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The BRST Quantization of First-Order Systems

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Abstract. In this paper we study purely first-order systems in the framework of the BRST quantization. In this light, we show that: i) transforming the original first-order system into a family of first-class ones possessing a number of independent variables equal to the one of the original system, but in a larger phase-space and, ii) quantizing in the antifield BRST formalism this family, we reobtain the original path integral. The quantization procedure is illustrated in the case of geodesic motion in spinning space.

1 Introduction

The BRST quantization for theories with both first- and second-class constraints is completely elucidated [1] and mainly relies on the presence in the theory of the first-class constraints. The results from [1] coincide with the ones obtained by means of canonical methods [2]-[3]. For systems possessing only second-class constraints the BRST formalism cannot be applied directly, in this case being necessary to implement some gauge invariances. This may be achieved turning the original second-class system into a first-class one in the original [4] or in a larger phase-space [5]-[6], and further quantizing the resulting first-class system(s) in the BRST manner. The BRST quantization for purely first- or second-order systems in the original phase-space is realized in [4]. The approach of second-order systems in a larger phase-space has been extended to theories maintaining the reducibility relic of a certain first-class one [7]-[8]. The canonical quantization methods may be found in [2]-[3], [9]-[10].

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It hasn't been accomplished until now the BRST quantization in a larger phase-space for purely first-order systems, which possess in general only primary second-class constraints. This is the aim of our paper. More precisely, starting with a Lagrangian first-order system with only primary second-class constraints, we *i)* transform it into a family of first-class ones possessing the same number of independent variables as the original system, but in a larger phase-space, and *ii)* quantize this family in the light of the antifield BRST formalism. In this way, we shall show that our resulting path integral is identical with the one obtained in [1]-[4] in other ways. The ideas exposed in the general part of the paper are applied to the case of geodesic motion in spinning space. We mention that our procedure of converting the original system into a family of first-class systems is basically different from the one in [5]-[6]. In this paper we adopt for simplicity the notations of finite-dimensional analytical mechanics, but the analysis can be straightforwardly extended to field theory. Concerning the BRST quantization in the path integral formalism we follow the same lines as in [11].

Our starting point is the Lagrangian action

$$S_0\left[y^i\right]=\int dt\left(a_i\left(y\right)\dot{y}^i-V\left(y\right)\right),\quad i=1,2,\ldots,N, \tag{1.1}$$

describing a purely first-order system with all the variables of the same Grassmann parity, $\epsilon\left(y^i\right)=\epsilon$. The one-form potential $a_i\left(y\right)$ obviously has the same Grassmann parity as the original variables. For later convenience, we make the notations

$$\frac{\partial a_i(y)}{\partial y^j}=\frac{1}{2}\left(\frac{\partial a_i(y)}{\partial y^j}+(-)^{\epsilon+1}\frac{\partial a_j(y)}{\partial y^i}\right)+\frac{1}{2}\left(\frac{\partial a_i(y)}{\partial y^j}+(-)^{\epsilon}\frac{\partial a_j(y)}{\partial y^i}\right)\equiv\frac{1}{2}a_{ij}\left(y\right)+\frac{1}{2}\tilde{a}_{ij}\left(y\right).$$

The matrices a_{ij} and $\tilde{a}_{ij}$ have the symmetry properties $a_{ij}=(-)^{\epsilon+1}a_{ji}$, respectively $\tilde{a}_{ij}=(-)^{\epsilon}\tilde{a}_{ji}$. The kinetic terms in (1.1) corresponding to $\tilde{a}_{ij}$ reduce to an irrelevant total derivative which can be dropped out. This is why we shall neglect these terms and consider in the sequel only those terms including a_{ij}, whose elements possess symmetry properties opposite to those of the original variables, e.g. if the y^i's are commuting, then the a_{ij}'s are anti-commuting. Thus, from now on we take

$$\frac{\partial a_i(y)}{\partial y^j}=\frac{1}{2}\left(\frac{\partial a_i(y)}{\partial y^j}+(-)^{\epsilon+1}\frac{\partial a_j(y)}{\partial y^i}\right)\equiv\frac{1}{2}a_{ij}\left(y\right), \tag{1.2}$$

with

$$\det a_{ij}\left(y\right)\neq 0. \tag{1.3}$$

The canonical analysis of (1.1) yields the canonical Hamiltonian

$$H\left(p,y\right)=V\left(y\right), \tag{1.4}$$

and the primary constraints

$$G_i\equiv p_i-a_i\left(y\right)=0, \tag{1.5}$$

with $\epsilon\left(G_i\right)=\epsilon$, and the p_i's being the canonical momenta conjugated to the y^i's. The above constraints are second-class due to the fact that $\det\left(\left[G_i,G_j\right]\right)=\det\left(a_{ji}\left(y\right)\right)$ and because of

(1.3). As a consequence, the system has no secondary constraints, the consistency conditions of the primary constraints determining the corresponding Lagrange multipliers. It is clear that our system possesses N independent variables (co-ordinates and/or momenta) [11], every second-class constraint decreasing the number of independent variables with one. Here and in the following, the symbol $[,]$ denotes the Poisson bracket. For definiteness we employ the right derivative throughout this paper. This completes the canonical analysis of the original system.

2 The one-parameter family

The next step of our procedure consists in constructing a one-parameter family of first-class systems associated to the original system. In this end, we shall extend the original phase-space adding a number of new canonical pairs equal to the number of the above second-class constraints. More precisely, for every function G_i we introduce a canonical pair (z^i, π_i) with the same Grassmann parity, $\epsilon(z^i) = \epsilon(\pi_i) = \epsilon$. The basic idea of the construction relies on building a one-parameter family of first-class systems possessing also N independent variables, but in a larger phase-space with the local co-ordinates (y^i, p_i, z^i, π_i). In this respect, we consider the action

$$S_0\left[y^i, z^i\right] = \int dt \left(b_i(y,z)\dot{y}^i + \lambda \bar{a}_i(y,z)\dot{z}^i - H^*(y,z)\right), \tag{2.1}$$

where $b_i(y,z)$ and $\bar{a}_i(y,z)$ are both one-form potentials having the key properties

$$\det\left(\frac{\partial b_i}{\partial y^j} + (-)^{\epsilon+1}\frac{\partial b_j}{\partial y^i}\right) \neq 0, \det\left(\frac{\partial \bar{a}_i}{\partial z^j} + (-)^{\epsilon+1}\frac{\partial \bar{a}_j}{\partial z^i}\right) \neq 0, \tag{2.2}$$

λ being the non-vanishing parameter of the first-class family. At this stage, the form of action (2.1) is hypothetical, the existence and concrete form of $b_i(y,z)$, $\bar{a}_i(y,z)$ and $H^*(y,z)$ being respectively proved and indicated in the sequel. In fact, all these functions can be derived from the original second-class system, as it will be shown.

Initially, we shall prove that there exists an appropriate symplectic structure such that action (2.1) describes a family of systems with only first-class constraints and N independent variables. In this end, we start with the canonical analysis of (2.1), which outputs the primary constraints $\overline{G}_i \equiv p_i - b_i(y,z) = 0$ and $\tilde{G}_i \equiv \pi_i - \lambda\bar{a}_i(y,z) = 0$, with π_i's the canonical momenta of z^i's. There are no further constraints due to the key properties (2.2). For later convenience, we make the notation $\chi_a \equiv \left(\overline{G}_i, \tilde{G}_i\right)$. System (2.1) possesses exactly N independent variables if

$$rank\left[\chi_a, \chi_b\right] = N. \tag{2.3}$$

The solution of this last equation is

$$b_i(y,z) = \bar{a}_i(y,z) = a_i(y + \lambda z). \tag{2.4}$$

The last formulas prove the existence of $b_i(y,z)$'s and $\overline{a}_i(y,z)$'s having the key properties (2.2). Because of equation (2.3), from the entire set of $2N$ constraints $\left(\overline{G}_i, \tilde{G}_i\right)$, half are first- and half are second-class. Eliminating the constraints $\tilde{G}_i = 0$ with the help of the Dirac bracket [9] built with respect to themselves, we infer the searched for symplectic structure under the form

$$[A,B]^* = [A,B] + (-)^{\epsilon}\frac{1}{\lambda^2}\left[A,\tilde{G}_i\right] a^{ij}(y+\lambda z)\left[\tilde{G}_j, B\right], \tag{2.5}$$

where $a^{ij}(y+\lambda z)$ is the inverse of $a_{ij}(y+\lambda z)$. It is clear that $a^{ij}(y+\lambda z)$ is well-defined as $\det a_{ij}(y) \neq 0$ by hypothesis. Using the above Dirac bracket, we get that $\left[\overline{G}_i, \overline{G}_j\right]^* = 0$, so the remaining constraints are first-class in this symplectic structure .

With this symplectic structure at hand, we are able to prove the existence of a function $H^*(y,z) = V(y)+$"extra terms in y's and z's" such that

$$\left[H^*, \overline{G}_i\right]^* = 0, \quad \text{strongly}. \tag{2.6}$$

We take H^* of the form $H^*(y,z) = V(y) + \sum\limits_{k=1}^{\infty} u_{i_1\ldots i_k} z^{i_1}\ldots z^{i_k}$, where $u_{i_1\ldots i_k}$'s are functions only of y's. Thus, proving the existence of H^* reduces to proving the existence of $u_{i_1\ldots i_k}$'s. Introducing the prior form of H^* in (2.6) we obtain, after usual computation, $u_{i_1\ldots i_k} = \frac{\lambda^k}{k!}\frac{\overrightarrow{\partial}^k V(y)}{\partial y^{i_1}\ldots\partial y^{i_k}}$. In order to infer the concrete form of H^* we represent $V(y)$ as a series in the powers of y's: $V(y) = \sum\limits_{k=0}^{\infty}\frac{1}{k!}\left.\frac{\overrightarrow{\partial}^k V(y)}{\partial y^{i_1}\ldots\partial y^{i_k}}\right|_{y^i=0} y^{i_1}\ldots y^{i_k}$. Introducing the last formula in H^* we finally deduce

$$H^*(y,z) = V(y+\lambda z). \tag{2.7}$$

Now, it is clear that $H^*(y,z)$ satisfying (2.6) really exists because $V(y)$ is assumed from the start to exist.

Formula (2.7) is similar to the one derived in [12] excepting the presence of the parameter λ which, as it will be seen, is crucial in our quantization procedure.

3 The BRST quantization of the first-class family

Following the lines of Sec. 2, we associated to the original system a one-parameter family of first-class systems described by the Hamiltonian action

$$S_0^T\left[y^i, p_i, z^i, v^i\right] = \int dt\left(p_i\dot{y}^i + \lambda a_i(y+\lambda z)\dot{z}^i - V(y+\lambda z) - v^i\overline{G}_i\right). \tag{3.1}$$

Action (3.1) is invariant under the following gauge transformations (generated in the Dirac bracket): $\delta_\epsilon F = \left[F, \overline{G}_i\right]^*\epsilon^i$, $\delta_\epsilon v^i = \dot{\epsilon}^i$, with ϵ^i arbitrary functions of Grassmann parity ϵ. The next point of our analysis is concerned with proving the formula

$$Z_\Psi = \int \mathcal{D}y^i \left(\det a_{ij}(y)\right)^{1/2} \exp\left(iS_0\left[y^i\right]\right), \tag{3.2}$$

where Z_Ψ denotes the path integral derived within the BRST quantization using the gauge-fixing fermion Ψ and corresponding to action (3.1). If proved, formula (3.2) reveals the meaning of applying the BRST formalism to first-order systems with only primary second-class constraints, establishing the equivalence at the path integral level between the original system and the first-class systems derived previously.

We begin with solving the master equation [11] for action (3.1). Its solution reads

$$S = S_0^T + \int dt \left((-)^\epsilon y_i^* \eta^i + (-)^{\epsilon+1} \frac{1}{\lambda} z_i^* \eta^i + v_i^* \dot{\eta}^i + \overline{\eta}_i^* B^i \right), \tag{3.3}$$

where the star variables denote the antifields of the corresponding variables, the η^i's are the minimal ghosts and $(\overline{\eta}^{i*}, \overline{\eta}_i, B^{i*}, B_i)$ form the non-minimal sector [11]. In order to find an appropriate gauge-fixing fermion, Ψ, we observe that (2.1) and (3.1) reduces to (1.1) if $z^i = 0$. Then, it is natural to choose a gauge-fixing fermion implementing exactly these canonical gauge conditions, namely

$$\Psi = \lambda \overline{\eta}_i z^i. \tag{3.4}$$

Eliminating in the usual manner the antifields from (3.3) with the help of (3.4), we infer the gauge-fixed action

$$S_\Psi = S_0^T + \int dt \left((-)^{\epsilon+1} \overline{\eta}_i \eta^i + \lambda B_i z^i \right). \tag{3.5}$$

Action (3.5) is invariant under the BRST transformations: $sy^i = (-)^\epsilon \eta^i$, $sz^i = (-)^{\epsilon+1} \frac{1}{\lambda} \eta^i$, $s\eta^i = 0$, $s\overline{\eta}^i = B^i$. Because $s\left(y^i + \lambda z^i\right) = 0$ for any $\lambda \neq 0$, the previous BRST invariances are not affected if we take λ as a solution of the equation

$$\lambda = \left(\det\, a_{ij}\left(y + \lambda z\right)\right)^{-1/2}. \tag{3.6}$$

With this choice of the parameter, the gauge-fixed action (3.5) remains in the zeroth order cohomological group, $H^0(s)$, [11]. Actually, there is no inconsistency in the fact that now we consider the parameter as a gauge-invariant function because S_Ψ remains in $H^0(s)$ with both λ constant or the above gauge-invariant function. Introducing the last value of λ in (3.5) and integrating in its correspondent path integral over all the variables excepting the y^i's, we obtain exactly (3.2).

The above discussion emphasizes the crucial role of λ, as well as of the BRST invariance, in our quantization procedure. Indeed, for any other choice we wouldn't have obtained the correct local measure in (3.2). In this way, we established that our manner of quantizing first-order systems leads to the same path integral as in [1]-[4]. By contrast with the case of second-order systems, where the value of the parameter can be fixed [7]-[8] comparing the path integrals respectively obtained in the extended and Lagrangian formalisms, here we cannot act anymore in the same way. This is because in the case under study, there are no secondary constraints, so the total formalism is directly equivalent to the Lagrangian one.

4 Example: geodesic motion in spinning space

Let's apply our formalism in the case of geodesic motion in spinning space, described by the Lagrangian action

$$S_0\left[x,\psi\right]=\int_{(1)}^{(2)}d\tau\left(\frac{1}{2}g_{\mu\nu}\left(x\right)\dot{x}^{\mu}\dot{x}^{\nu}+\frac{i}{2}g_{\mu\nu}\left(x\right)\psi^{\mu}\frac{D\psi^{\nu}}{D\tau}\right),\tag{4.1}$$

where the local co-ordinates x^{μ} are bosonic (commuting) and the ψ^{μ}'s are fermionic (anti-commuting). The second set of variables describes the spin degrees of freedom. We denote by an overdot the ordinary proper-time derivative $d/d\tau$, and by $\frac{D\psi^{\mu}}{D\tau}$ the covariant derivative of ψ^{μ}, which is defined as

$$\frac{D\psi^{\mu}}{D\tau}=\dot{\psi}^{\mu}+\dot{x}^{\lambda}\Gamma^{\mu}_{\lambda\nu}\psi^{\nu}.\tag{4.2}$$

The canonical analysis of action (4.1) furnishes the fermionic primary constraints

$$G_{\mu}\equiv\pi_{\mu}-\frac{i}{2}g_{\mu\nu}\psi^{\nu}=0,\tag{4.3}$$

and the canonical Hamiltonian

$$H\left(x,p,\psi\right)=\frac{1}{2}g^{\mu\nu}\left(p_{\mu}-\frac{i}{2}\Gamma_{\mu\rho;\lambda}\psi^{\lambda}\psi^{\rho}\right)\left(p_{\nu}-\frac{i}{2}\Gamma_{\nu\xi;\sigma}\psi^{\sigma}\psi^{\xi}\right),\tag{4.4}$$

with π_{μ} the canonical momenta conjugated to the odd variables ψ^{μ} and p_{ν} the momenta conjugated to the even variables x^{μ}. It is clear that the consistency conditions of the primary constraints give no further constraints, but merely determine the corresponding Lagrange multipliers. Actually, the G_{μ}'s are second-class due to the relations

$$\det\left(\left[G_{\mu},G_{\nu}\right]\right)=\det\left(ig_{\mu\nu}\left(x\right)\right)\neq0.\tag{4.5}$$

With these points made clear, we observe that the geodesic motion in spinning space is first-order with respect to the odd variables ψ^{μ}, the associated one-form potential being of the form $ig_{\mu\nu}\psi^{\nu}$. Now, we are able to emphasize the equivalencies with the general theory, namely

$$y^{i}\rightarrow\psi^{\mu},\ a_{i}\left(y\right)\rightarrow a_{\mu}\equiv ig_{\mu\nu}\left(x\right)\psi^{\nu},\ a_{ij}\rightarrow a_{\mu\nu}\equiv ig_{\mu\nu}\left(x\right).\tag{4.6}$$

It can be seen from the above equivalencies that in the case of our model the one-form potential is linear in the variables ψ^{μ}, so the matrix $ig_{\mu\nu}\left(x\right)$ does not depend on them. The starting action is regular (non-degenerate) in terms of the x^{μ}'s. We shall apply the theoretical part of the paper in connection with the first-order part of our model, and leave unchanged the regular part.

In order to construct the one-parameter family of first-class systems as in Sec. 2, we enlarge the original phase-space adding the fermionic canonical pairs $\left(\varphi^{\mu},\Pi_{\mu}\right)$ and take the new primary constraints

$$\overline{G}_{\mu}=G_{\mu}-\frac{i\lambda}{2}g_{\mu\nu}\varphi^{\nu},\tag{4.7}$$

$$\tilde{G}_\mu = \Pi_\mu - \frac{i\lambda}{2} g_{\mu\nu} \left(\psi^\nu + \lambda \varphi^\nu\right). \tag{4.8}$$

It can be easily checked that the $\tilde{G}_\mu$'s are second class, and thus we can eliminate them with the help of the Dirac bracket (constructed with respect to themselves). The fundamental Dirac brackets different from the Poisson ones read

$$[\varphi^\mu, \varphi^\nu]^* = \frac{i}{\lambda^2} g^{\mu\nu}, [\varphi^\mu, \pi_\nu]^* = \frac{1}{2\lambda}\delta^\mu{}_\nu, [\pi_\mu, \pi_\nu]^* = -\frac{i}{4} g_{\mu\nu}, \tag{4.9}$$

$$[p_\mu, \pi_\nu]^* = -\frac{i}{4}\frac{\partial g_{\nu\rho}}{\partial x^\mu}\left(\psi^\rho + \lambda\varphi^\rho\right), [p_\mu, \varphi^\nu]^* = \frac{1}{2\lambda} g^{\nu\rho}\frac{\partial g_{\rho\lambda}}{\partial x^\mu}\left(\psi^\lambda + \lambda\varphi^\lambda\right), \tag{4.10}$$

$$[p_\mu, p_\nu]^* = \frac{i}{4} g^{\alpha\beta}\frac{\partial g_{\alpha\lambda}}{\partial x^\mu}\frac{\partial g_{\beta\rho}}{\partial x^\nu}\left(\psi^\lambda + \lambda\varphi^\lambda\right)\left(\psi^\rho + \lambda\varphi^\rho\right). \tag{4.11}$$

Within this symplectic structure, the constraints $\overline{G}_\mu = 0$ become first-class. The Hamiltonian H^* is, accordingly to (2.7)

$$\begin{aligned} &H^*(x, p, \psi, \varphi) = \\ &\frac{1}{2} g^{\mu\nu}\left(p_\mu - \frac{i}{2}\Gamma_{\mu\rho;\lambda}\left(\psi^\lambda + \lambda\varphi^\lambda\right)\left(\psi^\rho + \lambda\,\varphi^\rho\right)\right) \times \\ &\left(p_\nu - \frac{i}{2}\Gamma_{\nu\xi;\sigma}\left(\psi^\sigma + \lambda\varphi^\sigma\right)\left(\psi^\xi + \lambda\varphi^\xi\right)\right), \end{aligned} \tag{4.12}$$

and it is indeed first-class with respect to the $\overline{G}_\mu$'s in the symplectic structure (4.9-4.11)

$$\left[H^*, \overline{G}_\mu\right]^* = 0. \tag{4.13}$$

Using the BRST formalism for the action

$$\begin{aligned} S_0^T[x, p, \psi, \pi, \varphi, v] = \int_{(1)}^{(2)} d\tau \Big(&\dot{x}^\mu p_\mu + \pi_\mu \dot{\psi}^\mu + \\ &i\lambda g_{\mu\nu}(x)\left(\psi^\mu + \lambda\varphi^\mu\right)\dot{\varphi}^\nu - H^* - v^\mu \overline{G}_\mu\Big) \end{aligned} \tag{4.14}$$

and making use of the gauge-fixing fermion

$$\Psi = \left(\det g_{\mu\nu}(x)\right)^{-1/2} \overline{\eta}_\mu \varphi^\mu, \tag{4.15}$$

we obtain, after eliminating some variables, the path integral

$$Z_\Psi = \int \mathcal{D}x^\mu \, \mathcal{D}\psi^\mu \left(\det g_{\mu\nu}(x)\right)^{1/2} \exp\left(iS_0[x, \psi]\right). \tag{4.16}$$

This is nothing but the path integral of the original system.

5 Conclusion

To conclude with, in this paper we showed that the quantization of purely first-order systems described by action (1.1) means the BRST quantization of action (3.1) using the gauge-fixing fermion (3.4) and the value of the parameter expressed by (3.6). At the Lagrangian level, the quantization of (1.1) implies the BRST quantization of the action

$$S_0\left[y^i, z^i\right] = \int dt\left(a_i(y+\lambda z)\left(\dot{y}^i+\lambda\dot{z}^i\right) - V(y+\lambda z)\right), \tag{5.1}$$

derived from (3.1) eliminating there the momenta and Lagrange multipliers on their equations of motion [13]. It is worth to note that our quantization method maintains the Lorentz covariance in the case of field theory (see action (5.1)) and remains valid when regarding a second-order system as a first-order one with the local generalized co-ordinates $y^i \equiv (q^\alpha, \overline{p}_\alpha)$, where the $\overline{p}_\alpha$'s are the momenta conjugated to the q^α's [10], [12]. Finally, it is remarkable that our results are in fact independent of the particular form of the $a_i(y)$'s. The theoretical results of the paper are illustrated in the case of geodesic motion in spinning space.

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