\ f(\sigma).$$

For convenience, the Mandelstam variables are used instead of x and y.

The contributions of the term $\tilde{A}$ of A^χ, defined in (A.1), to P_1, P_2, H_1^χ and H_1^χ are obtained in a similar way, and so are the contributions of the other non-polynomial terms of A^χ.

Appendix B An upper bound for the Taylor coefficients $C_{n,m}$

Replace (3.9) by the linear relation

$$\frac{\rho_x}{\rho_1} + \frac{\rho_y}{\rho_2} = 1, \tag{B.1}$$

$0 \leq \rho_x \leq \rho_1$, where ρ_1 and ρ_2 are such that $D_x(\rho_x) \times D_y(\rho_y)$ belongs to M_H if ρ_x and ρ_y obey (B.1). The Taylor coefficients $C_{n,m}$ defined in (4.7) have the upper bound

$$|C_{n,m}| < \mathrm{Inf} \frac{K}{\rho_x^n \rho_y^m} = \frac{K}{\rho_1^n \rho_2^m} \left(1 + \frac{m}{n}\right)^n \left(1 + \frac{n}{m}\right)^m . \tag{B.2}$$

If one chooses $\Lambda^2 = 16$ and $x_0 = -50$, one can take $\rho_1 = 12.3$ and $\rho_2 = 49.3$. This gives an extremely rapid decrease if m increases, n being fixed.

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