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Non-Commutative Quantum Dynamics in N Dimensions

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Abstract. In this paper the q-deformation of the N-dimensional quantum dynamics is described. This theory is shown to be consistent only when the deformation parameters satisfy the special relation

Since the non-commutative plane (so called quantum plane) [1] was introduced to the theoretical physics world , many theoretical physicists have been attempting to build up physical models based on this type of non-commutative geometry. The reference [2],[3] and [4] governs one dimensional quantum dynamics only. In commutative plane we can easily extend the results obtained in one dimension to more general cases - those in N dimensions. However it is not the case in quantum plane. In this paper we extend the result given in ref [4] to the N- dimensional case.

Now we build up the non-commutative quantum dynamics in N dimensions. Manin [5] showed that the quantum space is defined as the quotient of a non-commutative algebra by the two sided ideal. Arafe'va and Volovich [2] introduced the non-commuting mass to discuss the one dimensional quantum dynamics.

Now we formulate the non-commutative quantum dynamics in N dimensions by means of the q-deformation of the algebra of observables; q-deformed algebra H is assumed to be a quotient algebra of free algebra A by the two-sided ideal J;

$$H = A(I, x_1, \cdots, x_N, p_1, \cdots, p_N, \Lambda_1, \cdots, \Lambda_N, K_1, \cdots, K_N)$$

$$mod \;\; J(I, x_1, \cdots, x_N, p_1, \cdots, p_N, \Lambda_1, \cdots, \Lambda_N, K_1, \cdots, K_N),$$

where the two-sided ideal J is defined by the following Bethe ansatz

$$x_i p_i = q^2 p_i x_i + i\hbar q \Lambda_i^2 + \lambda \Sigma_{j=i+1}^{N} x_j p_j, \tag{1}$$

$$x_i p_j = q p_j x_i, \quad (i \neq j) \tag{2}$$

$$x_i x_j = q^{-1} x_j x_i, \quad (i < j) \tag{3}$$

$$p_i p_j = q p_j p_i, \quad (i < j) \tag{4}$$

$$x_i \Lambda_j = \xi_{ij} \Lambda_j x_i, \tag{5}$$

$$p_i \Lambda_j = \sigma_{ij} \Lambda_j p_i, \tag{6}$$

$$x_i K_j = \tau_{ij} K_j x_i, \tag{7}$$

$$p_i K_j = \epsilon_{ij} K_j p_i, \tag{8}$$

$$\Lambda_i K_j = \eta_{ij} K_j \Lambda_i, \tag{9}$$

$$\Lambda_i \Lambda_j = \Lambda_j \Lambda_i, \tag{10}$$

$$K_i K_j = K_j K_i, \tag{11}$$

where $\lambda = q^2 - 1$ and Λ_i and K_i are assumed to be additional hermitian generators of the extended non-commutative algebra in N dimensions.

From eq.(5) and eq.(6) with $j = i$, we have[1]

$$x_i \Lambda_i = \xi_i \Lambda_i x_i, \tag{12}$$

$$p_i \Lambda_i = \sigma_i \Lambda_i p_i. \tag{13}$$

Multiplying eq.(13) by eq.(12), side by side, and using eq.(5) and eq.(6) leads to

$$x_i p_i \Lambda_i^2 = (\xi_i \sigma_i)^2 \Lambda_i^2 x_i p_i. \tag{14}$$

Similarly, Multiplying eq.(12) by eq.(13), side by side, and using eq.(5) and eq.(6) leads to

$$p_i x_i \Lambda_i^2 = (\xi_i \sigma_i)^2 \Lambda_i^2 p_i x_i. \tag{15}$$

Subtracting eq.(15) multiplied by q^2 from eq.(14), we obtain the following two relations;

$$\sigma_i \xi_i = 1, \tag{16}$$

$$\sigma_{ji} \xi_{ji} = 1, \quad (j > i). \tag{17}$$

Multipying eq.(6) by eq.(5), side by side , and using eq.(5) and eq.(6), we obtain

$$x_i p_i \Lambda_j^2 = (\sigma_{ij} \xi_{ij})^2 \Lambda_j^2 x_i p_i, \tag{18}$$

[1] Here we adopted the following notation; $A_{ii} = A_i, \quad for \; A = \xi, \sigma, \tau, \epsilon \; and \; \eta.$

$$p_i x_i \Lambda_j^2 = (\sigma_{ij}\xi_{ij})^2 \Lambda_j^2 p_i x_i, \tag{19}$$

Subtracting eq.(18) multiplied by q^2 from eq.(19), we get

$$\sigma_{ij}\xi_{ij} = 1, \quad (i \neq j). \tag{20}$$

Then the eq.(16), eq.(17) and eq.(20) are expressed by the following one relation;

$$\sigma_{ij} = \xi_{ij}^{-1}. \tag{21}$$

The next step is to determine τ_{ij}, ϵ_{ij} and η_{ij} in terms of ξ_{ij}. From the two relations

$$x_i \Lambda_i = \xi_i \Lambda_i x_i,$$

$$p_i K_i = \epsilon_i K_i p_i,$$

we get the following two equations;

$$x_i p_i \Lambda_i K_i = \epsilon_i \tau_i \Lambda_i K_i x_i p_i, \tag{22}$$

$$p_i x_i K_i \Lambda_i = \epsilon_i \tau_i K_i \Lambda_i p_i x_i. \tag{23}$$

Subtracting eq.(23) multiplied by q^2 from eq.(22) and using $\Lambda_i K_i = \eta_i K_i \Lambda_i$, we obtain

$$\eta_i = \sqrt{\epsilon_i \tau_i} \tag{24}$$

$$\eta_i^2 = \epsilon_{ji}\tau_{ji}, \quad (j > i). \tag{25}$$

From the two relations

$$x_i \Lambda_i = \xi_i \Lambda_i x_i,$$

$$p_i K_j = \epsilon_{ij} K_j p_i,$$

we have

$$\eta_{ij} = \sqrt{\epsilon_{ij}\tau_{ij}}, \tag{26}$$

$$\eta_{ij} = \sqrt{\epsilon_{kj}\tau_{kj}} \ \ (k > i). \tag{27}$$

From the two relations

$$x_i \Lambda_j = \xi_{ij} \Lambda_j x_i,$$

$$p_i K_i = \epsilon_i K_i p_i,$$

we have

$$\eta_i = \sqrt{\epsilon_{ji}\tau_{ji}}, \ (j > i). \tag{28}$$

Consider the following extended Hamiltonian

$$H = \Sigma_{i=1}^N p_i^2 K_i^2 + V(x_1, K_1, \Lambda_1, \cdots, x_N, K_N, \Lambda_N). \tag{29}$$

Now we demand that all Λ_i's and K_i's are constant of motion;

$$\dot{\Lambda}_i = \frac{i}{\hbar}[H, \Lambda_i] = 0, \tag{30}$$

$$\dot{K}_i = \frac{i}{\hbar}[H, K_i] = 0. \tag{31}$$

First let us restrict our concern to the free theory ($V = 0$).Then eq.(30) and eq.(31) give

$$\sigma_{ij} = \eta_{ji}, \quad for\ all\ j. \tag{32}$$

$$\epsilon_{ji} = 1, \quad for\ all\ i, j. \tag{33}$$

Therefore we get

$$\xi_{ij} = \xi_i, \quad for\ i < j, \tag{34}$$

$$\xi_{ji} = \xi_j, \quad for\ i < j, \tag{35}$$

$$\tau_{ij} = \tau_j = \xi_j^{-2}, \ (i < j), \tag{36}$$

$$\tau_{ji} = \tau_i = \xi_i^{-2}, \ (i < j), \tag{37}$$

$$\eta_{ij} = \xi_j^{-1}, \ (i < j), \tag{38}$$

$$\eta_{ji} = \xi_i^{-1}, \ (i < j), \tag{39}$$

If we summarize all the results , we obtain

$$x_i \Lambda_j = \xi_i \Lambda_j x_i, \tag{40}$$

$$p_i \Lambda_j = \xi_i^{-1} \Lambda_j p_i, \tag{41}$$

$$x_i K_j = \xi_j^{-2} K_j x_i, \tag{42}$$

$$p_i K_j = K_j p_i, \tag{43}$$

$$\Lambda_i K_j = \xi_j^{-1} K_j \Lambda_i, \tag{44}$$

$$\Lambda_i \Lambda_j = \Lambda_j \Lambda_i, \tag{45}$$

$$K_i K_j = K_j K_i, \tag{46}$$

Now let us consider the general case with the non-vanishing potential term; Then , in order to make Λ_i's and K_i's remain constant of motion, we demand that the potential V should commute with Λ_i's and K_i's, which results in the following scale invariant properties of the potential;

$$V(\xi_1 x_1, \xi_1 K_1, \Lambda_1, \cdots, \xi_k x_k, \xi_k K_k, \Lambda_k, \cdots, \xi_N x_N, \xi_N K_N, \Lambda_N)$$
$$= V(x_1, K_1, \Lambda_1, \cdots, x_k, K_k, \Lambda_k, \cdots, x_N, K_N, \Lambda_N), \tag{47}$$

$$V(\xi_i x_1, K_1, \xi_i \Lambda_1, \cdots, \xi_i x_k, K_k, \xi_i \Lambda_k, \cdots, \xi_i x_N, K_N, \xi_i \Lambda_N)$$
$$= V(x_1, K_1, \Lambda_1, \cdots, x_k, K_k, \Lambda_k, \cdots, x_N, K_N, \Lambda_N), \ for\ all\ i = 1, 2, \cdots, N. \tag{48}$$

Finally we arrive at the Heisenberg equation of motion for non-commutative quantum dynamics in N dimensions;

$$\dot{x}_i = \frac{i}{\hbar}[H, x_i]$$

$$= [\frac{i}{\hbar}(\xi_i^4 - q^4)p_i^2 x_i + q(q^2+\xi_i^2)p_i\Lambda_i^2 - \frac{i}{\hbar}\lambda(q^2+1)p_i\Sigma_{m=i+1}^N x_m p_m]K_i^2$$

$$+ \frac{i}{\hbar}\Sigma_{k\neq i}(\xi_k^4 - q^2)p_k^2 x_i K_k^2, \tag{49}$$

where we damand[2] that

$$\xi_i = (\Pi_{j=1}^N \xi_j)^{\frac{1}{N}} q^{\frac{1}{N}(i-\frac{N+1}{2})}, \tag{50}$$

in order to impose the condition that

$$[V, x_i] = 0.$$

Similarly we can obtain the equation of motion for the momenta p_i's;

$$\frac{\hbar}{i}\dot{p}_i = [H, p_i]$$

$$= [\Sigma_{k=1}^N p_k^2 K_k^2 + V, p_i]$$

$$= \Sigma_{k=1}^{i-1}[p_k^2 K_k^2, p_i] + \Sigma_{k=i+1}^N [p_k^2 K_k^2, p_i] + [V, p_i]$$

$$= (q^2-1)p_i\Sigma_{k=1}^{i-1}p_k^2 K_k^2$$

$$+ (q^{-2}-1)p_i\Sigma_{k=i+1}^N p_k^2 K_k^2 + [V, p_i] \tag{51}$$

If we use the following identity

$$x_i^{n_i}p_i = q^{2n_i}p_i x_i^{n_i} + i\hbar q(\xi_i^2)^{n_i-1}[n_i]_{(\frac{q}{\xi_i})^2}\Lambda_i^2 x_i^{n_i-1}$$

$$+ \lambda[n_i]_{q^2}x_i^{n_i-1}\Sigma_{j>i}x_j p_j, \tag{52}$$

then the commutator between potential and momentum p_i is computed as

$$[V, p_i]$$

$$= p_i\{V(qx_1,\cdots,qx_{i-1},q^2x_i,qx_{i+1},\cdots,qx_N)$$

$$-V(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_N)\}$$

$$+i\hbar q\Lambda_i^2\frac{\partial}{\partial_{(\frac{q}{\xi_i})^2}x_i}V(\xi_1x_1,\cdots,\xi_{i-1}x_{i-1},\xi_i^2x_i,qx_{i+1},\cdots,qx_N,)$$

$$+ \lambda\Sigma_{j=i+1}^N x_j p_j\frac{\partial}{\partial_{q^2}x_i}V(x_1,\cdots,x_{i-1},x_i,qx_{i+1},\cdots,qx_N) \tag{53}$$

where the derivation of eq.(53) is given in Appendix B and we write V as follows;

$$V(x_1,\cdots,x_n,\cdots,x_N)$$

$$= V(x_1,K_1,\Lambda_1,\cdots,x_n,K_n,\Lambda_n,\cdots,x_N,K_N,\Lambda_N),$$

[2] The derivation of the condition (50) is given in Appendix A

and the deformed partial derivative is given by

$$\frac{\partial}{\partial_k x_i} V(x_1, \cdots, x_i, \cdots, x_N)$$
$$= \frac{V(x_1, \cdots, kx_i, \cdots, x_N) - V(x_1, \cdots, x_i, \cdots, x_N)}{x_i(k-1)}. \tag{54}$$

In this paper we discuss the N-dimensional quantum dynamics with some non-commuting extended variables. In order to get the unitary time evolution of observables we deformed the algebra of observables leaving Heisenberg equations as equations of motion unchanged. In time evolution of position, we had to demand that the deformation parameters, ξ_i's, fulfill the special relation given in eq.(50). We hope that this procedure will be connected with the N-dimensional version of Arafe'va-Volovich model in the near future.

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Appendix A

In this appendix we derive the eq.(50). Let us first start with

$$V(\cdots, x_k, K_k, \Lambda_k, \cdots)x_i$$
$$= x_i V(q^{-1}x_1, \xi_1^2 K_1, \xi_i^{-1}\Lambda_1, \cdots,$$
$$x_i, \xi_i^2 K_i, \xi_i^{-1}\Lambda_i, qx_{i+1}, \xi_{i+1}^2 K_{i+1}, \xi_i^{-1}\Lambda_{i+1},$$
$$\cdots, qx_N, \xi_N^2 K_N, \xi_i^{-1}\Lambda_N)$$
$$= x_i V(q^{-1}\xi_1^{-1}x_1, \xi_1 K_1, \xi_i^{-1}\Lambda_1, \cdots,$$
$$\xi_i^{-1}x_i, \xi_i K_i, \xi_i^{-1}\Lambda_i, q\xi_{i+1}^{-1}x_{i+1}, \xi_{i+1}K_{i+1}, \xi_i^{-1}\Lambda_{i+1},$$
$$\cdots, q\xi_N^{-1}x_N, \xi_N K_N, \xi_i^{-1}\Lambda_N)$$
$$= x_i V(q^{-1}\xi_1^{-1}\xi_i^2 x_1, \xi_1 K_1, \Lambda_1, \cdots,$$
$$\xi_i x_i, \xi_i K_i, \Lambda_i, q\xi_{i+1}^{-1}\xi_i^2 x_{i+1}, \xi_{i+1}K_{i+1}, \Lambda_{i+1},$$
$$\cdots, q\xi_N^{-1}\xi_i^2 x_N, \xi_N K_N, \Lambda_N)$$
$$= x_i V(q^{-1}\xi_1^{-2}\xi_i^2 x_1, K_1, \Lambda_1, \cdots,$$
$$x_i, K_i, \Lambda_i, q\xi_{i+1}^{-2}\xi_i^2 x_{i+1}, K_{i+1}, \Lambda_{i+1},$$
$$\cdots, q\xi_N^{-2}\xi_i^2 x_N, K_N, \Lambda_N)$$

Therefore , in order for the potential to commute with the coordinate, it should satisfy the following relations;

$$\xi_i = \sqrt{q}\xi_j, \quad j = 1, 2, \cdots, i-1.$$
$$\xi_i = \frac{1}{\sqrt{q}}\xi_j, \quad j = i+1, \cdots, N.$$

Multiplying all the above mentioned relations leads to

$$\xi_i^N = (\Pi_{k=1}^N \xi_k) q^{i-\frac{N+1}{2}}.$$

Appendix B

In this appendix we prove the relation (53) for the monomials.[3] The general form of monomials is given by

$$V = \Pi_k x_k^{n_k} K_k^{m_k} \Lambda_k^{l_k}$$

Then we have

$$\begin{aligned}
&[\Pi_k x_k^{n_k} K_k^{m_k} \Lambda_k^{l_k}]p_i \\
&= \Pi_k x_k^{n_k} p_i K_k^{m_k} (\xi_i \Lambda_k)^{l_k} \\
&= (\Pi_{k=1}^{i-1} x_k^{n_k})(x_i^{n_i})(\Pi_{k=i+1}^N x_k^{n_k} p_i)\Pi_k K_k^{m_k}(\xi_i\Lambda_k)^{l_k} \\
&= (\Pi_{k=1}^{i-1} x_k^{n_k})(x_i^{n_i} p_i)(\Pi_{k=i+1}^N (qx_k)^{n_k})\Pi_k K_k^{m_k}(\xi_i\Lambda_k)^{l_k}
\end{aligned}$$

Here, using the identity (52), we have

$$[\Pi_k x_k^{n_k} K_k^{m_k} \Lambda_k^{l_k}]p_i = A + B + C,$$

where

$$\begin{aligned}
A &= (\Pi_{k=1}^{i-1} x_k^{n_k})(p_i(q^2 x_i)^{n_i})(\Pi_{k=i+1}^N (qx_k)^{n_k})\Pi_k K_k^{m_k}(\xi_i\Lambda_k)^{l_k} \\
&= p_i(\Pi_{k=1}^{i-1} (qx_k)^{n_k})(p_i(q^2 x_i)^{n_i})(\Pi_{k=i+1}^N (qx_k)^{n_k})\Pi_k K_k^{m_k}(\xi_i\Lambda_k)^{l_k} \\
&= p_i V(qx_1, \cdots, qx_{i-1}, q^2 x_i, qx_{i+1}, \cdots, qx_N) \\
B &= (\Pi_{k=1}^{i-1} x_k^{n_k})(i\hbar q(\xi_i^2)^{n_i-1}[n_i]_{(\frac{q}{\xi_i})^2}\Lambda_i^2 x_i^{n_i-1})(\Pi_{k=i+1}^N (qx_k)^{n_k})\Pi_k K_k^{m_k}(\xi_i\Lambda_k)^{l_k} \\
&= i\hbar\Lambda_i^2 \frac{\partial}{\partial_{(\frac{q}{\xi_i})^2} x_i} V(\xi_1 x_1, \cdots, \xi_{i-1}x_{i-1}, \xi_i^2 x_i, qx_{i+1}, \cdots, qx_N), \\
C &= \lambda(\Pi_{k=1}^{i-1} x_k^{n_k})[n_i]_{q^2} x_i^{n_i-1}\Sigma_{j=i+1}^N x_j p_j (\Pi_{k=i+1}^N (qx_k)^{n_k})\Pi_k K_k^{m_k}(\xi_i\Lambda_k)^{l_k} \\
&= \lambda\Sigma_{j=i+1}^N x_j p_j \frac{\partial}{\partial_{q^2} x_i} V(x_1, \cdots, x_{i-1}, x_i, qx_{i+1}, \cdots, qx_N),
\end{aligned}$$

which complete the deriavation of eq.(53).

[3] Here the general potential is composed of the monomials with an arbitrary degree, so we only have to prove that eq.(53) holds for the general form of monomials only.

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