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q-Oscillators with q a Root of Unity

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Abstract. In this paper the representation of q-deformed oscillator algebra with q a root of unity is discussed. In this case the ordinary oscillator algebra is shown to be obtained from the q-deformed oscillator algebra.

After Jimbo [1] and Drinfeld [2], quantum groups or q-deformations are widely investigated by a lot of authors. In spite of many papers in this direction, (as far as I know) nobody has found the meaning of quantum groups. Since Macfarlane [3] and Biedenharn [4] presented the q-deformed boson algebra, its mathematical structure has been studied by many theoretical physicists and mathematicians. However most of papers turned out to be purely mathematical and failed in giving the new physics. Recently Kobayashi and Suzuki [5] showed the possibility that we can find the geniune meaning of some quantum groups when the deformation parameter q is a root of unity.

In this paper we discuss the q-deformed oscillator algebra with q a root of unity and show that the infinite dimensional representation of q-deformed oscillator algebra is decomposed into the finite dimensional representation of q-deformed algebra and infinite dimensional representation of classical oscillator algebra.

Let us consider the q-oscillator algebra [6] generated by a and a_+ and N;

$$[N, a] = -a, \quad [N, a_+] = a_+, \quad aa_+ - qa_+a = 1, \tag{1}$$

where we call N the number operator and $a(a_+)$ play a lowering (raising) operator. Let us take q to be a root of unity such that $q = e^{2\pi i/p}, (p = 2, 3, \cdots)$. In this setting, the algebra

(1) goes to fermion algebra when $p = 2$, while it goes to boson algebra when p goes to 1. Thus it seems that algebra (1) describes a new kind of particle for arbitrary p. In eq.(1) the relation between N and a and a_+ is given by

$$a_+ a = [N], \quad a a_+ = [N+1] \tag{2}$$

where

$$[x] = \frac{q^x - 1}{q - 1}$$

When q goes to 1, the algebra goes to the ordinary boson case, so a_+ can be interpreted as the hermitian conjugate operator of a. But it is not true when the deformation parameter q is a complex number (root of unity). When q is a root of unity, a_+ is not a hermitian conjugate of a because the algebra (1) does not remain invariant under the hermitian conjugation. Let A^* be a hermitian conjugate of A. If we take the hermitian conjugate of eq.(1), we have

$$[N, a^*] = a^*, \quad [N, a_+^*] = -a_+^*, \quad a_+^* a^* - q^{-1} a^* a_+^* = 1, \tag{3}$$

where we set $N^* = N$. The eq.(3) does not seem to be consistent with eq.(1). Since both a^* and a_+ play a role of raising operator, we can say that there exists some transformation from a_+ to a^* such that

$$a_+ = G(N) a^*, \tag{4}$$

Now we will determine the concrete form of $G(N)$ so that a^* and a_+^* may satisfy the algebra (4). Inserting the relation (4) into (3), we get

$$G(N+1)^* G(N+1)^{-1} [N+1] - q^{-1} G(N)^* G(N)^{-1} [N] = 1 \tag{5}$$

This recurrence relation is easily solved. The function $G(N)$ should satisfy

$$\frac{G(N)^*}{G(N)} = \frac{[N]^*}{[N]} \tag{6}$$

The simplest choice for $G(N)$ becomes $G(N) = [N]$. So the transformation reads

$$a_+ = [N] a^*, \quad a_+^* = a[N]^* \tag{7}$$

Now let us consider the representation of this algebra. Introducing the Fock space basis $|n>$ of the number operator N by

$$N|n> = n|n>, \quad n = 0, 1, 2, \cdots \tag{8}$$

we have from the algebra (1) and relations (2)

$$a|n> = f(n)|n-1>, \quad a_+|n> = g(n)|n+1> \tag{9}$$

If we assume the existence of the ground state $|0>$ satisfying $a|0> = 0$, we should demand that $f(0) = 0$. From the fact that a^* is a hermitian adjoint of a and the transformation (4), we have the following representation for a and a_+

$$a|0> = 0, \quad a|n> = |n-1>, \quad (n \geq 1)$$

$$a_+|n>=[n+1]|n+1>, \quad (n \geq 0) \tag{10}$$

From the fact that $[p]=0$, we have $a_+|p-1>=0$. Thus $|p-1>$ is the highest state, so the spectrum becomes finite dimensional. For each p the maximal occupaption number becomes $p-1$ and the allowed states are

$$\{|0>,|1>,\cdots,|p-1>\}.$$

Therefore the representation for the opertors are given by

$$\begin{gathered} a|0>=0, \quad a|n>=|n-1>, \quad (n=1,2,\cdots,p-1) \\ a_+|n>=[n+1]|n+1>, \quad (n=0,1,\cdots,p-1) \\ a^*|n>=|n+1>, \quad (n=0,1,\cdots,p-2), \ \ and \ \ a^*|p-1>=0 \\ a_+^*|n>=[n]^*|n-1>, \quad (n=0,1,\cdots,p-1) \end{gathered} \tag{11}$$

Now we can obtain the p-dimensional matrix representation for each operator as follows;

$$a := \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{12}$$

$$a_+ := \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ [1] & 0 & & & 0 \\ \vdots & [2] & \ddots & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & \cdots & [p-1] & 0 \end{pmatrix} \tag{13}$$

$$a_+^* := \begin{pmatrix} 0 & [1]^* & \cdots & \cdots & 0 \\ 0 & 0 & [2]^* & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & [p-1]^* \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{14}$$

$$a^* := \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & 0 \\ \vdots & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix} \tag{15}$$

$$N := \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & p-1 \end{pmatrix} \tag{16}$$

It is easily checked that this representation fulfills the algebra (1) and algebra (3), which implies that both algebra is equivalent. Is there any other equivalent algebra ? Yes. We have the following two equivalent algebras;

$$aa^\star - a^\star a = |0 >< 0| - |p-1 >< p-1| \tag{17}$$

$$a_+^\star a_+ - a_+ a_+^\star = \frac{q^N}{1-q^{-1}} + \frac{q^{-N}}{1-q} \tag{18}$$

Now we will discuss the infinite dimensional reprentation of algebra (1) with q a root of unity. How can we obtain the infinite dimensional representation for algebra (1). From algebra (1) we know that

$$a_+|kp-1>=0 \quad for \quad k=0,1,\cdots \tag{19}$$

which implies that we can not obtain $|kp>$ state by acting the raising operator a_+ on $|kp-1>$ state. It seems that the highest state is just $|p-1>$ in this algebra. But if we can introduce new kind of operators as follows

$$A^+ = \frac{(a_+)^p}{[p]!} \quad and \quad A = a^p \tag{20}$$

we have

$$A^+|kp>=(k+1)|(k+1)p>$$
$$A|kp>=|(k-1)p> \quad (k=1,2,\cdots), \quad A|0>=0 \tag{21}$$

This shows that we can obtain $|kp>$ state by acting the new raising oprator A^+ on $|(k-1)p>$ state. Moreover these new operators A and A^+ satisfies the ordinary Boson algebra

$$AA^+ - A^+A = 1 \tag{22}$$

Thus the classical number operator N_c is defined by

$$N_c = A^+A = \frac{(a_+)^p a^p}{[p]!} \tag{23}$$

Using the algebra (1) and relation (2), we can relate the classical number operator N_c to the q-deformed number operator N through the following relation

$$N_c = \frac{[N][N-1]\cdots[N-p+1]}{[p]!} \tag{24}$$

It is easy to check that N_c is hermitian when N is hermitian and q is a root of unity. From the above fact we know that

$$\frac{(A^+)^{kp+r}}{[kp+r]!} = \frac{(A^+)^k}{k!}\frac{a_+^r}{[r]!} \quad (r = 0, 1, \cdots, p-1) \tag{25}$$

This means that the infinite dimensional representation of q-deformed oscillator algebra is decomposed into classical boson algebra and finite dimensional representation of q-deformed oscillator algebra. In this case highest weight modules are dipicted by the following diagram;

$$\begin{array}{ccccccccccccccc} \bullet & \rightleftharpoons & \circ & \rightleftharpoons & \circ & \rightleftharpoons & \cdots & \rightleftharpoons & \circ & \leftharpoondown & \bullet & \rightleftharpoons & \circ & \rightleftharpoons & \cdots \\ 0 & & 1 & & 2 & & & & p-1 & & p & & p+1 & & \\ \rightleftharpoons & \circ & \leftharpoondown & \bullet & \rightleftharpoons & \circ & \rightleftharpoons & \cdots & & & & & & & \\ & 2p-1 & & 2p & & 2p+1 & & & & & & & & & \end{array} \tag{26}$$

Here $\bullet$ and $\circ$ stand for classical state and q-deformed state, respectively. The arrow $\rightharpoonup$ corresponds to the action of a_+ and $\leftharpoondown$ does to a. Thus the calssical state $\bullet$ is obtained by acting the classical raising operator A^+ on the classical state.

In this paper we discussed the representation of q-deformed oscillator algebra with q a root of unity. It is shown that the infinite dimensional representation of q-deformed oscillator algebra is decomposed into the finite dimensional representation of q-deformed oscillator algebra and the classical oscillator algebra. This fact does not appear in tha case that q is real. Thus we can obtain the classical boson algebra from the q-deformed boson algebra.

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