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The Loop Algebra of Quantum Gravity

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Abstract. Certain topological aspects of the space of the classical T-observables as well as that of the space of quantum T-operators in the loop representation of quantum gravity will be analysed in some detail. It will be shown that the space of classical T-observables can be imbedded in a linear extension that admits a suitable equivalence relation such that the resulting quotient space supports a non-unital Hopf algebra structure. Finally, the classical loop phase space for general relativity will be constructed explicitly and a number of unusual features noted.

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1 Introduction

The classical T-observables introduced by Rovelli and Smolin [6] were applied to unravel the seemingly intractable problem of quantum gravity with some unexpected success. This led to the loop representation of quantum gravity which provided some intriguing insight into the quantum structure of space-time, with the prediction of an underlying discrete structure of space-time being one of its major highlights [5, p. 1661].

Let Σ denote a smooth, compact, connected Riemannian 3-manifold. Recall that the T^n-observables $T^n[\gamma, A, E](s^1, \ldots, s^n)$ were defined as

$$\mathrm{tr}(U_{\gamma,A}(s^2, s^1)E^{a_1}(\gamma(s^1))U_{\gamma,A}(s^3, s^2)E^{a_2}(\gamma(s^2)) \cdots U_{\gamma,A}(s^1, s^n)E^{a_n}(\gamma(s^n))),$$

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where $s^1, \ldots, s^n \in I \stackrel{\text{def}}{=} [0,1]$, γ is a piecewise smooth loop in Σ, $U_{\gamma,A}(t,s) \stackrel{\text{def}}{=} \mathcal{P}\, e^{\int_{\gamma(s)}^{\gamma(t)} A}$ is the spinor propagator of the Ashtekar connection 1-form A along the loop γ from $\gamma(s)$ to $\gamma(t)$ and E, a densitised triad of weight 1 on Σ, is the conjugate variable of A.

Observe from the definition of a T^n-observable that it is strictly not an n-contravariant tensor density in the conventional sense: it is not an element of $\mathbf{T}_x\Sigma \otimes \cdots \otimes \mathbf{T}_x\Sigma$ for any $x \in \Sigma$, where $\mathbf{T}_x\Sigma$ is the densitised tangent space of weight 1 at x. In a sense, it is non-local as a T^n-observable is essentially an element of the form

$$t^{a_1 \cdots a_n}(x_1, x_2, \ldots, x_n)\mathbf{e}_{a_1}(x_1) \otimes \cdots \otimes \mathbf{e}_{a_n}(x_n),$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a densitised frame on Σ and $t^{a_1 \cdots a_n}$ is a smooth symmetric function on Σ^n. And most certainly, the T-algebra $\mathcal{T}_{\mathrm{RS}} \stackrel{\text{def}}{=} \bigoplus_n \mathcal{T}^n_{\mathrm{RS}}$, where $\mathcal{T}^n_{\mathrm{RS}}$ is the set of T^n-observables, does not possess a linear structure. Indeed, each $\mathcal{T}^n_{\mathrm{RS}}$ itself does not support an obvious linear structure since different tensors belonging to different fibres in the tensor bundle over Σ cannot be added: for instance, $T^a[\gamma](s) + T^b[\eta](t) \notin \mathcal{T}_{\mathrm{RS}}$ whenever $\gamma(s) \neq \eta(t)$.

The loop space $\mathcal{M}_1$ of Σ in this paper will refer to the space consisting of piecewise smooth non-constant (parameterised) loops together with a zero loop, where the zero loop is just the point where all the constant loops in Σ are identified. It can be shown that this space is metrizable [7]. Explicitly, let $\tilde{\mathcal{L}}_\Sigma$ denote the set of piecewise smooth loops on Σ and let $\tilde{\pi} : \tilde{\mathcal{L}}_\Sigma \to \mathcal{M}_1$ be the natural map on $\tilde{\mathcal{L}}_\Sigma$ that identifies all constant loop with a single point 0_Σ in $\mathcal{M}_1$. The topology on $\tilde{\mathcal{L}}_\Sigma$ will be reviewed very briefly below and notations consistent with reference [7] will be used.[1]

1.1. Remarks. The reasons for introducing the loop topology below, aside from defining *continuous* loop functionals, are: (i) $\mathcal{M}_1$ becomes a second countable metrizable space that admits interesting non-trivial measures [7]. This is crucial in the construction of a physical inner product on the loop states defined by continuous loop functionals; (ii) the holonomy map for each fixed Ashtekar 1-form A given by $H[\gamma; A] = \mathcal{P}\, e^{\oint_\gamma A}$, is continuous on $\mathcal{M}_1$.

Given two loops $\gamma, \eta \in \tilde{\mathcal{L}}_\Sigma$, define $\hat{d}(\gamma,\eta) \stackrel{\text{def}}{=} \sup_{t \in I} d_q(\gamma(t), \eta(t))$, where d_q is a topological metric on Σ induced by fixing a Riemannian metric q on Σ. Note that for any two admissible Riemannian metrics q, q' on Σ, d_q is equivalent to $d_{q'}$ and hence the resulting topology induced by $\hat{d}$ is independent of the particular choice of admissible Riemannian metric on Σ. Second, fix a finite atlas $\mathfrak{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ on Σ and define $\hat{d}' : \tilde{\mathcal{L}}_\Sigma \times \tilde{\mathcal{L}}_\Sigma \to \mathbb{R}_+$ by

$$\hat{d}'(\gamma,\eta) \stackrel{\text{def}}{=} \text{ess } \sup\{\, \|D^\ell\gamma(t) - D^\ell\eta(t)\| : t \in I,\, \ell \geqq 1\,\},$$

where sup ranges over all the relevant (finite) charts, $D^\ell\gamma(t)$ denotes—in abused notation—the ℓth differential of γ at t, and *ess* means that the expression $\|D^\ell\gamma(t) - D^\ell\eta(t)\|$ is defined on I *apart* from a *finite* number (possibly zero) of points $\{\, t_1, \ldots, t_n\,\} \subset I$ wherein γ or η are *not* differentiable. Evidently, $\rho(\gamma,\eta) \stackrel{\text{def}}{=} \hat{d}(\gamma,\eta) + \hat{d}'(\gamma,\eta)$ defines a metric on $\tilde{\mathcal{L}}_\Sigma$. It can

[1] In the cited reference, $\mathcal{M}_1$ is denoted by $\mathcal{L}_\Sigma$. This space, endowed with the quotient topology, *not* a topological manifold.

be shown that the topology on $\tilde{\mathcal{L}}_\Sigma$ does not depend on the choice of finite atlas and hence is well-defined—for more details, refer to reference [7]. From here on, $\tilde{\mathcal{L}}_\Sigma$ will be endowed with the ρ-topology and $\mathcal{M}_1$ with the quotient topology. As mentioned earlier, $\mathcal{M}_1$ is metrizable; let ρ^* denote the topological metric compatible with its quotient topology. This metric will be used in section 5 below.

It is the primary purpose of this paper to study the topological structure of $\mathcal{T}_{\rm RS}$ and to construct a suitable linear extension of the classical T-observables. This, in turn, will lead to the explicit construction of a loop phase space for general relativity, yielding deeper insights into quantum gravity—more will be said below. The loop Poisson structure was analysed in detail by Rovelli and Smolin [6].

In the next section, some preliminary results needed to probe the structure of $\mathcal{T}^n$ will be developed. It will be seen that the space of classical T-observables (the *loop algebra*) can be endowed with a Frechét structure. In section 3, the algebra of the classical T-observables will be studied and it will be shown that by introducing an equivalence relation such that the resulting induced Poisson structure on the quotient space of the T-algebra is associative, the quotient space admits a non-unital Hopf algebra structure. New properties of the quantum loop algebra will be briefly described in section 4 and the relation between the classical Poisson structure and the quantum commutation structure will be neatly captured from a slightly different perspective. In section 5, the classical loop phase space for general relativity will be constructed. The structure of the phase space is the countable topological sum of spaces that do not admit manifold structures and it is speculated that this could well lead to some insight regarding the discrete nature of space-time as predicted by the quantum loop representation theory pioneered by Rovelli and Smolin.

2 Some Preliminary Discussions

The purpose of this section is to briefly sketch the space into which the classical loop algebra of quantum gravity is imbedded. In fact, a rather nice and simple topology can be constructed to give it a second countable Frechét structure. First of all, recall briefly the concept of an n-point tensor density. Let $\mathbf{v}_1, \mathbf{v}_2$ be two densitised vector field (of weight 1) on Σ. Then, the *2-point tensor density* $\mathbf{v}_1 \otimes \mathbf{v}_2$ is defined by

$$(\mathbf{v}_1 \otimes \mathbf{v}_2)(x,y) \equiv \mathbf{v}_1(x) \otimes \mathbf{v}_2(y) \stackrel{\rm def}{=} v_1^a(x) v_2^b(y) \mathbf{e}_a(x) \otimes \mathbf{e}_b(y),$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a densitised frame on Σ and $x, y \in \Sigma$ are distinct points.

Let Σ_+^2 denote a connected subset of Σ^2 such that (i) $\forall (x^1, x^2) \in \Sigma_+^2$, $x^1 \neq x^2$ and (ii) if $(x^1,x^2) \in \Sigma_+^2$, then $(x^2, x^1) \notin \Sigma_+^2$. Denote the topological closure of Σ_+^2 by $\overline{\Sigma}_+^2$. It is easy to extend the theory of tensor bundles to construct the triple $(E[\Sigma^2], \pi_2, \overline{\Sigma}_+^2)$, where $E[\Sigma^2] \stackrel{\rm def}{=} \bigcup_{x \in \overline{\Sigma}_+^2} E_x[\Sigma^2]$ with $E_{(x^1,x^2)}[\Sigma^2] \cong \mathbf{T}_{x^1}\Sigma \otimes \mathbf{T}_{x^2}\Sigma$, $\pi_2(E_x[\Sigma^2]) = x$ is the natural projection, and the 2-point tensor density field $\mathbf{v}$ is the cross section of the triple: $\pi_2 \circ \mathbf{v} = \mathrm{id}_{\overline{\Sigma}_+^2}$. Recall that $\mathbf{T}_x\Sigma$ denotes the weight 1 densitised tangent space at x.

If A is a set of linearly independent vectors that spans V over $\mathbb{C}$, denote this by $\operatorname{span}_{\mathbb{C}} A$. Let $D_{\overline{\Sigma}^2_+}$ be a countable dense subset of $\overline{\Sigma}^2_+$; that is, $\overline{D}_{\overline{\Sigma}^2_+} = \overline{\Sigma}^2_+$, and set

$$\mathcal{F}_2 \stackrel{\text{def}}{=} \{\, S \subset D_{\overline{\Sigma}^2_+} : |S| < \aleph_0 \,\},$$

where $|S|$ denotes the cardinality of S. Fix a densitised frame $V = \{\mathbf{e}_i\}$ on Σ and let $E_\infty[\Sigma^2] = \operatorname{span}_{\mathbb{C}}\{\, \mathbf{e}_i(x) \otimes \mathbf{e}_j(y) \mid (x,y) \in D_{\overline{\Sigma}^2_+},\ i = 1,2,3\}$. For each $D \in \mathcal{F}_2$, let $E[D;\mathcal{F}_2] = \operatorname{span}_{\mathbb{C}}\{\, \mathbf{e}_i(x) \otimes \mathbf{e}_j(y) \mid (x,y) \in D,\ i,j = 1,2,3\}$. Then, clearly there exists an inclusion $i_D : E[D;\mathcal{F}_2] \hookrightarrow E_\infty[\Sigma^2]$ for each $D \in \mathcal{F}_2$ given by

$$(\mathbf{v}^1, \ldots, \mathbf{v}^n) \mapsto (\mathbf{v}^1, \ldots, \mathbf{v}^n, 0, 0, \ldots).$$

Let $E_\infty[\Sigma^2]$ be endowed with the topology such that each inclusion i_D is continuous, where $E[D;\mathcal{F}_2]$ is given a norm topology for each $D \in \mathcal{F}_2$. In particular, directing $\mathcal{F}_2$ by set inclusion $\subseteq$, it is not difficult to see that $E_\infty[\Sigma^2]$ is the direct limit of the system $(E[D;\mathcal{F}_2], p^D_{D'}, \mathcal{F}_2)$, where $p^D_{D'} : E[D;\mathcal{F}_2] \hookrightarrow E[D';\mathcal{F}_2]$ is defined by

$$(\mathbf{v}^1, \ldots, \mathbf{v}^n) \mapsto (\mathbf{v}^1, \ldots, \mathbf{v}^n, \overbrace{0, \ldots, 0}^{m \text{ times}}),$$

$|D| = n$ and $|D'| = n + m$.

Now, observe that the densitised frame V induces a canonical isomorphism from $\mathbf{T}_x\Sigma$ onto $\mathbf{T}_y\Sigma$ for each pair $x, y \in \Sigma$ given by $\mathbf{e}_i(x) \mapsto \mathbf{e}_i(y)$. Note also that for any vector density $\mathbf{v}(x) \in \mathbf{T}_x\Sigma$, there exists a sequence $\{x_n\}_n$ in D_Σ, where D_Σ is a dense countable subset of Σ, such that $\lim_{n\to\infty} x_n = x$ and hence, $\lim_{n\to\infty} \mathbf{v}(x_n) = \mathbf{v}(x)$ by continuity. So, let $\hat{E}_\infty[\Sigma^2]$ denote the sequential completeness of $E_\infty[\Sigma^2]$ and call this space the *associated infinite dimensional linear space* of $E[\Sigma^2]$.

The above construction can be generalised easily to yield $E[\Sigma^n]$ and $\hat{E}_\infty[\Sigma^n]$ and hence, *n-point contravariant n-tensor density fields* on Σ. Let $E_\Sigma = \bigoplus_{n=1}^\infty \hat{E}_\infty[\Sigma^n]$ be the direct sum of $\hat{E}_\infty[\Sigma^n]$'s, where $\hat{E}_\infty[\Sigma^0] \stackrel{\text{def}}{=} \mathbb{C}$. Then, $(E_\Sigma, \otimes)$ forms the *multi-point tensor density algebra* of Σ. In fact, from the construction, E_Σ is seen to be a commutative $\mathbb{C}$-algebra. A Frechét topology can be endowed on E_Σ and this will be done below.

Let $\mathfrak{C} = \{\mathfrak{A}_i : i \in \mathbb{N}\}$ be a countable set of finite atlases $\mathfrak{A}_i$ of Σ, where $\mathfrak{C}$ is fixed once and for all. Then $\mathfrak{C}$ induces a Frechét structure on $\hat{E}_\infty[\Sigma^n]$ in the following way. First, some notations will be introduced. Fix a finite atlas $\mathfrak{A}$ of Σ and an n-tuple $(x^1, \ldots, x^n) \in \overline{\Sigma}^n_+$. Let $C_\alpha(x^1, \ldots, x^n)$ denote a particular choice of distinct charts $U_1, \ldots, U_m$, $m \leqq n$, about $x_1, \ldots, x_n$ respectively, and let $S_{x^1 \ldots x^n}(\mathfrak{A})$ be the set of $C_\alpha(x^1, \ldots, x^n)$'s, where $C_\alpha(x^1, \ldots, x^n)$ will be identified with the set $\{U_1, \ldots, U_n\}$. Then, $S_{x^1 \ldots x^n}(\mathfrak{A})$ is a finite set by definition.

Second, it will be shown that each finite atlas $\mathfrak{A}_i$ induces a metric d_i on $\hat{E}_\infty[\Sigma^n]$. For each pair of elements $\mathbf{v}(x^1, \ldots, x^n), \bar{\mathbf{v}}(\bar{x}^1, \ldots, \bar{x}^n) \in \hat{E}_\infty[\Sigma^n]$, define $d_i(\mathbf{v}(x^1, \ldots, x^n), \bar{\mathbf{v}}(\bar{x}^1, \ldots, \bar{x}^n))$ by

$$\max\{|v^{a_1 \ldots a_n}_\alpha(x_1, \ldots, x_n) - \bar{v}^{a_1 \ldots a_n}_\alpha(\bar{x}_1, \ldots, \bar{x}_n)| : a_1, \ldots, a_n = 1,2,3\},$$

where the coordinate representation of $\mathbf{v}(x^1, \ldots, x^n)$, denoted by $v^{a_1 \ldots a_n}_\alpha(x_1, \ldots, x_n)$, runs over all elements $C_\alpha(x^1, \ldots, x^n)$ in $S_{x^1 \ldots x^n}(\mathfrak{A})$ and, in a similar way, $\bar{v}_\alpha(\bar{x}^1, \ldots, \bar{x}^n)$ runs over all elements in $S_{\bar{x}^1 \ldots \bar{x}^n}(\mathfrak{A})$. It is clear from the definition of $\hat{E}_\infty[\Sigma^2]$ that this metric is complete.

2.1. Proposition. *The pair $(\mathfrak{C}, \{d_k\}_{k=1}^{\infty})$ induces a Frechét structure on $\hat{E}_\infty[\Sigma^n]$.*

Proof. For each element $\mathbf{v}(x^1, \ldots, x^n)$ in $\hat{E}_\infty[\Sigma^n]$, define a $\mathfrak{C}$-open ball $B_{\varepsilon_1 \ldots \varepsilon_n}(\mathbf{v}(x^1, \ldots, x^n))$ about $\mathbf{v}(x^1, \ldots, x^n)$ by

$$\{\, \mathbf{u}(y^1, \ldots, y^n) : d_{i_j}(\mathbf{u}(y^1, \ldots, y^n), \mathbf{v}(x^1, \ldots, x^n)) < \varepsilon_j,\ j = 1, \ldots, n \,\},$$

where $\varepsilon_1, \ldots, \varepsilon_n > 0$. Since this topology is compatible with the metric $d \stackrel{\text{def}}{=} \sum_k \frac{1}{k^2} \frac{d_k}{1+d_k}$, only completeness need be established. However, as each metric d_k is itself complete on $\hat{E}_\infty[\Sigma^n]$, it follows at once that d is complete. □

2.2. Proposition. *The Frechét structure on $\hat{E}_\infty[\Sigma^n]$ is independent of the choice of $\mathfrak{C}$ and hence is well-defined.*

Proof. Let $\mathfrak{C}'$ be another countable set of finite atlases on Σ distinct from $\mathfrak{C}$. Then, it is an easy matter to see that the identity map $\mathrm{id} : (\hat{E}_\infty[\Sigma^n], \mathfrak{C}) \to (\hat{E}_\infty[\Sigma^n], \mathfrak{C}')$ is bicontinuous. □

2.3. Theorem. *E_Σ admits a Frechét structure and is second countable.*

Proof. E_Σ is the countable direct sum of second countable Frechét spaces. □

It will be clear in the following section that the loop algebra trivially inherits a Frechét structure from E_Σ. From that, a new way of looking at the Poisson structure on the extended loop algebra will be described and, what is more, it is shown to support a non-unital Hopf structure after quotiening part of the space away. Interestingly enough, in §4, it will be seen that the quantum T-algebra actually covers the extended classical loop algebra.

3 The Classical T-algebra

Let γ be a loop in Σ and define

$$P_n^\alpha(\gamma) = \{\, (s_\alpha^1, \ldots, s_\alpha^n) \in I^n \mid \gamma(s_\alpha^i) \neq \gamma(s_\alpha^j) \text{ for } i \neq j \,\},$$

where $0 \leqq s_\alpha^1 < \cdots < s_\alpha^n < 1$, and let $\mathcal{P}_n(\gamma)$ denote the set of $P_n^\alpha(\gamma)$'s. Set $\mathfrak{L}_\Sigma^n = \bigcup_{\gamma \in \mathcal{L}_\Sigma} \{\gamma\} \times \mathcal{P}_n(\gamma)$ and let $\mathcal{A}$ denote the space of Ashtekar connection 1-forms. Recall also that the Ashtekar phase space of general relativity is the following infinite dimensional manifold: $\Gamma_\mathcal{A} \stackrel{\text{def}}{=} \{\, (A, E) \mid C_i(A, E) = 0,\ i = 1, 2, 3 \,\}$, where C_1 is the Gauss constraint, C_2 is the diffeomorphism constraint, C_3 is the Hamiltonian constraint and E is a densitised $\mathfrak{su}(2)$-soldering form on Σ. See reference [1] for more details.

Define a map $T^n : \mathfrak{L}_\Sigma^n \times \Gamma_\mathcal{A} \to \hat{E}_\infty[\Sigma^n]$ by

$$((\gamma, s^1, \ldots, s^n), (A, E)) \mapsto T^n[\gamma, A, E](s^1, \ldots, s^n).$$

Then, the T^n-algebra $\mathcal{T}^n$ is defined to be the tensor subalgebra in $\hat{E}_\infty[\Sigma^n]$ generated by $T^n(\mathfrak{L}_\Sigma^n \times \Gamma_\mathcal{A}) \cup T^i(\mathfrak{L}_\Sigma^i \times \Gamma_\mathcal{A}) \otimes T^j(\mathfrak{L}_\Sigma^j \times \Gamma_\mathcal{A}) \cup \cdots \cup T^{i_1}(\mathfrak{L}_\Sigma^{i_1} \times \Gamma_\mathcal{A}) \otimes \cdots \otimes T^{i_p}(\mathfrak{L}_\Sigma^{i_p} \times \Gamma_\mathcal{A}) \cup \cdots \cup$

$T^1(\mathfrak{L}_\Sigma \times \Gamma_{\mathcal{A}})\otimes\cdots\otimes T^1(\mathfrak{L}_\Sigma \times \Gamma_{\mathcal{A}})$ (n factors), where $i+j=n,\ldots,i_1+\cdots+i_p=n$, $1\leqq p<n$. Finally, set $\mathcal{T}^0\equiv\mathbb{C}$. The graded sum $\mathcal{T}\stackrel{\text{def}}{=}\bigoplus_{n=0}^{\infty}\mathcal{T}^n$ is closed relative to $\otimes$ and it is called the *extended loop algebra* of quantum gravity. By theorem 2.3, $\mathcal{T}$ supports a Frechét structure.

Some comments are now due. First, note in passing that because $U_{\gamma,A}$ is invariant under any orientation preserving reparametrisations of γ, given any $T^1[\gamma,A,E](s)$ and $s\in I$, there exists a unique reparametrisation γ' of γ such that $T^1[\gamma,A,E](s)\equiv T^1[\gamma',A,E](0)$. More generally, given $T^n[\gamma,A,E](s^1,\ldots,s^n)$, there exists a unique reparametrisation γ' of γ such that $T^n[\gamma,A,E](s^1,\ldots,s^n)\equiv T^n[\gamma',A,E](0,t^1,\ldots,t^{n-1})$. Hence, in all that follows, it may be assumed without any loss of generality that $s^1\equiv 0$ and in particular, let $T^1[\gamma,A,E]$ denote $T^1[\gamma,A,E](0)$. Second, let $\mathcal{T}_{\text{RS}}$ denote the loop "algebra" as defined by Rovelli and Smolin [6]. From the above construction, $\mathcal{T}_{\text{RS}}\subset\mathcal{T}$.

The space $\mathcal{T}_{\text{RS}}$ is strictly not an algebra although a Poisson structure can be endowed on it: $\mathcal{T}_{\text{RS}}$ is not a linear space since, for example, given a pair of non-intersecting loops (γ,η), $\gamma(I)\cap\eta(I)=\varnothing$, $T^{a_1\ldots a_n}[\gamma,A,E]+T^{b_1\ldots b_m}[\eta,A,E]\notin\mathcal{T}_{\text{RS}}$ $\forall n,m$; indeed, the quantity is not even defined. Second, $\mathcal{T}$ is in some sense a genuine enlargement of $\mathcal{T}_{\text{RS}}$—by this is meant that elements such as $T^n[\gamma,A,E]\otimes T^m[\gamma',A',E']$ also belongs to $\mathcal{T}$ although the physical significance of such elements are yet to be determined. However, consider $-T^1[\,\cdot\,,A,E]\otimes T^1[\,\cdot\,,A,E]$. This quantity determines a densitised Riemannian 3-metric $\det\bar{q}\cdot\bar{q}$ on Σ for some $\bar{q}$. Hence, in this enlarged loop algebra, a Riemannian metric on Σ can be recovered from a T-observable without shrinking loops down to a point. Lastly, let $\bar{\mathcal{T}}_{\text{RS}}$ be spanned by $\mathcal{T}_{\text{RS}}$ in $\mathcal{T}$ and let $\mathcal{T}^n_{\text{RS}}$ denote the set of T^n-observables. Then, $\mathcal{T}^n_{\text{RS}}=T^n(\mathfrak{L}^n_\Sigma\times\Gamma_{\mathcal{A}})$ and $\bar{\mathcal{T}}_{\text{RS}}$ is not closed under $\otimes$. It is not difficult to verify that $\mathcal{T}^n_{\text{RS}}$ is path connected in $\bar{\mathcal{T}}_{\text{RS}}$ for each n.

Note in passing that a Poisson structure cannot be naïvely imposed on $\mathcal{T}$ simply because $\{\tau_n,\tau_m\}$ is a $\mathcal{T}$-valued distribution, where $\tau_n,\tau_m\in\mathcal{T}$. The precise sense of the word distribution will be clarified below. An alternative way of imposing the Poisson structure on the loop algebra will be given in this section; this differs from the conventional current algebra type structure that defines the loop algebra. For more details regarding the full Poisson algebraic relationship between elements in $\mathcal{T}_{\text{RS}}$, see reference [6].

First, observe that if ω is a 1-form on Σ, then

$$\int_0^1 \mathrm{d}t\int_\Sigma \mathrm{d}^3x\,\omega_a(x)\dot{\eta}^a(t)\delta^3(x,\eta(t))=\oint_\eta\omega. \tag{3.1}$$

Hence, it is clear from equation (3.1) that the singularity $\Delta^a[\gamma,\eta]\stackrel{\text{def}}{=}\int_0^1\mathrm{d}t\,\delta^3(\gamma(s),\eta(t))\dot{\eta}^a(t)$ that appears in the Poisson brackets of the T-observables can be regularised by a 1-form on setting $\gamma(s)=x$ and then integrating over x. Second, given two loops γ and η, and assuming that the loop composition is defined, let $(\gamma*\eta)^+\stackrel{\text{def}}{=}\gamma*\eta$, $(\gamma*\eta)^-\stackrel{\text{def}}{=}\gamma*\eta_-$ and $\eta_-(t)\stackrel{\text{def}}{=}\eta(1-t)$, where

$$\gamma *_s \eta(t)\stackrel{\text{def}}{=}\begin{cases}\gamma(2t) & \text{for } 0\leqq t\leqq s,\\ \eta(2t-1) & \text{for } s\leqq t\leqq 1,\end{cases}$$

and $\gamma*\eta\stackrel{\text{def}}{=}\gamma*_{\frac{1}{2}}\eta$. Then, from the definition of the Poisson brackets of the canonical pair (A,E)—$\{A_a{}^{AB}(x),E^b{}_{CD}(y)\}=-\frac{\mathrm{i}}{\sqrt{2}}\delta^b_a\delta_C{}^{(A}\delta_D{}^{B)}\delta^3(x,y)$—and hence from the

Poisson brackets of the T-observables [6, p. 102]

$$\begin{aligned}(3.2)\qquad &\{T^{a_1\ldots a_n}[\gamma](s^1,\ldots,s^n),T^{b_1\ldots b_m}[\eta](t^1,\ldots,t^m)\}\\ &= -\mathrm{i}\sum_{k=1}^{n}\sum_{\epsilon}(-1)^{|\epsilon|}\Delta^{a_k}[\gamma,\eta](s^k)T^{a_1\ldots\hat{a}_k\ldots a_n b_1\ldots b_m}[(\gamma *_{s^k}\eta)^{\epsilon}]\\ &\quad+\mathrm{i}\sum_{k=1}^{m}\sum_{\epsilon}(-1)^{|\epsilon|}\Delta^{b_k}[\eta,\gamma](t^k)T^{b_1\ldots\hat{b}_k\ldots b_m a_1\ldots a_n}[(\eta *_{t^k}\gamma)^{\epsilon}],\end{aligned}$$

it follows that $\{T^m[\gamma,A,E],T^n[\eta,A',E']\}\equiv 0$ unless A and A' belong to the same SU(2) gauge orbit irrespective of its conjugate E, where E lies in the "fibre" over A, $\hat{a}_k$ denotes the deletion of index a_k and likewise for $\hat{b}_k$; $\epsilon\in\{+,-\}$ denotes the two possible ways of rearranging $\gamma *_{s^k}\eta$ defined above and $|+|=0$, $|-|=1$. In view of this fact, the Poisson brackets of the two T-observables may be simplified to $\{T^n[\gamma],T^m[\eta]\}$ should no confusion arise, where it is understood that the Ashtekar 1-forms that define $T^n[\gamma]$ and $T^m[\eta]$ are SU(2) equivalent.

Let $\Lambda^1(\Sigma)$ denote the space of smooth 1-forms on Σ and set $P(T^m[\gamma],T^n[\eta]) = \{T^m[\gamma],T^n[\eta]\}$. Given $\omega\in\Lambda^1(\Sigma)$, define

$$P(T^m[\gamma],T^n[\eta])(\omega)\stackrel{\text{def}}{=}\int_{\Sigma}\mathrm{d}^3x\,\omega\cdot\{T^n[\gamma],T^m[\eta]\}.$$

Then, from equations (3.1) and (3.2), $P(T^n[\gamma],T^m[\eta])(\omega)\in\mathcal{T}_{\mathrm{RS}}$ for each $\omega\in\Lambda^1(\Sigma)$. Hence, P maps $\mathcal{T}_{\mathrm{RS}}\times\mathcal{T}_{\mathrm{RS}}$ into a $\mathcal{T}_{\mathrm{RS}}$-valued distribution space of $\Lambda^1(\Sigma)$. This provides an alternative description of the loop algebra; and the smearing procedure here turns out to be remarkably simpler when compared to that given in references [2, 6].

Currently in the literatures on the loop representation of quantum gravity, the T-observables are treated as some kind of distribution in the sense that they need to be smeared. In this paper, the burden of distribution is shifted onto the Poisson structure—this seems a more natural and much neater way of interpreting the (classical) T-observables for the simple reason that $T^k[\gamma]$'s are not functions. And what is more, the space of T-observables is not a current algebra: that is, the space cannot be decomposed into $C^\infty(\Sigma)\otimes\mathfrak{su}^{\mathbf{C}}(2)$ or even into $\bigoplus_n\Gamma^\infty(\Sigma,\mathbf{T}^n\Sigma)\otimes\mathfrak{su}^{\mathbf{C}}(2)$, where $\Gamma^\infty(\Sigma,\mathbf{T}^n\Sigma)$ is the space of smooth cross sections of the densitised contravariant n-tensor bundle over Σ and $\mathfrak{su}^{\mathbf{C}}(2)$ is the complexification of $\mathfrak{su}(2)$. From this perspective, it would seem unnatural that the T-observables, which are not even functions, need smearing. It therefore appears more appropriate that the Poisson structure be smeared instead: in other words, $\mathcal{T}_{\mathrm{RS}}$ admits only a distributional Poisson structure. In fact, with a little bit of tedious algebra, it can be shown that the Poisson brackets do not satisfy the Jacobian identity and hence the Poisson structure does not even support a Lie algebra structure. The term distribution will now be made precise.

Let $\Lambda^1(\Sigma)'$ denote the topological dual of $\Lambda^1(\Sigma)$. Set $\mathcal{X}_{\mathrm{RS}}=\Lambda^1(\Sigma)'\otimes\mathcal{T}_{\mathrm{RS}}$. Elements of $\mathcal{X}_{\mathrm{RS}}$ are called $\mathcal{T}_{\mathrm{RS}}$*-valued* $\Lambda^1(\Sigma)$*-distributions* on Σ. It is clear that given any pair $T^n[\gamma],T^m[\eta]\in\mathcal{T}_{\mathrm{RS}}$, $P(T^n[\gamma],T^m[\eta])\in\mathcal{X}_{\mathrm{RS}}$. Hence, $P:\mathcal{T}_{\mathrm{RS}}\times\mathcal{T}_{\mathrm{RS}}\to\mathcal{X}_{\mathrm{RS}}$ and P :

$\mathcal{T}_{\mathrm{RS}} \times \mathcal{T}_{\mathrm{RS}} \times \Lambda^1(\Sigma) \to \mathcal{T}_{\mathrm{RS}}$. The P-structure can be extended to the entire loop algebra $\mathcal{T}$. To see this, set $\mathcal{X} = \Lambda^1(\Sigma)' \otimes \mathcal{T}$ and define $\bar{P} : \mathcal{T} \times \mathcal{T} \to \mathcal{X}$ by

$$\bar{P}(T^n[\gamma] \otimes T^m[\eta], T^k[\xi]) \stackrel{\text{def}}{=} T^n[\gamma] \otimes P(T^m[\eta], T^k[\xi]) + P(T^n[\gamma], T^k[\xi]) \otimes T^m[\eta].$$

It is easy to see from the definition of $\bar{P}$ that $\bar{P}|\mathcal{T}_{\mathrm{RS}} = P$. The pair $(\mathcal{T}, \bar{P})$ is called a $\Lambda^1(\Sigma)$-*distributional loop algebra*, where $\bar{P}$ determines its algebra structure.

This section will conclude with the construction of a non-unital Hopf algebra structure on a quotient space of $\mathcal{T}$. This will hopefully provide a preliminary link between quantum groups and quantum gravity. Because the P-algebra on $\mathcal{T}$ is not associative, it is not possible to construct such a structure on it directly. To get around this, consider the subspace $\mathcal{J}_{\mathrm{RS}} \subset \bar{\mathcal{T}}_{\mathrm{RS}}$ spanned by elements of the form

$$\begin{aligned}
&P(T^n[\gamma], P(T^m[\eta], T^k[\xi])(\omega))(\omega) - P(P(T^n[\gamma], T^m[\eta])(\omega), T^k[\xi])(\omega),\\
&P(T^{n_1}[\gamma_1], P(T^{n_2}[\gamma_2], P(T^{n_3}[\gamma_3], T^{n_4}[\gamma_4])(\omega))(\omega))(\omega) -\\
&P(T^{n_1}[\gamma_1], P(P(T^{n_2}[\gamma_2], T^{n_3}[\gamma_3])(\omega), T^{n_4}[\gamma_4])(\omega))(\omega), \ldots
\end{aligned}$$

and so forth, where $\omega \in \Lambda^1(\Sigma)$. It is easy to see that $\mathcal{J}_{\mathrm{RS}}$ defines a P-ideal in $\bar{\mathcal{T}}_{\mathrm{RS}}$.

Set $\mathfrak{T}_{\mathrm{RS}} \stackrel{\text{def}}{=} \bar{\mathcal{T}}_{\mathrm{RS}}/\mathcal{J}_{\mathrm{RS}}$ and let $\pi : \bar{\mathcal{T}}_{\mathrm{RS}} \to \mathfrak{T}_{\mathrm{RS}}$ be the canonical projection. The P-structure on $\bar{\mathcal{T}}_{\mathrm{RS}}$ can be projected onto $\mathfrak{T}_{\mathrm{RS}}$ by setting $P_I(\pi(T^n[\gamma]), \pi(T^m[\eta])) \stackrel{\text{def}}{=} \pi \circ P(T^n[\gamma], T^m[\eta])$. It is routine to verify that P_I is indeed well-defined and hence π is a P_I-epimorphism. Moreover, it follows from the construction that $\mathfrak{T}_{\mathrm{RS}}$ is an associative P_I-algebra.

3.1. Proposition. *$\mathfrak{T}_{\mathrm{RS}}$ admits a non-unital, Hopf algebra structure.*

Proof. Define multiplication $\mu : \mathfrak{T}_{\mathrm{RS}} \otimes \mathfrak{T}_{\mathrm{RS}} \to \mathfrak{T}_{\mathrm{RS}}$ by $\mu : a \otimes b \mapsto P_I(a, b)$. Then, the associativity of P_I implies that μ is also associative. Define comultiplication $\delta : \mathfrak{T}_{\mathrm{RS}} \to \mathfrak{T}_{\mathrm{RS}} \otimes \mathfrak{T}_{\mathrm{RS}}$ by $T^n[\gamma] \mapsto T^n[\gamma] \otimes T^n[\gamma]$. It is trivial to check that $\delta \otimes (\mathrm{id} \circ \delta) = (\mathrm{id} \otimes \delta) \circ \delta$. Next, define the counit to be the constant map $\epsilon : \mathfrak{T}_{\mathrm{RS}} \to \mathbb{C}$ by $T^n[\gamma] \mapsto 1$. Then, ϵ satisfies the counit axiom trivially: $(\mathrm{id} \otimes \epsilon) \circ \delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \delta$. The pair (μ, ϵ) defines a non-unital bialgebra structure on $\mathfrak{T}_{\mathrm{RS}}$ as the following connection axioms can be verified easily: $\delta \circ \mu = (\mu \otimes \mu) \circ \sigma_{23} \circ (\delta \otimes \delta)$ and $\epsilon(P_I(T^n[\gamma], T^m[\eta])) = 1 = \epsilon(T^n[\gamma])\epsilon(T^m[\eta])$, where $\sigma_{23} : \mathfrak{T}_{\mathrm{RS}} \otimes \mathfrak{T}_{\mathrm{RS}} \otimes \mathfrak{T}_{\mathrm{RS}} \otimes \mathfrak{T}_{\mathrm{RS}} \to \mathfrak{T}_{\mathrm{RS}} \otimes \mathfrak{T}_{\mathrm{RS}} \otimes \mathfrak{T}_{\mathrm{RS}} \otimes \mathfrak{T}_{\mathrm{RS}}$ is the flip homomorphism given by $a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto a_1 \otimes a_3 \otimes a_2 \otimes a_4$. Finally, define the antipode $S : \mathfrak{T}_{\mathrm{RS}} \to \mathfrak{T}_{\mathrm{RS}}$ by $T^n[\gamma] \mapsto -T^n[\gamma]$. Once again, it is an easy matter to show that $\mu \circ (S \otimes \mathrm{id}) \circ \delta = \mu \circ (\mathrm{id} \otimes S) \circ \delta$ and the proposition is thus established. □

This proposition gives a tentative hint that there might perhaps be an underlying connection between quantum groups and the loop algebra of quantum gravity.

4 Quantum Loop Algebra

In this section, elementary topological and algebraic properties of the quantum loop algebra will be derived. Before doing so, observe that the quantum loop algebra $\hat{\mathcal{T}}_{\mathrm{RS}}$, where its elements are linear maps acting on the space of multi-loop functionals ψ obtained from $T^n[\gamma] \mapsto \hat{T}^n[\gamma]$, trivially admits a linear structure and, furthermore, $\hat{\mathcal{T}}_{\mathrm{RS}}$ possesses a ring structure under composition: $(\hat{T}^n[\gamma] \circ \hat{T}^m[\eta])(\psi) \stackrel{\text{def}}{=} \hat{T}^n[\gamma](\hat{T}^m[\eta](\psi))$. The map $\lambda : \bar{\mathcal{T}}_{\mathrm{RS}} \longrightarrow \hat{\mathcal{T}}_{\mathrm{RS}}$ given by $T^n[\gamma] \mapsto \hat{T}^n[\gamma]$ is clearly a linear monomorphism and there exists a natural projection $\pi_{\mathrm{RS}} : \hat{\mathcal{T}}_{\mathrm{RS}} \longrightarrow \mathcal{T}$ given by $T_1 \circ T_2 \mapsto T_1 \otimes T_2$.

In all that follows, $\hat{\mathcal{T}}_{\mathrm{RS}}$ will be endowed with the weakest topology such that λ defines a topological imbedding and π_{RS} a continuous surjection. Observe trivially that if $i_{\mathrm{RS}} : \bar{\mathcal{T}}_{\mathrm{RS}} \hookrightarrow \mathcal{T}$ is the inclusion map, then $i_{\mathrm{RS}} = \pi_{\mathrm{RS}} \circ \lambda$. Hence, π_{RS} is also open by definition and the topology thus endowed on $\hat{\mathcal{T}}_{\mathrm{RS}}$ is Frechét. Now, let $\hat{\mathcal{T}}^n_{\mathrm{RS}}$ be spanned by elements of the form:

$$\hat{T}^n[\gamma], \ldots, \hat{T}^{i_1}[\gamma_1] \circ \cdots \circ \hat{T}^{i_p}[\gamma_p], \ldots, \hat{T}^1[\gamma'_1] \circ \cdots \circ \hat{T}^1[\gamma'_n] \ (n \text{ factors})$$

$\forall\, i_1 + \cdots + i_p = n$, $1 \leqq p < n$, and define an equivalence relation $\sim$ on $\hat{\mathcal{T}}^n_{\mathrm{Rs}}$ as follows:

$$\hat{T}^{i_1}[\gamma_1] \circ \cdots \circ \hat{T}^{i_p}[\gamma_p] \sim \hat{T}^{\sigma(i_1)}[\gamma_{\sigma(1)}] \circ \cdots \circ \hat{T}^{\sigma(i_p)}[\gamma_{\sigma(p)}]$$

for each permutation σ of $\{1, \ldots, p\}$. Let $\hat{\mathcal{T}}^n_{\mathrm{Rs}}/\sim$ denote the quotient space and $\hat{\pi}_n : \hat{\mathcal{T}}^n_{\mathrm{RS}} \longrightarrow \hat{\mathcal{T}}^n_{\mathrm{RS}}/\sim$ the natural map. Then, it can be easily checked that $(\hat{\mathcal{T}}^n_{\mathrm{RS}}, \hat{\pi}_n)$ is the covering space of $\hat{\mathcal{T}}^n_{\mathrm{RS}}/\sim$. However, since by definition, $\mathcal{T}^n \cong \hat{\mathcal{T}}^n_{\mathrm{RS}}/\sim$, it follows at once that $\hat{\mathcal{T}}^n_{\mathrm{RS}}$ covers $\mathcal{T}^n$ and hence $(\hat{\mathcal{T}}_{\mathrm{RS}}, \pi_{\mathrm{RS}})$ is a covering space of $\mathcal{T}$. This in itself raises a very intriguing observation: that the quantum observable space is a covering space for the classical observable space in the loop representation of quantum gravity.

Let ξ be a piecewise smooth loop in Σ with $n > 0$ self-intersections at points $\xi(s^1), \ldots, \xi(x^n)$, where $0 \leqq s^1 < \cdots < s^n < 1$ and $\xi(s^i)$ need not all be distinct. For a given point $\xi(s^i)$, it will be shown below that ξ can be uniquely decomposed into two loops ξ_1, ξ_2 attached at $\xi(s^i)$. This rather obvious fact will give rise to the construction of a mapping from the classical loop algebra into the quantum loop algebra valued distribution space.

4.1. Lemma. *Let ξ be a loop possessing a finite number of self-intersections $\xi(s^1), \ldots, \xi(s^n)$. Then, each point $\xi(s^i)$ determines a unique decomposition (up to orientation-preserving reparametrisation) of ξ into ξ_1 and ξ_2.*

Proof. If the origin of ξ is shifted from 0 to some point $u \in I$, then denote this reparameterised loop by ξ_u, where

$$\xi_u(t) \stackrel{\text{def}}{=} \begin{cases} \xi(t+u) & 0 \leqq t \leqq 1-u, \\ \xi(t-1+u) & 1-u \leqq t \leqq 1. \end{cases}$$

Now, fix s^i and shift the origin of ξ to $\xi(s^i)$: $\xi \to \xi_{s^i}$. Define $\xi_1(t) \stackrel{\text{def}}{=} \xi_{s^i}(2t^i t)$ on $[0, \frac{1}{2}]$, where $\xi_{s^i}(0) = \xi_{s^i}(t^i)$, and $\xi_2(t) \stackrel{\text{def}}{=} \xi_{s^i}(2(1-t^i)t + 2t^i - 1)$ on $[\frac{1}{2}, 1]$. Then, $\xi \equiv \xi_1 *_{s^i} \xi_2$ by construction. The uniqueness (up to reparametrisation) is obvious. $\square$

Using lemma 4.1, it is possible to construct a sequence of maps $\mathbf{f}_n$ each of which maps $\bar{\mathcal{T}}_{\text{RS}}$ into a $\bar{\mathcal{T}}_{\text{RS}}$-valued distribution space. To begin with, let $\mathbf{f}_0$ be the identity map on $\bar{\mathcal{T}}_{\text{RS}}$. Given $T^n[\gamma](s^1, \ldots, s^n)$, define $\mathbf{f}_1(T^n[\gamma](s^1, \ldots, s^n))$ by

$$\begin{aligned}&\sum_{i=1}^{n} \Delta[\gamma_1, \gamma_2](s^i) T^{n-1}[\gamma_1 *_{s^i} \gamma_2](s^1, \ldots, \hat{s}^i, \ldots, s^n)\\ &- \sum_{i=1}^{n} \Delta[\gamma_1, \gamma_{2-}](s^i) T^{n-1}[\gamma_1 *_{s^i} \gamma_{2-}](s^1, \ldots, \hat{s}^i, \ldots, s^n),\end{aligned}$$

where $\hat{s}^i$ denotes the deletion of s^i, (γ_1, γ_2) is the unique decomposition of γ determined by lemma 4.1 and γ_{2-} is the reverse orientation of γ_2. Observe from the definition that if γ has no self-intersection, then $\mathbf{f}_1(T^n[\gamma](s^1, \ldots, s^n)) = 0$.

Given two loops γ and η which intersects at at least two points $\gamma(s^i) = \eta(t^i)$ for $i = 1, 2$, define $\gamma *_{s^1 s^2} \eta$ to be the set of four possible loops $(\gamma *_{s^1 s^2} \eta)^{++}, (\gamma *_{s^1 s^2} \eta)^{--}, (\gamma *_{s^1 s^2} \eta)^{-+}$ and $(\gamma *_{s^1 s^2} \eta)^{+-}$ obtained from all the possible ways of joining and breaking at s^1 and s^2. They are defined explicitly below. First, suppose for simplicity that when γ an η are projected down on a 2-plane, they have opposite orientations: γ is oriented counter-clockwise whilst η is oriented clockwise. Set $\bar{s}^2 = s^2 - s^1$ and $\bar{t}^2 = t^2 - t^1$, where $s^1 < s^2$ and $t^1 < t^2$. Then,

$$(\gamma *_{s^1 s^2} \eta)^{++}(t) \stackrel{\text{def}}{=} \begin{cases} \gamma_{s^1}(4\bar{s}_2) & 0 \leqq t \leqq \frac{1}{4}, \\ \eta_{t^1}(4(1-\bar{t}^2)t + 2\bar{t}^2 - 1) & \frac{1}{4} \leqq t \leqq \frac{1}{2}, \\ \gamma_{s^1}(4(\bar{s}^2 - 1)t - 2\bar{s}^2 + 3) & \frac{1}{2} \leqq t \leqq \frac{3}{4}, \\ \eta_{t^1}(4(1-\bar{t}^2)t + 4\bar{t}^2 - 3) & \frac{3}{4} \leqq t \leqq 1, \end{cases}$$

and $\epsilon(++) = 3$, where $\epsilon(++)$ denotes that a total of 3 segments in $(\gamma *_{s^1 s^2} \eta)^{++}$ required their orientations reversed,

$$(\gamma *_{s^1 s^2} \eta)^{--}(t) \stackrel{\text{def}}{=} \begin{cases} \gamma_{s^1}(4\bar{s}^2 t) & 0 \leqq t \leqq \frac{1}{4}, \\ \eta_{t^1}(4(1-\bar{t}_2)t + 2\bar{t}_2 - 1) & \frac{1}{4} \leqq t \leqq \frac{1}{2}, \\ \gamma_{s^1}(4(\bar{s}^2 - 1)t + 3 - 2\bar{s}^2) & \frac{1}{2} \leqq t \leqq \frac{3}{4}, \\ \eta_{t^1}(4(1-\bar{t}^2) + 4\bar{t}^2 - 3) & \frac{3}{4} \leqq t \leqq 1, \end{cases}$$

where $\epsilon(--)$, denoting the number of segments in $(\gamma *_{s^1 s^2} \eta)^{--}$, equals 2,

$$(\gamma *_{s^1 s^2} \eta)^{-+}(t) \stackrel{\text{def}}{=} \begin{cases} \gamma_{s^1}(2t) & 0 \leqq t \leqq \frac{1}{2}, \\ \eta_{t^1}(2t - 1) & \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

with $\epsilon(-+) = 0$ and

$$(\gamma *_{s^1 s^2} \eta)^{+-} \stackrel{\text{def}}{=} \begin{cases} \gamma_{s^1}(3\bar{s}^2 t) & 0 \leqq t \leqq \frac{1}{3}, \\ \eta_{t^2}(3t-1) & \frac{1}{3} \leqq t \leqq \frac{2}{3}, \\ \gamma_{s^1}(3(1-\bar{s}^2)t + 3\bar{s}^2 - 2) & \frac{2}{3} \leqq t \leqq 1, \end{cases}$$

where $\epsilon(+-) = 0$. Likewise, if γ and η are oriented in the same direction, the four ways of breaking and joining them can also be worked out analogously.

The map $\mathbf{f}_2 : \bar{\mathcal{T}}_{\text{RS}} \to (\Lambda^1(\Sigma)^2)' \otimes \bar{\mathcal{T}}_{\text{RS}}$, where $(\Lambda^1(\Sigma)^2)'$ is the topological dual of $\Lambda^1(\Sigma) \times \Lambda^1(\Sigma)$, is defined by assigning $\mathbf{f}_2(T^n[\gamma](s^1, \ldots, s^n))$ the value

$$\begin{aligned} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Delta[\gamma_1, \gamma_2](s^i, s^j)((-1)^{\epsilon(++)} T^{n-1}[(\gamma_1 *_{s^i s^j} \gamma_2)^{++}](s^1, \ldots, s^n)_{s^i s^j} + \\ (-1)^{\epsilon(--)} T^{n-1}[(\gamma_1 *_{s^i s^j} \gamma_2)^{--}](s^1, \ldots, s^n)_{s^i s^j} + \\ (-1)^{\epsilon(-+)} T^{n-1}[(\gamma_1 *_{s^i s^j} \gamma_2)^{-+}](s^1, \ldots, s^n)_{s^i s^j} + \\ (-1)^{\epsilon(+-)} T^{n-1}[(\gamma_1 *_{s^i s^j} \gamma_2)^{+-}](s^1, \ldots, s^n)_{s^i s^j}), \end{aligned}$$

where $\Delta[\gamma_1, \gamma_2](s^i, s^j) \stackrel{\text{def}}{=} \Delta[\gamma_1, \gamma_2](s^i)\Delta[\gamma_1, \gamma_2](s^j)$, γ_1 and γ_2 are the unique decomposition of γ afforded by lemma 4.1 and $(s^1, \ldots, s^n)_{s^i s^j}$ denotes the $n-2$-tuple where s^i and s^j are deleted from the n-tuple $(s^1, \ldots, s^n)$ for type-setting convenience.

Along a similar vein, the map $\mathbf{f}_n : \bar{\mathcal{T}}_{\text{RS}} \to (\Lambda^1(\Sigma)^n)' \otimes \bar{\mathcal{T}}_{\text{RS}}$ can be defined by "splitting" the resultant self-intersecting loop at n intersecting points simultaneously (after applying lemma 4.1 to decompose the loop). For instance, setting $\bar{s}^i = s^i - s^1$ and $\bar{t}^i = t^i - t^1$ and assuming that γ and η are oppositely oriented when projected down onto a 2-plane, $(\gamma *_{s^1 \ldots s^n} \eta)^{-\cdots-}(t) \stackrel{\text{def}}{=} \xi_1 * \xi_2$, where

$$\xi_1(t) \stackrel{\text{def}}{=} \begin{cases} \gamma_{s^1}((n-1)\bar{s}^n t) & 0 \leqq t \leqq \frac{1}{n-1}, \\ \eta_{t^1}((n-1)(\bar{t}^{n-1} - \bar{t}^n)t + 2\bar{t}^n - \bar{t}^{n-1}) & \frac{1}{n-1} \leqq t \leqq \frac{2}{n-1}, \\ \gamma_{s^1}((n-1)(\bar{s}^{n-2} - \bar{s}^{n-1})t + 3\bar{s}^{n-1} - 2s^{n-2}) & \frac{2}{n-1} \leqq t \leqq \frac{3}{n-1}, \\ \vdots & \\ \eta_{t^1}((n-1)(1-\bar{t}^2)t + 1 - (n-1)(1-\bar{t}^2)) & \frac{n-2}{n-1} \leqq t \leqq 1, \end{cases}$$

ξ_2 is defined analogously with γ and η interchanged and $\epsilon(-\cdots-) = n-1$ in this example. Recall that $\gamma * \eta \stackrel{\text{def}}{=} \gamma *_{\frac{1}{2}} \eta$.

Hence, the quantum commutator as defined by Rovelli and Smolin [6] can be expressed in terms of the linear operators $\mathbf{f}_n$ and the classical Poisson brackets by

$$[\hat{T}^n[\gamma], \hat{T}^m[\eta]] \stackrel{\text{def}}{=} \sum_{i=1}^{m} \hbar^i \lambda \circ \mathbf{f}_{i-1} \circ \{T^n[\gamma], T^m[\eta]\}.$$

In short, the quantum commutator maps $\hat{\mathcal{T}}_{\text{RS}} \times \hat{\mathcal{T}}_{\text{RS}}$ into $\bigoplus_n (\Lambda^1(\Sigma)^n)' \otimes \bar{\mathcal{T}}_{\text{RS}}$. It is not difficult to see that $\overline{[\hat{T}^n[\gamma], \hat{T}^m[\eta]]} = [\bar{\hat{T}}^n[\gamma], \bar{\hat{T}}^m[\eta]]$, as $\overline{P_I(T^n[\gamma], T^m[\eta])} = P_I(\bar{T}^n[\gamma], \bar{T}^m[\eta])$.

5 Classical Loop Phase Space

In this section, the classical loop phase space for general relativity will be constructed: this is motivated by the desire to express the reality conditions in the loop variables. Insight into the loop phase space for general relativity should hopefully illuminate this matter further—work on the loop reality conditions is currently in progress. The work outlined in this section yields new results and new insights into the loop formulation of gravity. In the following, let $\theta^1, \theta^2, \theta^3$ denote some fixed linearly independent nowhere vanishing 1-forms on Σ.

Let $W_x[\Sigma^2] = \operatorname{span}\{\, \theta^i(x^1) \wedge \theta^j(x^2) \mid 1 \leqq i < j \leqq 3 \,\}$, where $x = (x^1, x^2) \in \overline{\Sigma}^2_+$ and construct $\hat{W}_\infty[\Sigma^2]$ in the same way as $\hat{E}_\infty[\Sigma^2]$ was constructed in §2 and set $W_\Sigma = \bigoplus_{x \in \Sigma^2} W_x[\Sigma^2]$, where $\hat{W}_\infty[\Sigma^0] \stackrel{\text{def}}{=} \mathbb{C}$. Then, $(W_\Sigma, \wedge)$ is the *multi-point exterior algebra* of Σ. The topology on W_Σ is constructed in an identical way to that of E_Σ in §2. That is, it is defined by a sequence of metrics $\{d_n^*\}_{n=1}^\infty$, where d_n^* defines the topology on $\hat{W}_\infty[\Sigma^n]$. Given $\gamma \in \mathcal{M}_1$, recall from §3 that $P_n^\alpha(\gamma) = \{\, s_\alpha^1, \ldots, s_\alpha^n \,\}$ denotes a partition of I, where $0 \leqq s_\alpha^1 < \cdots < s_\alpha^n < 1$, such that $\gamma(s_\alpha^i) \neq \gamma(s_\alpha^j)$ $\forall i \neq j$, and $\mathcal{P}_n(\gamma)$ denotes the set of all such partitions $P_n^\alpha(\gamma)$. Let $M_n(\gamma) = \bigcup\{\, P_n^\alpha(\gamma) \times W_{x_\alpha}[\Sigma^n] : P_n^\alpha(\gamma) \in \mathcal{P}_n(\gamma)\}$, where $x_\alpha = (\gamma(s_\alpha^1), \ldots, \gamma(s_\alpha^n))$ and let $M_n \stackrel{\text{def}}{=} \bigcup_{\gamma \in \mathcal{M}_1} \{\gamma\} \times M_n(\gamma)$. A topology on M_n will be constructed as follows. Given a point $(\gamma, s^1, \ldots, s^n, \omega_1 \wedge \cdots \wedge \omega_n)$ in M_n, define a neighbourhood $N_{\varepsilon_1 \varepsilon_2 \varepsilon_3}(\gamma, s^1, \ldots, s^n, \omega_1 \wedge \cdots \wedge \omega_n)$ about it to consist of all points $(\eta, t^1, \ldots, t^n, \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_n)$ that satisfy (i) $\rho^*(\gamma, \eta) < \varepsilon_1$, (ii) $|t^i - s^i| < \varepsilon_2$, and (iii) $\|\omega_1(\gamma(s^1)) \wedge \cdots \wedge \omega_n(\gamma(s^n)) - \bar{\omega}(\eta(t^1)) \wedge \cdots \wedge \bar{\omega}(\eta(t^n))\| < \varepsilon_3$. Recall that ρ^* is a metric on $\mathcal{M}_1$ mentioned in §1.

Now, given $\mathbf{v}_1(x^1) \otimes \cdots \otimes \mathbf{v}_n(x^n) \in \hat{E}_\infty[\Sigma^n]$, define

$$
\begin{aligned}
&\langle\, \mathbf{v}_1(x^1) \otimes \cdots \otimes \mathbf{v}_n(x^n) \,, \omega(y^1) \wedge \cdots \wedge \omega_n(y^n) \,\rangle_\epsilon \\
&\stackrel{\text{def}}{=} \prod_{i=1}^n \frac{\langle\, \mathbf{v}_i(x^i) \,, \omega_i(x^i) \,\rangle_\epsilon + \langle\, \mathbf{v}_i(y^i) \,, \omega_i(y^i) \,\rangle_\epsilon}{2},
\end{aligned}
$$

where ϵ is a fixed nowhere vanishing 3-form on Σ (chosen once and for all), $\langle\, \mathbf{v}_i(x^i) \,, \omega_i(x^i) \,\rangle_\epsilon \stackrel{\text{def}}{=} \langle\, \mathbf{v}_i(x^i)_\epsilon \,, \omega_i(x^i) \,\rangle$ and $\mathbf{v}_i(x^i)_\epsilon$ denotes the dedensitisation of $\mathbf{v}_i(x^i)$. For more details regarding the dedensitisation of tensor densities, see reference [2, p. 312]. Hence, with respect to the scalar product $\langle\,\cdot\,,\cdot\,\rangle_\epsilon$, $T^n[\gamma, A, E](s^1, \ldots, s^n)$ is a continuous map from M_n into $\mathbb{C}$.

In the light of the preceeding discussion, it seems natural to define the *loop phase space* for general relativity to be the topological sum $M \stackrel{\text{def}}{=} \bigoplus_{n=0}^\infty M_n$ since by construction, $\bar{\mathcal{T}}_{\mathsf{RS}} \subset C(M, \mathbb{C}) = \bigoplus_{n=0}^\infty C(M_n, \mathbb{C})$, where $M_0 \stackrel{\text{def}}{=} \mathcal{M}_1 \times \mathbb{C}$. That M is indeed the phase space for general relativity in the loop representation will be justified below. First of all, recall the definition of the area derivative from references [3, 4]. Let $\gamma^\delta = \gamma *_{s'} \delta *_s \gamma$, where γ is any loop and δ is a loop null-homotopic to $\gamma(s) = \gamma(s')$. Set $\sigma^{ab}(\delta) = \oint_\delta \delta^a \dot{\delta}^b$ and denote $|\delta|$ to be the area of δ with respect to any fixed Riemannian metric q on Σ.

Then, given any loop functional ψ,

$$\frac{\delta\psi[\gamma]}{\delta\sigma^{ab}(x)} = \lim_{|\delta|\to 0}\frac{\psi[\gamma^\delta]-\psi[\gamma]}{\sigma^{ab}(x)},$$

where $x = \gamma(s)$. This can clearly be extended to matrix valued loop functionals in the obvious way. For simplicity, denote the area derivative with respect to γ at $\gamma(s)$ by $D_\gamma(s)$.

Observe that a map $\Psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ on the space of $n \times n$ matrices trivially induces a map $\tilde{\Psi}$ given by $\tilde{\Psi}(\operatorname{tr} U) \stackrel{\text{def}}{=} \operatorname{tr}(\Psi \cdot U)$. Hence, a map $D^1_\gamma(s)$ on $\bar{\mathcal{T}}_{\text{RS}}$ can be defined via its action on matrices given below:

$$\begin{aligned} D^1_\gamma(s^i) :& U_\gamma(s^2,s^1)E(\gamma(s^1))U_\gamma(s^3,s^2)E(\gamma(s^2))\ldots U_\gamma(s^1,s^n)E(\gamma(s^n)) \mapsto \\ & U_\gamma(s^2,s^1)E(\gamma(s^1))\ldots D_\gamma(s^i)U_\gamma(s^{i+1},s^i)E(\gamma(s^i))\ldots U_\gamma(s^1,s^n)E(\gamma(s^n)), \end{aligned}$$

and $D^2_\gamma(s^i,s^j) \stackrel{\text{def}}{=} D^1_\gamma(s^i)\otimes D^1_\gamma(s^j)$, $s^i < s^j$, acts on

$$\begin{aligned} U_\gamma(s^2,s^1)E(\gamma(s^1))\ldots U_\gamma(s^{i+1},s^i)E(\gamma(s^i))\ldots U_\gamma(s^{j+1},s^j)E(\gamma(s^j))\ldots \\ U_\gamma(s^1,s^n)E(\gamma(s^n)) \end{aligned}$$

in the following way:

$$\begin{aligned} U_\gamma(s^2,s^1)E(\gamma(s^1))\ldots D_\gamma(s^i)U_\gamma(s^{i+1},s^i)E(\gamma(s^i))\ldots D_\gamma(s^j)U_\gamma(s^{j+1},s^j)E(\gamma(s^j))\ldots \\ U_\gamma(s^1,s^n)E(\gamma(s^n)). \end{aligned}$$

More generally, $D^n_\gamma(s^{i_1},\ldots,s^{i_n}) \stackrel{\text{def}}{=} D^1_\gamma(s^{i_1})\otimes\cdots\otimes D^1_\gamma(s^{i_n})$, $s^{i_1} < \cdots < s^{i_n}$, can be defined analogously. Then, as mentioned previously, these mappings can be carried over to $T^n[\gamma,A,E]$ trivially by assigning $D^m_\gamma(s^{i_1},\ldots,s^{i_m})T^n[\gamma,A,E](s^1,\ldots,s^n)$, where $m \leqq 2n$, the following value:

$$\begin{aligned} \operatorname{tr}(U_\gamma(s^2,s^1)E(\gamma(s^1))\ldots D_\gamma(s^{i_1})U_\gamma(s^{i_1+1},s^{i_1})E(\gamma(s^{i_1}))\ldots \\ D_\gamma(s^{i_m})U_\gamma(s^{i_m+1},s^{i_m})E(\gamma(s^{i_1}))\ldots U_\gamma(s^1,s^n)E(\gamma(s^n))). \end{aligned}$$

It is easy to see from the definition of the area derivative and from the following expansion $U_\delta[A] = 1 + \iint_{S(\delta)} F + o(|\delta|)$ for $|\delta|$ taken to be sufficiently small, where $S(\delta)$ is a surface in Σ bounded by δ, F is the gauge curvature 2-form of the Ashtekar 1-form A, that $D^1_\gamma(s^1)T^1[\gamma,A,E] = \operatorname{tr}(F(\gamma(s^1))E(\gamma(s^1)))$ and $D^1_\gamma(s^1)T^2[\gamma,A,E](s^1,s^2) = \operatorname{tr}(F(\gamma(s^1))E(\gamma(s^1))E(\gamma(s^2)))$. Likewise, $D^n_\gamma(s^1,\ldots,s^n)\cdot T^1[\gamma,A,E](s^1)\otimes T^2[\gamma,A,E](s^2,\bar{s}^2)\otimes\cdots\otimes T^1[\gamma,A,E](s^n)$ is equal to

$$\operatorname{tr}(F(\gamma(s^1))E(\gamma(s^1)))\cdot\operatorname{tr}(F(\gamma(s^2))E(\gamma(s^2))E(\gamma(\bar{s}^2)))\ldots\operatorname{tr}(F(\gamma(s^n))E(\gamma(s^n))),$$

and so forth. Hence, taking the limit, for example,

$$\lim_{s^2\to s^1} D^1_\gamma(s^1)T^2[\gamma,A,E](s^1,s^2) = \operatorname{tr}(F(\gamma(s^1)E(\gamma(s^1))E(\gamma(s^1))),$$

yields the desired constraint equations—the limit is well-defined as γ is continuous. This suggests the following way of constructing constraint surfaces in M.

Given $T^{a_1\dots a_n}[\gamma](s^1,\dots,s^n)$, define an n-point n-form $\omega^\gamma_{a_1\dots a_n}(x^1,\dots,x^n)$ by $q_{a_1b_1}(x^1)\dots q_{a_nb_n}(x^n)T^{b_1\dots b_n}_\epsilon[\gamma](s^1,\dots,s^n)$, where $x^i=\gamma(s^i)$ for each i, $T^n_\epsilon[\gamma]$ is the dedensitised T-variable $T^n[\gamma]$ and the Riemannian 3-metric q is defined by $\det q\cdot q = -\operatorname{tr}(E_\gamma\cdot E_\gamma)$ with E_γ defining the T^n-observable $T^n[\gamma]$. Then, it is clear that the constraint surface $S_n\subset M_n$ is spanned by n-point n-forms ω^γ defined by the exterior products of $\omega^\gamma_a(\gamma(s))$ and $\omega^\gamma_{ab}(\gamma(s^1),\gamma(s^2))$ that lie in the kernels of the two equations C_{1n} and C_{2n} given by

$$C_{1n}(x)(\gamma,\omega^\gamma)=\lim_{s^2,\dots,s^n\to s^1}C_{1n}(\gamma,s^1,s^2,\dots,s^n,\omega^\gamma)$$
$$C_{2n}(x)(\gamma,\omega^\gamma)=\lim_{s^3,\dots,s^m\to s^1}C_{2n}(\gamma,s^1,s^3,\dots,s^m,\omega^\gamma)$$

where $x=\gamma(s^1)$,

$$C_{1n}(\gamma,s^1,s^2,\dots,s^n,\omega^\gamma)\stackrel{\text{def}}{=}D^n_\gamma(s^1,s^2,\dots,s^n)\omega^\gamma,$$
$$C_{2n}(\gamma,s^1,s^3,\dots,s^m,\omega^\gamma)\stackrel{\text{def}}{=}D^m_\gamma(s^1,s^3,\dots,s^m)\omega^\gamma,$$

and

$$m=\begin{cases} n & \text{if } n \text{ is odd,}\\ n-1 & \text{if } n \text{ is even.}\end{cases}$$

Alternatively, instead of taking limits, the constraints may be viewed as distributional type constraints—for instance, of the form

$$D^1_\gamma(s^1)T^2[\gamma](s^2,s^1)\delta^3(\gamma(s^2),\gamma(s^1)).$$

Then, integrating out $x=\gamma(s^2)$ reduces to the desired Hamiltonian constraint. The interesting point to note with this formulation is that the constraints become distributional, which is unlike any classical dynamical systems encountered. However, what is to be emphasised here is that the constraint surfaces yielding general relativity can be found in M and moreover, are defined by an infinite set of constraint equations (for a fixed Cauchy data) belonging to M_n for each n.

6 Discussion

It is clear from the above construction that M is indeed the required classical loop phase space for general relativity. A number of comments will now be made. First, the classical phase space for general relativity differs from the phase space of traditional classical mechanics in that it is not a smooth symplectic manifold (which, in many cases, are merely cotangent bundles over appropriate configuration spaces). In fact, M_n, for each n, is not even a topological manifold! In a loose sense, the set of pairs $(M_n,\mathcal{M}_1)$ may be considered as a generalisation of cotangent bundles that define a large class of phase spaces; that is, $\pi_n : M_n \to \mathcal{M}_1$ given by $(\gamma, s^1, \ldots, s^n, \omega) \mapsto \gamma$, defines a natural projection with the fibre over γ being $\pi_n^{-1}(\gamma) = M_n(\gamma)$, and the wavefunctionals are defined via the pull-back π_n^* of π_n for each n.

The non-smoothness of the phase space seems to give an intuitive support that quantum space-time is discrete, or at least, the smooth picture of space-time must break down at some point. Another striking point to note is that the phase space consists of disconnected subspaces: it is the topological sum of M_n's. In hindsight, this arises from the construction of the T-observables. These two observations might perhaps contribute to the overall understanding of the prediction of a discrete picture of quantum space-time in the loop representation at the Planck scale.

In conclusion, the classical loop algebra was shown to be imbeddable in a linear extension which supports a distributional Poisson structure. Furthermore, it was demonstrated that by defining a suitable quotient relation on $\bar{\mathcal{T}}_{\mathrm{RS}}$ so that the resulting quotient space is P-associative, the quotient space supports a non-unital Hopf structure in an almost trivial manner. This leads to a tentative speculation that there might be an underlying connection between the algebra of loop observables and quantum groups after all. Furthermore, as a result of the construction of the P-structure, a much simpler smearing procedure for eliminating the δ-singularities inherent in the P-structure was given. An explicit dependence on the P-structure in the quantum commuatation relation was derived and the quantum loop algebra turned out to be the covering space for the classical loop algebra. Finally, the classical loop phase space for general relativity was explicitly constructed. And therein, a number of distinct features that are not present in traditional classical mechanics were duely noted. One of which is the distinct lack of a manifold structure and furthermore, the constraint surfaces were defined by a countably infinite set of equations, each surface lying in M_n, for each n.

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