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# On the One-Dimensional Scattering by Time-Periodic Potentials: General Theory and Application to Two Specific Models

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*Abstract.* A comprehensive introduction to the basic formalism of the one-dimensional scattering by time-periodic short-ranged potentials is presented. The fundamental objects of the theory (transmission and reflexion probabilities, sidebands and time delays) are defined, and a generalized Born expansion derived. Particular emphasis is given to the connection between the time-dependent approach and the quasi-stationary one. In particular, the independence of the scattering process of the choice of time-origin, in the limit of a monoenergetic wave packet, is clearly established. The generalized Born expansion is applied to two archetypical models: the square barrier with modulated height (the celebrated Büttiker-Landauer model) and the square barrier with oscillating position. For these two models, the full transmission probability is calculated up to the first non-vanishing correction in the time-dependent perturbation.

## 1 Introduction

Roughly speaking, time-dependent potentials arise in physics when a small part of a system is singled out, and its action on the larger part neglected. If the motion of the larger part is known, one obtains an approximate description of the small subsystem in terms of an effective non-conservative force field. Examples include interaction of electromagnetic waves with matter, thermal fluctuations, chemical reactions at surfaces, coupling of electrons with

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optical phonons and electrons transport in presence of external oscillating voltages [1]-[10].

By now, the general description of scattering by time-dependent potentials is fairly well understood mathematically [11]-[20]. However, as far as we know, one cannot find in the physical literature a sufficiently simple and complete description of the basic formalism for such systems. The purpose of the first part of the present paper is precisely to fill this gap. For this, we shall limit ourselves to the case of time-periodic potentials and of a single spatial dimension.

We give in Section 2 the formulation of the scattering problem for perturbations periodic in time. The concept of quasi-energy will be explained. Transmission and reflexion probabilities and sideband intensities will be clearly defined. The notion of stationarity will also be discussed, showing that if the incoming packet is sharply peaked in energy, the process becomes insensitive to the choice of time-origin. In Section 3, we establish the connection between the time-dependent approach, described in Section 2, and the quasi-stationary one which deals with the scattering of non-normalizable plane waves. We shall also obtain an heuristic definition of the relevant time delays involved in the description of the scattering process. Section 4 is devoted to the derivation of the perturbation expansion for the scattering matrix, generalizing the usual Born series to the case of time-periodic potentials.

The second part of our work consists of a study of two specific models: the Büttiker-Landauer model [1]-[6], consisting of a square barrier with time-modulated amplitude, and the square barrier whose mean position oscillates [5]-[10]. To do this, we shall use the perturbation expansion we derived in Section 4, to obtain, in Section 5, explicit formulae for the intensities of the transmitted and reflected sidebands at the leading order in the perturbation.

More interestingly, we shall obtain explicit expressions for the first non-vanishing correction to the full transmission probability. This will require second order perturbation expansion which, we believe, is performed here for the first time for these two models. The graphical solutions of Section 6 will show that for both models, the full transmission probability is dominated, in the tunnelling regime, by its first-order-sideband contribution (provided the barrier is sufficiently opaque). The variation of the transmission probability, as a function of the modulation frequency, shows resonances corresponding to maximums of the transmission time delay. These resonances arise approximately at frequencies at which the energy of the first-order-sideband becomes equal to the resonance energies above the top of the barrier, in agreement with the numerical study [21]. We conclude our study by presenting some final comments in Section 7. In particular, the role played by the Büttiker-Landauer time in the description of the crossover regime will be examined.

## 2 Formulation of the scattering problem

We consider a quantum mechanical system with Hamiltonian

$$H(t) = H_0 + V(t), \tag{1}$$

acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, where $d$ is the spatial dimension (later we shall set $d = 1$). The first term in the Hamiltonian, $H_0 = p^2/2m$, is the kinetic energy, with $p$ the momentum operator and $m$ the mass of the particle. The second term, $V(t)$, is the potential energy, which may depend explicitly on time. We shall assume that $V(t)$ is local and short-range. Local means that it can be written

$$V(t) = \int d^d x \, v(x,t) \, |x\rangle\langle x|, \tag{2}$$

with $v(x,t)$ a real function of $x$ and $t$. Short-range means that the function $v(x,t)$ goes to zero sufficiently rapidly (faster than $1/|x|$) as $|x| \to \infty$. To write (2), we have introduced the improper eigenvectors $q|x\rangle = x|x\rangle$, $x \in I\!R^d$, of the position operator $q$, obeying the completeness and orthogonality relationships $\int d^d x \, |x\rangle\langle x| = I$ and $\langle x|x'\rangle = \delta(x - x')$. In the following, we shall assume for simplicity that $v(x,t)$ is bounded everywhere, with (compact) support in a ball of radius $r$.

When one allows the Hamiltonian to become time-dependent, many complications arise. In general $[H(t), H(t')] \neq 0$ for $t \neq t'$, so that the unitary evolution operator $U(t, t_0)$, solution of the Schrödinger equation

$$i\hbar \frac{d}{dt} U(t,t_0) = H(t) U(t,t_0), \qquad U(t_0,t_0) = I, \tag{3}$$

is no longer given simply in terms of an exponential $e^{-\frac{i}{\hbar}H(t-t_0)}$, as it is the case for a static Hamiltonian $H$, but by its Dyson expansion (at least when it converges)

$$U(t,t_0) = e^{-\frac{i}{\hbar}H_0 t} \left[ I + \sum_{n=1}^{\infty} U_{\mathrm{I}}^{(n)}(t,t_0) \right] e^{\frac{i}{\hbar}H_0 t_0}, \tag{4}$$

with

$$U_{\mathrm{I}}^{(n)}(t,t_0) = \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, V_{\mathrm{I}}(t_1) \cdots V_{\mathrm{I}}(t_{n-1}) V_{\mathrm{I}}(t_n) \tag{5}$$

and

$$V_{\mathrm{I}}(t) = e^{\frac{i}{\hbar}H_0 t} V(t) e^{-\frac{i}{\hbar}H_0 t}. \tag{6}$$

Contrary to the static case, the evolution is no more invariant under time translations, the energy of the system is in general not conserved and, in particular, one has to abandon the notion of stationary state.

However, this does not mean that one has to renounce scattering theory. Indeed, the important point for the characterization of scattering states, which leave any bounded region in configuration space as $t \to \pm\infty$, is not that the potential may depends or not explicitly in time, but that it decreases sufficiently rapidly in space i.e., that it is short-range.

More precisely, let $|\psi(t_0)\rangle \in \mathcal{H}$ be the pure state describing the system at time $t_0$. At time $t$ the system will be described by the state $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$. If the initial condition $|\psi(t_0)\rangle$, at time $t_0$, is of scattering type, then $|\psi(t)\rangle$ will behave in the distant future, and has behaved in the remote past, according to the free evolution. In other terms, there exist free evolving outgoing and incoming states $|\varphi_\pm(t)\rangle = e^{-\frac{i}{\hbar}H_0(t-t_0)}|\varphi_\pm(t_0)\rangle$, such that the difference

$$|\psi(t)\rangle - |\varphi_\pm(t)\rangle = U(t,t_0)|\psi(t_0)\rangle - e^{-\frac{i}{\hbar}H_0(t-t_0)}|\varphi_\pm(t_0)\rangle \tag{7}$$

tends to zero (in the Hilbert space norm) as $t \to \pm\infty$.

Multiplying (7) from the left by $U^\dagger(t,t_0)$, one finds that the asymptotic conditions hereabove are equivalent to the existence (as strong limits) of the wave operators[2]

$$\Omega_\pm(t_0) = \lim_{t\to\pm\infty} U^\dagger(t,t_0)e^{-\frac{i}{\hbar}H_0(t-t_0)}. \tag{8}$$

According to (7) and (8), the scattering state at time $t_0$ is related to the incoming and outgoing states at time $t_0$, by the equalities

$$|\psi(t_0)\rangle = \Omega_\pm(t_0)|\varphi_\pm(t_0)\rangle. \tag{9}$$

This yields the correspondence

$$|\varphi_+(t_0)\rangle = \Omega_+^\dagger(t_0)\Omega_-(t_0)|\varphi_-(t_0)\rangle \tag{10}$$

between the outgoing and the incoming state, and defines the unitary scattering operator[3]

$$S(t_0) = \Omega_+^\dagger(t_0)\Omega_-(t_0) \tag{11}$$

for the initial condition at time $t = t_0$.

The main difference between the discussion hereabove and the standard presentation for time-independent potentials, lies in the fact that the scattering process now depends on the choice of the initial time $t_0$, since the evolution is not invariant under time translations i.e., $U(t,t_0) \neq U(t+t_1, t_0+t_1)$ if $t_1 \neq 0$. According to (8),(9), the wave operators for different initial times are related by

$$\Omega_\pm(t_0) = U(t_0,t_1)\Omega_\pm(t_1)e^{\frac{i}{\hbar}H_0(t_0-t_1)}, \tag{12}$$

which in turn yields for the scattering operators

$$S(t_0) = e^{-\frac{i}{\hbar}H_0(t_0-t_1)}S(t_1)e^{\frac{i}{\hbar}H_0(t_0-t_1)}. \tag{13}$$

Relations (12) and (13) are, respectively, the generalization of the usual intertwining property and of the energy conservation law. This becomes more transparent if one differentiates both equations with respect to the initial condition $t_0$, yielding (the prime denotes differentiation with respect to the argument)

$$i\hbar\Omega'_\pm(t_0) = H(t_0)\Omega_\pm(t_0) - \Omega_\pm(t_0)H_0 \tag{14}$$

and

$$i\hbar S'(t_0) = [H_0, S(t_0)]\,. \tag{15}$$

Clearly, for a static Hamiltonian the left hand sides of (14) and (15) are zero, and one recovers the usual form of the commutation properties of wave and scattering operators. In the following, we shall set $\Omega_\pm \equiv \Omega_\pm(0)$ and $S \equiv S(0)$.

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[2] As strong limits of unitary operators, the wave operators are isometries i.e., $\Omega_\pm^\dagger(t_0)\Omega_\pm(t_0) = I$. However, in general they are not unitary i.e., $\Omega_\pm(t_0)\Omega_\pm^\dagger(t_0) \neq I$, according to the fact that bound-states may exist in the theory.

[3] The scattering operator is unitary if the theory is asymptotically complete [20]. Roughly speaking, the theory is complete if all states in $\mathcal{H}$ are either scattering states or bound states i.e., if there are no states trapped by the interaction, nor states that, although propagating away from any bounded region as $t \to \pm\infty$, do not do it according to the law characterizing free particles [17]. In this paper we are only interested in time-periodic short-ranged potentials, for which the completeness of the theory has been demonstrated [12].

## 2.1 Time-periodic potentials and the law of quasi-energy conservation

From now on, we shall assume that the time-dependence of the potential is periodic with period $T = 2\pi/\omega$ i.e., $V(t) = V(t+T)$. Then, it is clear from (3)-(6) that $U(t+T,T) = U(t,0)$, so that $\Omega_\pm(T) = \Omega_\pm(0) \equiv \Omega_\pm$ and $S(T) = S(0) \equiv S$. According to (13), it follows that the scattering operator $S$ commutes with the free evolution over one period:

$$[S, e^{-\frac{i}{\hbar}H_0 T}] = 0. \tag{16}$$

The commutation relation (16) is the precise law of quasi-energy conservation saying that, while $H_0$ may not be conserved by scattering, the energy can be changed only by discrete quanta $n\hbar\omega$, $n = 0, \pm1, \pm2, \ldots$ To see this more precisely, we shall limit ourselves to the case of only one spatial dimension ($d = 1$). We introduce the simultaneous improper eigenvectors of $H_0$ and $\hat{p} = p/|p|$,

$$H_0|E,\sigma\rangle = E|E,\sigma\rangle, \quad \hat{p}|E,\sigma\rangle = \sigma|E,\sigma\rangle, \tag{17}$$

with $E \in [0,\infty)$ and $\sigma \in \{-1,1\}$. They satisfy the relations of completeness

$$\sum_{\sigma=\pm1}\int_0^\infty dE\,|E,\sigma\rangle\langle E,\sigma| = I, \tag{18}$$

and of orthogonality

$$\langle E,\sigma|E',\sigma'\rangle = \delta(E-E')\delta_{\sigma,\sigma'}, \tag{19}$$

and have for spatial wave functions the plane waves ($\hbar k = \sqrt{2mE}$)

$$\langle x|E,\sigma\rangle = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{m}{\hbar k}}\,e^{i\sigma kx}. \tag{20}$$

Taking the improper matrix elements of the commutation relation (16), one finds

$$\langle E',\sigma'|[S, e^{-\frac{i}{\hbar}H_0T}]|E,\sigma\rangle = \langle E',\sigma'|S|E,\sigma\rangle\left(e^{-\frac{i}{\hbar}ET} - e^{-\frac{i}{\hbar}E'T}\right) = 0. \tag{21}$$

Equation (21) implies that the kernel $\langle E',\sigma'|S|E,\sigma\rangle$ is zero except when $e^{-\frac{i}{\hbar}ET} = e^{-\frac{i}{\hbar}E'T}$ or in other words, $E' - E = n\hbar\omega$ with $n = 0, \pm1, \pm2, \ldots$ Therefore, writing the energy $E \in [0,\infty)$ as the sum $E = \varepsilon + n\hbar\omega$, with $n \geq 0$ the entire part of $E/\hbar\omega$ and $\varepsilon \in [0,\hbar\omega)$ the quasi-energy (i.e., the energy modulo $\hbar\omega$), the equality (21) becomes (we set $|\varepsilon,n,\sigma\rangle \equiv |\varepsilon + n\hbar\omega,\sigma\rangle$)

$$\langle\varepsilon',n',\sigma'|S|\varepsilon,n,\sigma\rangle\left(e^{-\frac{i}{\hbar}\varepsilon T} - e^{-\frac{i}{\hbar}\varepsilon' T}\right) = 0. \tag{22}$$

Hence, since the difference in the brackets is zero only if $\varepsilon = \varepsilon'$, the kernel in (22) has the form

$$\langle\varepsilon',n',\sigma'|S|\varepsilon,n,\sigma\rangle = \langle n',\sigma'|S(\varepsilon)|n,\sigma\rangle\,\delta(\varepsilon-\varepsilon'), \tag{23}$$

showing that the quasi-energy $\varepsilon$ is conserved during the scattering process.

The operator $S(\varepsilon)$ is called the scattering matrix or scattering operator on the quasi-energy shell. According to the fact that each quasi-energy state is doubly degenerate (there are only two angles $\sigma = \pm1$ in one dimension), it is natural to divide $S(\varepsilon)$ into four blocks:

$$S(\varepsilon) = \begin{pmatrix} T(\varepsilon) & \tilde{R}(\varepsilon) \\ R(\varepsilon) & \tilde{T}(\varepsilon) \end{pmatrix}, \tag{24}$$

where $T(\varepsilon) = \langle +|S(\varepsilon)|+\rangle$ refers to states coming from the left being finally transmitted, $R(\varepsilon) = \langle -|S(\varepsilon)|+\rangle$ to states coming from the left being finally reflected, and similarly for $\tilde{T}(\varepsilon) = \langle -|S(\varepsilon)|-\rangle$ and $\tilde{R}(\varepsilon) = \langle +|S(\varepsilon)|-\rangle$, but for particles incident from the right.

The unitarity of the scattering operator $S$ induces the unitarity of the scattering matrix $S(\varepsilon)$ i.e.,

$$S^{\dagger}(\varepsilon)S(\varepsilon) = I, \qquad S(\varepsilon)S^{\dagger}(\varepsilon) = I. \tag{25}$$

More specifically, using the decomposition (24), the first identity in (25) yields

$$\begin{aligned} T^{\dagger}(\varepsilon)T(\varepsilon) + R^{\dagger}(\varepsilon)R(\varepsilon) = I \\ \tilde{T}^{\dagger}(\varepsilon)\tilde{T}(\varepsilon) + \tilde{R}^{\dagger}(\varepsilon)\tilde{R}(\varepsilon) = I \\ T^{\dagger}(\varepsilon)\tilde{R}(\varepsilon) + R^{\dagger}(\varepsilon)\tilde{T}(\varepsilon) = 0, \end{aligned} \tag{26}$$

and similarly for the second one. In particular, the first equality in (26) gives

$$\sum_{m\geq 0} \left[ |\langle m|T(\varepsilon)|n\rangle|^2 + |\langle m|R(\varepsilon)|n\rangle|^2 \right] = 1, \tag{27}$$

for all $n \geq 0$.

## 2.2 Transmission and reflexion probabilities

Transmission probability has an easily visualisable meaning. For a particle coming from the left, it may be defined as the probability of finding the scattering particle on the right hand side of the potential as $t \to \infty$. To put the above sentence in the appropriate mathematical language, we first need to introduce the projection operator

$$F_{[b,\infty)} = \int_b^{\infty} dx\, |x\rangle\langle x|, \tag{28}$$

onto the set of states localized in the spatial interval $[b, \infty)$, and the projection operators

$$F_{\pm} = \int_0^{\infty} dE\, |E, \pm\rangle\langle E, \pm| \equiv |\pm\rangle\langle\pm|, \tag{29}$$

onto the set of states with positive $(+)$ and negative $(-)$ momentum. Then, the precise statement that the scattering particle approaches the potential from the left is $F_+|\varphi\rangle = |\varphi\rangle$ i.e., that its incoming momentum is positive (to simplify the notation, we have set $|\varphi\rangle \equiv |\varphi_-(0)\rangle$ for the incoming state at time $t = 0$). The transmission probability, $\mathcal{P}_{\text{tr}}(\varphi)$, for a particle coming from the left, with incoming state $|\varphi\rangle$, may thus be defined by the limit

$$\begin{aligned} \mathcal{P}_{\text{tr}}(\varphi) &= \lim_{t\to\infty} \langle\psi(t)|\, F_{[b,\infty)}\, |\psi(t)\rangle \\ &= \lim_{t\to\infty} \langle e^{-\frac{i}{\hbar}H_0 t} S\varphi|\, F_{[b,\infty)}\, |e^{-\frac{i}{\hbar}H_0 t} S\varphi\rangle \\ &= \langle S\varphi|\, F_+\, |S\varphi\rangle. \end{aligned} \tag{30}$$

The second equality in (30) follows from the asymptotic condition (7). The last equality conforms to the intuition that for large positive times the probability that a free evolving particle will be found in the interval $[b, \infty)$ is the same as the probability that its momentum is positive.[4] The result is independent of $b$, as is clear from the fact that the scattering state propagates away from any bounded region as $t \to \infty$.

According to quasi-energy conservation (23) and to the decomposition (24), one finds that (30) reduces to

$$\mathcal{P}_{\rm tr}(\varphi) = \sum_{n,m,s\geq 0} \int_0^{\hbar\omega} d\varepsilon\, \varphi_n^*(\varepsilon)\langle s|T(\varepsilon)|n\rangle^* \langle s|T(\varepsilon)|m\rangle \varphi_m(\varepsilon), \tag{31}$$

where we have used the completeness and orthogonality relations (18),(19), rewritten into the form

$$\sum_{\sigma=\pm 1}\sum_{n\geq 0}\int_0^{\hbar\omega} d\varepsilon\, |\varepsilon, n, \sigma\rangle\langle\varepsilon, n, \sigma| = I, \tag{32}$$

and

$$\langle \varepsilon, n, \sigma|\varepsilon', n', \sigma'\rangle = \delta(\varepsilon - \varepsilon')\delta_{n,n'}\delta_{\sigma,\sigma'}, \tag{33}$$

and we have set $\varphi_n(\varepsilon) \equiv \langle\varepsilon, n, +|\varphi\rangle$.

Similar considerations lead, of course, to the definition of the reflexion probability for a particle coming from the left, which is simply obtained by replacing $T(\varepsilon)$ by $R(\varepsilon)$ in (31). In the same way, for a particle coming from the right, one has to use $\tilde{T}(\varepsilon)$ and $\tilde{R}(\varepsilon)$ instead of $T(\varepsilon)$ and $R(\varepsilon)$.

## 2.3 Stationary process

Since in experiments one does not usually know the exact form of the incoming packet, one may look for more realistic expressions than (31), for transmission and reflexion probabilities, which are independent of any of the details of preparation of the particles in the incoming beam. Indeed, one may think of the incoming beam as constituted of a succession of incoming wave packets with a small time lag, that are scattered independently.

According to the discussion above, the scattering with a time-dependent potential is sensitive to the choice of time origin (i.e., to the choice of the initial condition), since particles entering the interaction region at different times will not experience the same configuration of the potential.[5] More specifically, using (13), one finds that the transmission probability (31), corresponding to an initial condition at time $t = 0$, transforms to

$$\mathcal{P}_{\rm tr}(\varphi, t_0) = \sum_{n,m,s\geq 0} e^{i(m-n)\omega t_0} \int_0^{\hbar\omega} d\varepsilon\, \varphi_n^*(\varepsilon)\langle s|T(\varepsilon)|n\rangle^* \langle s|T(\varepsilon)|m\rangle \varphi_m(\varepsilon), \tag{34}$$

if the initial condition is shifted at time $t = t_0 \neq 0$.

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[4] From a mathematical point of view, the equality follows from the Dollard decomposition of the free evolution. We refer the interested reader to [22].

[5] The evolution operator $U^{t_1}(t, t_0)$, associated with the time-translated potential $V(t + t_1)$, coincides with the evolution operator $U(t + t_1, t_0 + t_1)$, associated with the potential $V(t)$, since both operators obey the same differential equation with the same initial condition at $t = t_0$. Therefore, according to definitions (8),(11), the scattering operator $S^{t_1}(t_0)$ for the potential $V(t + t_1)$ and with initial condition at time $t = t_0$, is the same as the scattering operator $S(t_0 + t_1)$ for the potential $V(t)$ and with initial condition at time $t = t_0 + t_1$. This is the precise version of the intuitive statement saying that two incoming packets with a time lag $t_1$ will feel the external potential with a time difference $t_1$. See also the discussion in [23].

However, for an incoming packet with a small spread in energy, one loses the control on the time at which the particle is prepared. Therefore, if the energy spread $\Delta E$ is sufficiently narrow to give a time-dispersion $\Delta t \simeq \hbar/\Delta E$,[6] larger than the period of the potential, one expect the transmission and reflexion probabilities to become independent of the choice of the time of preparation of the particle.

To show that this is indeed the case, one simply has to observe that if the incoming state has its energy support in an interval $\Delta E < \hbar\omega$, i.e., it is of the form $\varphi_{n'}(\varepsilon) = \delta_{n,n'} g(\varepsilon)$, with $g$ a given function of quasi-energy $\varepsilon \in [0, \hbar\omega)$, then (34) reduces to

$$\mathcal{P}_{\rm tr}(\varphi, t_0) = \sum_{m\geq 0} \int_0^{\hbar\omega} d\varepsilon' \; |\langle m|T(\varepsilon')|n\rangle|^2 \, |g(\varepsilon')|^2. \tag{35}$$

This expression is manifestly independent of the initial condition $t_0$. In other words, the process becomes stationary for incoming packets with a spread in energy less than $\hbar\omega$, in the sense that it becomes independent of the choice of time-origin or, equivalently, of the choice of the time of preparation of the incoming state.

Furthermore, under the additional assumption that the elements of the scattering matrix $S(\varepsilon)$ are slowly varying functions of quasi-energy $\varepsilon$, and that $g(\varepsilon')$ is sharply peaked at about $\varepsilon' = \varepsilon$, we have the approximation

$$\begin{aligned} \mathcal{P}_{\rm tr}(\varphi, t_0) &\approx \int_0^{\hbar\omega} d\varepsilon' \, |g(\varepsilon')|^2 \sum_{m\geq 0} |\langle m|T(\varepsilon)|n\rangle|^2 \\ &= \sum_{m\geq 0} |\langle m|T(\varepsilon)|n\rangle|^2 \equiv \mathcal{P}_{\rm tr}(E), \end{aligned} \tag{36}$$

where the last equality follows from the fact that the incoming state is normalized to unity, and we have set $E = \varepsilon + n\hbar\omega$ for the incoming energy.

For the reflexion probability, one finds in the same way

$$\mathcal{P}_{\rm ref}(\varphi, t_0) \approx \sum_{m\geq 0} |\langle m|R(\varepsilon)|n\rangle|^2 \equiv \mathcal{P}_{\rm ref}(E), \tag{37}$$

and, as a consequence of unitarity (27), the probability is conserved: $\mathcal{P}_{\rm tr}(E) + \mathcal{P}_{\rm ref}(E) = 1$.

In summary, equations (36) and (37) show that, in the limit of initial packets sharply peaked in energy, the process becomes stationary and transmission and reflexion probabilities depend only upon the incoming energy i.e., they are free of any wave packet structure considerations.

## 2.4 Sidebands

According to equations (36) and (37) the physical interpretation of the scattering matrix elements is straightforward: $|\langle n'|T(\varepsilon)|n\rangle|^2$ is the probability that a particle with incoming kinetic energy $E = \varepsilon + n\hbar\omega$, will be finally transmitted with energy $E' = \varepsilon + n'\hbar\omega$, and similarly for the elements of $R(\varepsilon)$ for the case of reflexion.

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[6]We recall that the proper sense of the time-energy uncertainty relation $\Delta E \Delta t \simeq \hbar$ for a free particle is that one is unable to say when it will cross a given point with a precision greater than $\Delta t \simeq \hbar/\Delta E$ [24].

From a physical point of view, it is however more natural to fix the incoming energy $E$ (instead of the quasi-energy $\varepsilon$), and to discuss only in terms of energy transfer. This leads us to the definition of the so-called intensities for the transmitted sidebands. These are the probabilities,

$$\mathcal{P}^m_{\rm tr}(E) = |\langle n+m|T(\varepsilon)|n\rangle|^2 , \tag{38}$$

for an incoming particle of energy $E$, of being transmitted with a transfer of exactly $m$ quanta of energy $\hbar\omega$ with the external field, $m = 0, \pm 1, \pm 2, \ldots$ Clearly, a similar definition holds for the case of reflexion.

The probability of a transfer of $m$ quanta with the external field, irrespective of the fact that the particle is ultimately observed as transmitted or reflected, is given by the sum

$$\mathcal{P}^m(E) = \mathcal{P}^m_{\rm tr}(E) + \mathcal{P}^m_{\rm ref}(E) = |\langle n+m|T(\varepsilon)|n\rangle|^2 + |\langle n+m|R(\varepsilon)|n\rangle|^2 . \tag{39}$$

It is worth noting that, by definition, sidebands are zero for $m < -n$, since there is no scattering for a particle with negative outgoing energy. The case $m = 0$, which corresponds to the case of no transfer with the external field, will be referred to as the elastic channel in the sequel.

## 2.5 Invariance principles

We conclude this Section by discussing time reversal and parity invariance. For a static potential, time reversal invariance implies that the transmission amplitudes from the left and from the right coincide. What happens for a time-periodic potential? To find the answer let us first recall that the time reversal operator $\mathcal{T}$ is the anti-unitary operator obeying $\mathcal{T}|x\rangle = |x\rangle$ and $\mathcal{T}|E,\sigma\rangle = |E,-\sigma\rangle$. Since the local potential is real, the total Hamiltonian commutes with $\mathcal{T}$. Thus, according to Dyson's series (4)-(6), and taking into account the anti-unitarity of $\mathcal{T}$, one finds that $\mathcal{T}U^\dagger(t,t_0) = U_{\rm R}(-t,-t_0)\mathcal{T}$, where $U_{\rm R}$ denotes the evolution operator for the time-reversed potential $V_{\rm R}(t) = V(-t)$. Then, assuming $V(t) = V(-t)$, one has $S = \mathcal{T}^\dagger S^\dagger \mathcal{T}$, from which it follows that

$$\langle n|T(\varepsilon)|m\rangle = \langle n|\tilde{T}^\dagger(\varepsilon)|m\rangle^* = \langle m|\tilde{T}(\varepsilon)|n\rangle \tag{40}$$

and

$$\langle n|R(\varepsilon)|m\rangle = \langle n|\tilde{R}^\dagger(\varepsilon)|m\rangle^* = \langle m|\tilde{R}(\varepsilon)|n\rangle . \tag{41}$$

In other words, for time-reversal invariant potentials $V(t) = V(-t)$, the probability for a particle coming from the left with energy $\varepsilon + n\hbar\omega$ to be transmitted (respectively, reflected) with energy $\varepsilon + m\hbar\omega$, is the same as the probability for a particle coming from the right with energy $\varepsilon + m\hbar\omega$ to be transmitted (respectively, reflected) with energy $\varepsilon + n\hbar\omega$ (the same will be true for the associated time-delays, see next Section). Considering equation (13) and the remark in Footnote 5, one can note that the same conclusion also holds for potentials which satisfy time reversal invariance only in the limited sense $V(t) = V(-t+t_1)$, for a given $t_1$.[7]

[7] In that case one has, instead of (40),

$$\langle n|T(\varepsilon)|m\rangle = \langle m|\tilde{T}(\varepsilon)|n\rangle e^{2i(n-m)\omega t_1}$$

and similarly for the reflexion amplitudes.

For rotational (i.e., parity) invariant potentials $v(x,t) = v(-x,t)$, one finds in the same way that the scattering operator commutes with the unitary parity operator obeying $\mathcal{P}|x\rangle = |-x\rangle$ and $\mathcal{P}|E,\sigma\rangle = |E,-\sigma\rangle$, implying $T(\varepsilon) = \tilde{T}(\varepsilon)$ and $R(\varepsilon) = \tilde{R}(\varepsilon)$.

Finally, if the potential is $\mathcal{P}$ and $\mathcal{T}$ invariant, then $\langle n|T(\varepsilon)|m\rangle = \langle m|T(\varepsilon)|n\rangle$ and $\langle n|R(\varepsilon)|m\rangle = \langle m|R(\varepsilon)|n\rangle$, i.e., scattering amplitudes remain unaffected if one permutes the incoming with the outgoing energy.

# 3 Connection with the quasi-stationary approach and time delays

In this Section we establish the connection between the time-dependent approach, presented in the previous Section, and the more commonly used quasi-stationary approach, which describes the scattering of (non-normalizable) plane waves in terms of a system of coupled stationary Schrödinger equations, called the quasi-stationary Schrödinger equation. This connection is based on a stationary phase argument which also provide us with a definition of time delays.

To derive the quasi-stationary Schrödinger equation, we introduce the Fourier series

$$V(t) = \sum_n V_n e^{-in\omega t}, \quad \Omega_-(t) = \sum_n \Omega_n e^{-in\omega t} \tag{42}$$

for the potential $V(t)$ and for the wave operator $\Omega_-(t)$. Inserting (42) into (14) and comparing the Fourier coefficients one finds the operatorial identity

$$\sum_s \left(\delta_{n,s} H_0 + V_{n-s}\right) \Omega_s = \Omega_n \left(H_0 + n\hbar\omega I\right). \tag{43}$$

Formally, we multiply (43) from the left by $\langle x|$, and from the right by $|E,\sigma\rangle$, to obtain the system of coupled stationary Schrödinger equations

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n^\sigma(E,x) + \sum_s v_{n-s}(x)\psi_s^\sigma(E,x) = (E + n\hbar\omega)\psi_n^\sigma(E,x) \tag{44}$$

for the kernels $\psi_n^\sigma(E,x) = \sqrt{2\pi\hbar}\sqrt{\frac{\hbar k}{m}}\langle x|\Omega_n|E,\sigma\rangle$. The system (44) is the so-called quasi-stationary Schrödinger equation.

The solution of (44) corresponding to a plane wave $e^{ikx}$ of energy $E = \hbar^2k^2/2m$, incident from the left ($\sigma = 1$), is distinguished by the boundary conditions (we assume for simplicity that the potential is of compact support in the interval $[-r,r]$)

$$\psi_n^+(E,x) = \begin{cases} \delta_{n,0}\, e^{ikx} + R_n(E)\, e^{-ik_n x} & x < -r \\ T_n(E)\, e^{ik_n x} & x > r, \end{cases} \tag{45}$$

where we have defined $\hbar k_n = \sqrt{2m(E + n\hbar\omega)}$. Similarly, for a plane wave $e^{-ikx}$ incident from the right ($\sigma = -1$), we have the boundary conditions

$$\psi_n^-(E,x) = \begin{cases} \tilde{T}_n(E)\, e^{-ik_n x} & x < -r \\ \delta_{n,0}\, e^{-ikx} + \tilde{R}_n(E)\, e^{ik_n x} & x > r. \end{cases} \tag{46}$$

The amplitudes $T_n(E)$, $R_n(E)$, $\tilde{T}_n(E)$ and $\tilde{R}_n(E)$ are uniquely determined by (44)-(46). Note that in the definition of $k_n$, we chose the branch of $\sqrt{E+n\hbar\omega}$ so that if $E+n\hbar\omega = |E+n\hbar\omega|e^{i\alpha}$, then $\sqrt{E+n\hbar\omega} = \sqrt{|E+n\hbar\omega|}e^{i\alpha/2}$. This means that when $E+n\hbar\omega < 0$, then $k_n = i\sqrt{|E+n\hbar\omega|}$, corresponding to exponentially decaying functions as $x \to \pm\infty$, the so-called quasi-bound states.

Relating the above amplitudes to the elements of the scattering matrix gives the connection between the quasi-stationary approach and the time-dependent one (we shall follow here essentially the treatment presented in [15]). For this, we assume that the incoming wave has its energy support in the interval $[n\hbar\omega, (n+1)\hbar\omega)$, and describes a particle approaching the potential from the left i.e., $\varphi_m^\sigma(\varepsilon) = \langle \varepsilon, m, \sigma|\varphi\rangle = \delta_{n,m}\delta_{\sigma,1}g(\varepsilon)$, with $g(\varepsilon)$ a sufficiently smooth function of quasi-energy $\varepsilon \in [0, \hbar\omega)$. Then, by means of (9),(12),(32),(33) and (42), the scattering spatial wave function reads

$$\begin{aligned}\psi(t,x) &= \langle x|\psi(t)\rangle = \langle x|U(t,0)\Omega_-|\varphi\rangle \\ &= \langle x|\Omega_-(t)e^{-\frac{i}{\hbar}H_0 t}|\varphi\rangle = \sum_{s'}\langle x|\Omega_{s'}e^{-\frac{i}{\hbar}H_0 t}|\varphi\rangle e^{-is'\omega t} \\ &= \frac{1}{\sqrt{2\pi\hbar}}\int_0^{\hbar\omega} d\varepsilon \sqrt{\frac{m}{\hbar\kappa_n}}\sum_{s'}\psi_{s'}^+(\varepsilon+n\hbar\omega,x)e^{-\frac{i}{\hbar}[\varepsilon+(n+s')\hbar\omega]t}g(\varepsilon) \\ &= \frac{1}{\sqrt{2\pi\hbar}}\int_0^{\hbar\omega} d\varepsilon \sqrt{\frac{m}{\hbar\kappa_n}}\sum_{s}\psi_{s-n}^+(\varepsilon+n\hbar\omega,x)e^{-\frac{i}{\hbar}[\varepsilon+s\hbar\omega]t}g(\varepsilon), \end{aligned} \tag{47}$$

where $\psi_{s-n}^+(\varepsilon+n\hbar\omega,x) = \sqrt{2\pi\hbar}\sqrt{\frac{\hbar\kappa_n}{m}}\langle x|\Omega_{s-n}|\varepsilon,n,+\rangle$, $\hbar\kappa_n = \sqrt{2m(\varepsilon+n\hbar\omega)}$, and for the last equality we have made the change of variables $s = s'+n$. In the same way, the spatial wave functions of the free evolving incoming and outgoing states are

$$\varphi(t,x) = \langle x|e^{-\frac{i}{\hbar}H_0 t}|\varphi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_0^{\hbar\omega} d\varepsilon \sqrt{\frac{m}{\hbar\kappa_n}} e^{-\frac{i}{\hbar}[(\varepsilon+n\hbar\omega)t-\hbar\kappa_n x]}g(\varepsilon) \tag{48}$$

and

$$\begin{aligned}(S\varphi)(t,x) &= \langle x|e^{-\frac{i}{\hbar}H_0 t}S|\varphi\rangle \\ &= \frac{1}{\sqrt{2\pi\hbar}}\int_0^{\hbar\omega} d\varepsilon \sum_{\sigma=\pm1}\sum_{s\geq 0}\sqrt{\frac{m}{\hbar\kappa_s}}\langle s,\sigma|S(\varepsilon)|n,+\rangle e^{-\frac{i}{\hbar}[(\varepsilon+s\hbar\omega)t-\hbar\sigma\kappa_s x]}g(\varepsilon) \\ &= \frac{1}{\sqrt{2\pi\hbar}}\int_0^{\hbar\omega} d\varepsilon \sum_{s\geq 0}\sqrt{\frac{m}{\hbar\kappa_s}}\Big[\langle s|T(\varepsilon)|n\rangle e^{-\frac{i}{\hbar}[(\varepsilon+s\hbar\omega)t-\hbar\kappa_s x]} \\ &+ \langle s|R(\varepsilon)|n\rangle e^{-\frac{i}{\hbar}[(\varepsilon+s\hbar\omega)t+\hbar\kappa_s x]}\Big]g(\varepsilon). \end{aligned} \tag{49}$$

By the stationary phase argument, $\varphi(t,x)$ differs appreciably from zero as $t \to -\infty$ only for those values of $(t,x)$ for which the argument of the exponential in (48) is stationary with respect to energy variations. More precisely, the incoming packet is essentially localized at time $t$ ($t \to -\infty$) at a point $x$ given by the condition

$$\frac{d}{d\varepsilon}\left[(\varepsilon+n\hbar\omega)t - \hbar\kappa_n x\right] = 0 \tag{50}$$

i.e., $x = \frac{\hbar\kappa_n}{m}t$, showing that the particle was on the far left of the potential in the distant past.

In the same way, applying the stationary phase argument to the outgoing state $(S\varphi)(t, x)$ for $t \to \infty$, yields the conditions

$$\frac{d}{d\varepsilon}\left[(\varepsilon + s\hbar\omega)t - \hbar\kappa_s x - \arg\langle s|T(\varepsilon)|n\rangle\right] = 0, \quad s = 0, 1, 2, \ldots \tag{51}$$

and

$$\frac{d}{d\varepsilon}\left[(\varepsilon + s\hbar\omega)t + \hbar\kappa_s x - \arg\langle s|R(\varepsilon)|n\rangle\right] = 0, \quad s = 0, 1, 2, \ldots, \tag{52}$$

showing that the outgoing wave is split into an infinite number of parts, each of which is essentially localized at time $t$ ($t \to \infty$) at points

$$x = \frac{\hbar\kappa_s}{m}\left(t - \hbar\frac{d}{d\varepsilon}\arg\langle s|T(\varepsilon)|n\rangle\right), \quad s = 0, 1, 2, \ldots \tag{53}$$

in the right hand side of the potential (transmission contributions), and at points

$$x = -\frac{\hbar\kappa_s}{m}\left(t - \hbar\frac{d}{d\varepsilon}\arg\langle s|R(\varepsilon)|n\rangle\right), \quad s = 0, 1, 2, \ldots \tag{54}$$

in the left hand side of the potential (reflexion contributions).

Next, the asymptotic relations (7) are used to compare the scattering wave (47) with the incoming (48) and outgoing (49) waves for $t \to -\infty$ and $t \to \infty$, respectively. To this end, it is necessary to study the asymptotic of $\psi(t, x)$ as $t \to \pm\infty$. By (45), the terms with $s < 0$ in (47) (the quasi-bound states) do not contribute since they are exponentially decreasing as $x \to \pm\infty$. Again by the stationary phase argument, it is easy to check that only the terms of the form $\delta_{s,n}e^{i\kappa_n x}$ contribute in the limit $t \to -\infty$ of (47), whereas the others contribute only as $t \to \infty$. Comparing these asymptotic with those obtained hereabove for the incoming and outgoing states, one finds that they coincide if one sets

$$T_{s-n}(\varepsilon + n\hbar\omega) = \sqrt{\frac{\kappa_n}{\kappa_s}}\,\langle s|T(\varepsilon)|n\rangle \tag{55}$$

and

$$R_{s-n}(\varepsilon + n\hbar\omega) = \sqrt{\frac{\kappa_n}{\kappa_s}}\,\langle s|R(\varepsilon)|n\rangle. \tag{56}$$

Relations (55) and (56) establish the proper connection between the amplitudes delivered by the quasi-stationary approach and the elements of the scattering matrix. Similar relations hold, with obvious modifications, for the case of a particle incident from the right.

## 3.1 Time delays

According to the analysis above, if the incoming wave is well peaked in energy, the outgoing scattering state is a superposition of an infinite number of waves characterized by the free velocities $v_s = \hbar\kappa_s/m$, $s = 0, 1, 2, \ldots$ Since the latter are essentially localized at points (53) for the transmitted parts, and points (54) for the reflected ones, it is natural to interpret

$$\tau^{\mathrm{tr}}_{s,n}(\varepsilon) = \hbar\frac{d}{d\varepsilon}\arg\langle s|T(\varepsilon)|n\rangle \tag{57}$$

and

$$\tau^{\mathrm{ref}}_{s,n}(\varepsilon) = \hbar\frac{d}{d\varepsilon}\arg\langle s|R(\varepsilon)|n\rangle \tag{58}$$

in (53) and (54) as the time delays experienced by the transmitted and reflected particles with incoming energy $\varepsilon + n\hbar\omega$ and outgoing energy $\varepsilon + s\hbar\omega$.

Transmission and reflexion time delays, irrespective of the outgoing energy, may thus be defined by the conditional averages ($E = \varepsilon + n\hbar\omega$)

$$\tau^{\mathrm{tr}}(E) = \frac{1}{\mathcal{P}_{\mathrm{tr}}(E)}\sum_{s\geq 0}|\langle s|T(\varepsilon)|n\rangle|^2\ \tau^{\mathrm{tr}}_{s,n}(\varepsilon) \tag{59}$$

and

$$\tau^{\mathrm{ref}}(E) = \frac{1}{\mathcal{P}_{\mathrm{ref}}(E)}\sum_{s\geq 0}|\langle s|R(\varepsilon)|n\rangle|^2\ \tau^{\mathrm{ref}}_{s,n}(\varepsilon). \tag{60}$$

Finally, the global (i.e., unconditional) time delay is given by

$$\begin{aligned}\tau(E) &= \mathcal{P}_{\mathrm{ref}}(E)\,\tau^{\mathrm{ref}}(E) + \mathcal{P}_{\mathrm{tr}}(E)\,\tau^{\mathrm{tr}}(E) \\ &= -i\hbar\langle n,+|S^{\dagger}(\varepsilon)\frac{d}{d\varepsilon}S(\varepsilon)|n,+\rangle,\end{aligned} \tag{61}$$

where $\tau(\varepsilon) = -i\hbar S^{\dagger}(\varepsilon)dS(\varepsilon)/d\varepsilon$ is the quasi-energy version of the well known Eisenbud-Wigner time-delay operator [25].

# 4 Generalization of the Born series

In this Section, we derive the natural generalization of the Born series for the scattering operator on the quasi-energy shell. We begin by observing that, by means of (8) and (11), we may write the scattering operator as the (weak) limit of $U_{\mathrm{I}}(t,t_0) = e^{\frac{i}{\hbar}H_0 t}U(t,t_0)e^{-\frac{i}{\hbar}H_0 t_0}$ for $t_0 \to -\infty$ and $t \to \infty$. Writing $U_{\mathrm{I}}(t,t_0)$ as the integral of its derivative, one obtains for the scattering operator the integral representation

$$S = I - \frac{i}{\hbar}\int dt\, e^{\frac{i}{\hbar}H_0 t}V(t)\,\Omega_-(t)e^{-\frac{i}{\hbar}H_0 t}. \tag{62}$$

Taking the matrix elements of (62) with respect to the improper eigenvectors (17), we find

$$\begin{aligned}\langle\varepsilon',s',\sigma'|S|\varepsilon,s,\sigma\rangle &= \delta(\varepsilon'-\varepsilon)\delta_{s',s}\delta_{\sigma',\sigma} \\ &- \frac{i}{\hbar}\int dt\, e^{\frac{i}{\hbar}[\varepsilon'-\varepsilon+(s'-s)\hbar\omega]t}\langle\varepsilon',s',\sigma'|V(t)\Omega_-(t)|\varepsilon,s,\sigma\rangle.\end{aligned} \tag{63}$$

Inserting the Fourier representations (42) for $V(t)$ and $\Omega_-(t)$ and redefining the summation indices, we arrive at the relation

$$\langle \varepsilon', s', \sigma' | S | \varepsilon, s, \sigma \rangle = \langle s', \sigma' | S(\varepsilon) | s, \sigma \rangle \, \delta(\varepsilon' - \varepsilon) \tag{64}$$

with

$$\langle s', \sigma' | S(\varepsilon) | s, \sigma \rangle = \delta_{s',s}\delta_{\sigma',\sigma} - 2\pi i \sum_m \langle \varepsilon, s', \sigma' | V_{s'-m} \Omega_{m-s} | \varepsilon, s, \sigma \rangle . \tag{65}$$

According to (65), to determine the perturbation expansion for the elements of the scattering matrix (i.e., the Born series), it suffices to determine the perturbation expansion of $\Omega_{m-s}|\varepsilon, s, \sigma\rangle$. To do this, our starting point will be the Dyson expansion (4)-(6), which we rewrite, with the help of (42), as[8]

$$U_{\rm I}(t, t_0) = I + \sum_{n=1}^{\infty} U_{\rm I}^{(n)}(t, t_0) \tag{66}$$

with

$$U_{\rm I}^{(n)}(t, t_0) = \left(-\frac{i}{\hbar}\right)^n \sum_{\ell_1, \ldots, \ell_n} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n \, V_{\ell_1}(t_1) \cdots V_{\ell_n}(t_n) \, e^{-i\omega \sum_{j=1}^{n} \ell_j t_j} \tag{67}$$

and

$$V_\ell(t) = e^{\frac{i}{\hbar} H_0 t} V_\ell \, e^{-\frac{i}{\hbar} H_0 t} . \tag{68}$$

On the other hand, for the wave operator $\Omega_-(t)$, we have (in the strong sense)

$$\begin{aligned} \Omega_-(t) &= \lim_{t_0 \to -\infty} U^\dagger(t_0, t) e^{-\frac{i}{\hbar} H_0 (t_0 - t)} \\ &= e^{-\frac{i}{\hbar} H_0 t} \left[ \lim_{t_0 \to -\infty} U_{\rm I}(t, t_0) \right] e^{\frac{i}{\hbar} H_0 t} . \end{aligned} \tag{69}$$

Thus, we need to study the Dyson expansion (66)-(68) in the limit $t_0 \to -\infty$. In order to take this limit without breaking the convergence of the series, we have to introduce an adiabatic cut off of the interaction i.e., we have to replace $V(t)$ by $e^{\delta t} V(t)$, $\delta > 0$. The wave operator $\Omega_-(t, \delta)$, for the potential with cut off, is given by the series[9]

$$\Omega_-(t, \delta) = I + \sum_{n=1}^{\infty} \Omega_-^{(n)}(t, \delta), \tag{70}$$

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[8]Notice that each term of the series can be majorized by

$$\| U_{\rm I}^{(n)}(t, t_0) \| \leq \frac{1}{n!} \left( \frac{|t - t_0|}{\hbar} \sum_m \| V_m \| \right)^n ,$$

so that the series is norm-convergent for finite times $|t - t_0| < \infty$, and for bounded time-periodic potentials obeying $\sum_m \| V_m \| < \infty$.

[9]The series is norm-convergent for each $\delta > 0$, since we have the bounds

$$\| \Omega_-^{(n)}(t, \delta) \| \leq \frac{1}{n!} \left( \frac{e^{\delta t}}{\hbar \delta} \sum_m \| V_m \| \right)^n .$$

At the end of the calculation, we shall take the limit $\delta \to 0$. However, the convergence of the series will then no longer be guaranteed, since these bounds diverge in that limit.

with

$$\Omega_-^{(n)}(t,\delta) = \left(-\frac{i}{\hbar}\right)^n \sum_{\ell_1,\dots,\ell_n} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \, e^{-\frac{i}{\hbar}H_0 t} \times \\ \times \; [V_{\ell_1}(t_1)\cdots V_{\ell_n}(t_n)]\, e^{\frac{i}{\hbar}H_0 t} e^{-i\sum_{j=1}^n(\omega\ell_j+i\delta)t_j}. \tag{71}$$

Making in (71) the change of variables $t_m = \sum_{j=1}^m t'_j$, $m = 1,\dots,n$, it gives

$$\Omega_-^{(n)}(t,\delta) = \left(-\frac{i}{\hbar}\right)^n \sum_{\ell_1,\dots,\ell_n} \int_{-\infty}^{t} dt'_1 \int_{-\infty}^{0} dt'_2 \cdots \int_{-\infty}^{0} dt'_n \, e^{-\frac{i}{\hbar}H_0 t} \times \\ \times \; \left\{ \prod_{m=1}^{n} e^{\frac{i}{\hbar}\left[H_0 - \hbar\sum_{j=m}^n(\omega\ell_j+i\delta)\right]t'_m} V_{\ell_m} \right\} e^{\frac{i}{\hbar}H_0\left(t-\sum_{j=1}^n t'_j\right)}. \tag{72}$$

The integrals in (72) can be evaluated by using the formula ($\mathrm{Im} z > 0$)

$$\int_{-\infty}^{t} dt' \, e^{\frac{i}{\hbar}(H_0-z)t'} = i\hbar R_0(z) e^{\frac{i}{\hbar}(H_0-z)t}, \tag{73}$$

where we have defined the resolvent $R_0(z) = (z - H_0)^{-1}$. To remove the last exponential, we apply (72) to an improper eigenvector $|\varepsilon, s, \sigma\rangle$. Using (73) and the change of variables $s_j = s + \sum_{m=n-(j-1)}^n \ell_m$, $j = 1,\dots,n$, one finds

$$\Omega_-^{(n)}(t,\delta)|\varepsilon,s,\sigma\rangle = e^{n\delta t} \sum_{s_n} \Omega_{s_n-s}^{(n)}(\delta)|\varepsilon,s,\sigma\rangle e^{-i(s_n-s)\omega t} \tag{74}$$

with

$$\Omega_{s_n-s}^{(n)}(\delta)|\varepsilon,s,\sigma\rangle = \sum_{s_1,\dots,s_{n-1}} R_0[\varepsilon+\hbar(s_n\omega+i\delta)]V_{s_n-s_{n-1}}\cdots \\ \cdots V_{s_2-s_1} R_0[\varepsilon+\hbar(s_1\omega+i\delta)]V_{s_1-s}|\varepsilon,s,\sigma\rangle. \tag{75}$$

If we let $\delta \to 0$ in (74) we recover the Fourier series for $\Omega_-^{(n)}(t)|\varepsilon,s,\sigma\rangle$. Setting

$$G_0(E) \equiv \lim_{\substack{\delta\to 0\\ \delta>0}} R_0(E+i\delta), \tag{76}$$

we get for $\Omega_{m-s}|\varepsilon,s,\sigma\rangle$ the formal series

$$\Omega_{m-s}|\varepsilon,s,\sigma\rangle = \delta_{m,s}|\varepsilon,s,\sigma\rangle + \sum_{n=1}^{\infty} \Omega_{m-s}^{(n)}|\varepsilon,s,\sigma\rangle \tag{77}$$

with

$$\Omega_{m-s}^{(n)}|\varepsilon,s,\sigma\rangle = \sum_{\ell_1,\dots,\ell_{n-1}} G_0(\varepsilon+m\hbar\omega)V_{m-\ell_1}\cdots \\ \cdots V_{\ell_{n-2}-\ell_{n-1}} G_0(\varepsilon+\ell_{n-1}\hbar\omega)V_{\ell_{n-1}-s}|\varepsilon,s,\sigma\rangle. \tag{78}$$

Finally, inserting (78) into (65), we obtain for the elements of $S(\varepsilon)$ the perturbation expansion

$$S(\varepsilon) = I + \sum_{n=1}^{\infty} S^{(n)}(\varepsilon) \tag{79}$$

with

$$\begin{aligned}\langle s', \sigma'|S^{(n)}(\varepsilon)|s, \sigma\rangle &= -2\pi i \sum_{\ell_1,\ldots,\ell_{n-1}} \langle \varepsilon, s', \sigma'|V_{s'-\ell_1} G_0(\varepsilon + \ell_1\hbar\omega) V_{\ell_1-\ell_2}\cdots \\ &\quad \cdots G_0(\varepsilon + \ell_{n-1}\hbar\omega) V_{\ell_{n-1}-s}|\varepsilon, s, \sigma\rangle. \end{aligned} \tag{80}$$

Equation (80) is the natural generalization of the Born series for a potential periodic in time. In the special case of a static potential $V_{\ell_j-\ell_{j+1}} = \delta_{\ell_j,\ell_{j+1}} V$, one recovers the usual Born expansion as expected. In fact, it is possible to write (79),(80) in a form closer to the latter, by introducing the matrix notation

$$\bar{V} = \sum_{n,m} V_{n-m}\,|n\rangle\langle m|, \quad \bar{N} = \sum_{n} n\,|n\rangle\langle n| \tag{81}$$

which allows (80) to be rewritten compactly as

$$S^{(n)}(\varepsilon) = -2\pi i \langle \varepsilon|\bar{V}\left[G_0(\varepsilon + \bar{N}\hbar\omega)\bar{V}\right]^n|\varepsilon\rangle. \tag{82}$$

# 5 Application to two specific models

In this second part of the paper, we apply the general theory developed in Sections 1-4, to two basic one-dimensional models: the oscillating amplitude barrier, better known as the Büttiker-Landauer model, and the oscillating position barrier.

The oscillating amplitude barrier model has played a central role in the recent controversy over tunnelling times (see also Section 7), and may be seen as a simplistic model for tunnelling electrons coupled with optical phonons of fixed frequency. In its generalized form, it is described by the time-periodic potential

$$v(x,t) = v_0(x) + \lambda(t)v_1(x), \qquad \lambda\left(t + \frac{2\pi}{\omega}\right) = \lambda(t) \tag{83}$$

consisting of a static potential $v_0(x)$, locally modulated by a time-periodic perturbation $\lambda(t)v_1(x)$. The oscillating position barrier, on the other hand, is pertinent to the study of the a-c Stark effect or in the modeling of chemical reactions at surfaces. It is described by the potential

$$v(x,t) = v_0(x - a(t)), \qquad a\left(t + \frac{2\pi}{\omega}\right) = a(t). \tag{84}$$

For small modulations $\lambda(t)$ and oscillations $a(t)$, both models can be analyzed with the help of the perturbation theory for time-periodic potentials. For this, since we no longer take as the unperturbed Hamiltonian the free Hamiltonian $H_0$ but the total static Hamiltonian $H = H_0 + V_0$, we first need to modify the Born expansion we derived in Section 4.

Let $H(t) = H+W(t)$ be a time-periodic Hamiltonian, with $H = H_0+V_0$ time-independent and $W(t)$ of zero average i.e., $\int_0^{2\pi/\omega} dt\, W(t) = 0$. Writing

$$\begin{aligned} U_I(t,t_0) &= e^{\frac{i}{\hbar}H_0 t}U(t,t_0)e^{-\frac{i}{\hbar}H_0 t_0} \\ &= e^{\frac{i}{\hbar}H_0 t}e^{-\frac{i}{\hbar}Ht}\left(e^{\frac{i}{\hbar}Ht}U(t,t_0)e^{-\frac{i}{\hbar}Ht_0}\right)e^{\frac{i}{\hbar}Ht_0}e^{-\frac{i}{\hbar}H_0 t_0}, \end{aligned} \tag{85}$$

and taking the limits $t \to \infty$ and $t_0 \to -\infty$, one finds that the full scattering operator $S$ for the scattering system defined by the pair $(H(t), H_0)$, factorizes according to

$$S = \Omega_+^{0\,\dagger}\, \hat{S}\; \Omega_-^0, \tag{86}$$

where $\Omega_\pm^0$ are the wave operators belonging to the pair $(H, H_0)$, and $\hat{S}$ is the scattering operator for the system $(H(t), H)$. Clearly, the perturbation expansion of Section 4 holds for $\hat{S}$ (at least at a formal level), if one replaces $H_0$ by $H$ throughout. By the chain rule (86), one thus finds for the transmission amplitudes

$$\begin{aligned} \langle s'|T(\varepsilon)|s\rangle &= \langle s',+|S(\varepsilon)|s,+\rangle \\ &= \delta_{s',s}T(E) + \langle s'|T^{(1)}(\varepsilon)|s\rangle + \langle s'|T^{(2)}(\varepsilon)|s\rangle + \cdots \end{aligned} \tag{87}$$

where $T(E)$ is the transmission amplitude, at energy $E = \varepsilon + s\hbar\omega$, for the static potential $v_0(x)$ and

$$\langle s'|T^{(1)}(\varepsilon)|s\rangle = -2\pi i\, \langle \varepsilon, s', +|\Omega_+^{0\,\dagger} W_{s'-s}\Omega_-^0|\varepsilon, s, +\rangle, \tag{88}$$

$$\langle s'|T^{(2)}(\varepsilon)|s\rangle = -2\pi i \sum_\ell \langle \varepsilon, s', +|\Omega_+^{0\,\dagger} W_{s'-\ell}G(\varepsilon+\ell\hbar\omega)W_{\ell-s}\Omega_-^0|\varepsilon, s, +\rangle, \tag{89}$$

with $G(\varepsilon + \ell\hbar\omega)$ the Green operator for the total Hamiltonian $H$. Similar equations hold, of course, for the transmission amplitudes for a particle incident from the right and for the case of reflexion.[10]

We are now in a good position to calculate the first non-vanishing corrections to the transmission and reflexion amplitudes for the two mentioned models. For this, let us first introduce the relevant quantities pertaining to the unperturbed problem. We write $\psi^\pm(E,x) = \sqrt{2\pi\hbar}\sqrt{\frac{\hbar k}{m}}\langle x|\Omega_-|E,\pm\rangle$ for the two linearly independent scattering solutions of the stationary equation $H\psi^\pm = E\psi^\pm$, with boundary conditions (we assume that $v_0(x)$ is of compact support in the interval $[-r,r]$)

$$\psi^+(E,x) = \begin{cases} e^{ikx} + R(E)\,e^{-ikx} & x < -r \\ T(E)\,e^{ikx} & x > r, \end{cases} \tag{90}$$

for the incoming plane wave coming from the left, and

$$\psi^-(E,x) = \begin{cases} T(E)\,e^{-ikx} & x < -r \\ e^{-ikx} + \tilde{R}(E)\,e^{ikx} & x > r \end{cases} \tag{91}$$

[10]The following remark is in order. If the unperturbed potential supports a given number $N$ of bound-states of energy $E_j$, $j = 1,\ldots,N$, then the Green operator in the second order term (89) diverges for resonant quasi-energies $\varepsilon$ such that $\varepsilon + \ell\hbar\omega = E_j$, for a given $j$ (provided that $W_{s'-\ell}$ and $W_{\ell-s}$ are not zero). For these resonant energies a non trivial interaction arises between the incoming wave and a quasi-bound state of the time-dependent potential and the perturbation expansion for the scattering amplitudes becomes more complicated to describe [15]. In this work, we shall restrict ourselves to the case of non-resonant incoming energies.

for the one coming from the right, with

$$S(E) = \begin{pmatrix} T(E) & \tilde{R}(E) \\ R(E) & T(E) \end{pmatrix} \tag{92}$$

the associated $2 \times 2$ scattering matrix. We denote by $G_E(x, y) = \langle x|G(E)|y\rangle$ the Green function for the total Hamiltonian $H$, which can be entirely expressed in terms of the solutions $\psi^{\pm}(E, x)$ by the formula

$$G_E(x, y) = \frac{m}{i\hbar^2 kT(E)} \begin{cases} \psi^-(E, x)\psi^+(E, y) & y \geq x \\ \psi^+(E, x)\psi^-(E, y) & y \leq x. \end{cases} \tag{93}$$

## 5.1 The modulated barrier

We consider the potential (83), letting $v_0(x)$ be an arbitrary bounded function with support in the interval $[-r, r]$. We assume that $v_1(x) = \chi_{[-r,r]}(x)$, with $\chi_{[-r,r]}(x)$ the characteristic function of the interval $[-r, r]$ ($\chi_{[-r,r]}(x) = 1$ for $x \in [-r, r]$ and $\chi_{[-r,r]}(x) = 0$ otherwise) and, with no loss of generality, $\int_0^{2\pi/\omega} dt\, \lambda(t) = 0$.

### 5.1.1 Sidebands

Inserting the completeness relation $\int dx|x\rangle\langle x| = I$ into (88) and using the fact that $\langle E, +|\Omega_+^{0\dagger}|x\rangle = \langle x|\Omega_-^0|E, -\rangle$ (time-reversal invariance), one finds for the first order correction the expression ($n \neq 0$)

$$\begin{aligned} \langle s+n|T(\varepsilon)|s\rangle &= -2\pi i\lambda_n\langle E + n\hbar\omega, +|\Omega_+^{0\dagger}F_{[-r,r]}\Omega_-^0|E, +\rangle + O(\lambda^2) \\ &= -i\frac{\lambda_n}{\hbar^2}\frac{m}{\sqrt{kk_n}}\int_{-r}^{r} dx\, \psi^-(E + n\hbar\omega, x)\psi^+(E, x) + O(\lambda^2), \end{aligned} \tag{94}$$

where $E = \varepsilon + s\hbar\omega$, $\hbar k_n = \sqrt{2m(E + n\hbar\omega)}$, $\lambda_n$ denotes the Fourier coefficient of $\lambda(t)$, and $F_{[-r,r]} = \int_{-r}^{r} dx|x\rangle\langle x|$ is the projection operator onto the set of states localized in the spatial interval $[-r, r]$.

If $v_0(x)$ is a square barrier i.e., $v_0(x) = \mu_0\chi_{[-r,r]}(x)$ (with $\mu_0$ an arbitrary coupling), one can easily perform the integral (94) since in that case the solutions inside the barrier are known. However, even for an arbitrary $v_0(x)$, it is possible to integrate (94) and to express the result in terms of the elements of the scattering matrix (92). For this, one observes that the projection operator $F_{[-r,r]}$ can be written as the (strong) limit

$$F_{[-r,r]} = I - \lim_{b\to\infty}\left[F_{[-b,-r)} + F_{(r,b]}\right]. \tag{95}$$

Substituting (95) into (94), one finds

$$\begin{aligned} \langle s+n|T(\varepsilon)|s\rangle &= -2\pi i\lambda_n T(E)\delta(n\hbar\omega) \\ &+ i\frac{\lambda_n}{\hbar^2}\frac{m}{\sqrt{kk_n}}\lim_{b\to\infty}\left(\int_{-b}^{-r} dx + \int_{r}^{b} dx\right)\psi^-(E + n\hbar\omega, x)\psi^+(E, x) + O(\lambda^2). \end{aligned} \tag{96}$$

The first term in (96) is zero since $n \neq 0$. In the second term, the intervals of integration in (96) are outside the support of the potential so that one can simply replace $\psi^-(E+n\hbar\omega, x)$ and $\psi^+(E,x)$ by their asymptotic (90),(91) and the integration is straightforward. In the limit $b \to \infty$, it is not difficult to show that the $b$-dependent oscillating terms converge to the distribution $2\pi i\lambda_n T(E)\delta(n\hbar\omega)$, cancelling the first (in any case zero) term in (96). The final result is the formula ($n \neq 0$)

$$\begin{aligned}\langle s+n|T(\varepsilon)|s\rangle &= -\frac{\lambda_n}{\hbar^2}\frac{m}{\sqrt{kk_n}}\frac{1}{k_n-k}\Big\{T(E+n\hbar\omega)e^{i(k_n-k)r} - T(E)e^{-i(k_n-k)r}\\ &+ \left(\frac{k_n-k}{k_n+k}\right)e^{i(k_n+k)r}\Big[T(E+n\hbar\omega)R(E)\\ &+ T(E)\tilde{R}(E+n\hbar\omega)\Big]\Big\} + O(\lambda^2).\end{aligned} \tag{97}$$

For the case of reflexion from the left, a similar calculation gives

$$\begin{aligned}\langle s+n|R(\varepsilon)|s\rangle &= -\frac{\lambda_n}{\hbar^2}\frac{m}{\sqrt{kk_n}}\frac{1}{k_n-k}\Big\{R(E+n\hbar\omega)e^{i(k_n-k)r} - R(E)e^{-i(k_n-k)r}\\ &+ \left(\frac{k_n-k}{k_n+k}\right)e^{i(k_n+k)r}\Big[T(E+n\hbar\omega)T(E)\\ &+ R(E+n\hbar\omega)R(E) - e^{-2i(k_n+k)r}\Big]\Big\} + O(\lambda^2).\end{aligned} \tag{98}$$

### 5.1.2 Elastic channel

As shown above, the calculation of the first non-vanishing contributions to the amplitudes of the transmitted and reflected sidebands requires only first-order perturbation expansion. On the other hand, since by definition the time-average of the perturbation over one period is zero, to calculate the first non-vanishing correction for the elastic amplitudes $\langle s|T(\varepsilon)|s\rangle$ and $\langle s|R(\varepsilon)|s\rangle$ one has to resort to second order perturbation expansion ((88) is zero if $s = s'$, since $W_0 = 0$). By (87)-(89), we have for the case of transmission

$$\langle s|T(\varepsilon)|s\rangle = T(E) + \langle s|T^{(2)}(\varepsilon)|s\rangle + O(\lambda^3), \tag{99}$$

where $T(E)$ is the transmission amplitude for the static unperturbed potential at energy $E = \varepsilon + s\hbar\omega$, and

$$\begin{aligned}\langle s|T^{(2)}(\varepsilon)|s\rangle &= -2\pi i\sum_\ell |\lambda_{\ell-s}|^2\langle\varepsilon, s, +|\Omega_+^{0\,\dagger} F_{[-r,r]}G(\varepsilon+\ell\hbar\omega)F_{[-r,r]}\Omega_-^0|\varepsilon, s, +\rangle\\ &= -\frac{im}{\hbar^2 k}\sum_\ell |\lambda_{\ell-s}|^2\int_{-r}^{r}dx\int_{-r}^{r}dy\,\psi^-(E,x)G_{E+\ell\hbar\omega}(x,y)\psi^+(E,y),\end{aligned} \tag{100}$$

where $\psi^\pm(E,x)$ are the solutions (90),(91), $G_{E+\ell\hbar\omega}(x,y)$ is the Green function (93), and $\hbar k = \sqrt{2mE}$.

For simplicity, we restrict ourselves to the special case where $v_0(x)$ is a square barrier i.e., $v_0(x) = \mu_0\chi_{[-r,r]}(x)$. The solutions $\psi^\pm(E,x)$ inside the barrier are then of the form (see any good book of quantum mechanics)

$$\psi^\pm(E,x) = A(E)e^{\pm icx} + B(E)e^{\mp icx}, \tag{101}$$

where $\hbar c = \sqrt{2m(E-\mu_0)}$,

$$A(E) = -\frac{e^{-i(k+c)r}}{iD(E)}k(k+c), \quad B(E) = \frac{e^{-i(k-c)r}}{iD(E)}k(k-c), \tag{102}$$

and

$$D(E) = 2ick\cos(2cr) + (c^2+k^2)\sin(2cr). \tag{103}$$

The transmission and reflexion amplitudes are given by

$$T(E) = \frac{2ick}{D(E)}e^{-2ikr}, \quad R(E) = \frac{(k^2-c^2)\sin 2cr}{D(E)}e^{-2ikr}. \tag{104}$$

Inserting (101)-(104) into (100) and using the explicit formula (93) for the Green function, one is led to a long but straightforward calculation of which the result is

$$\langle s|T(\varepsilon)|s\rangle = T(E) + \sum_\ell |\lambda_\ell|^2 \frac{m^2}{\hbar^4 k k_\ell T(E+\ell\hbar\omega)} I_\ell(E) + O(\lambda^3), \tag{105}$$

where $\hbar k_\ell = \sqrt{2m(E+\ell\hbar\omega)}$, and the function $I_\ell(E)$ is given by the rather involved expression (for simplicity, we write $A$ for $A(E)$ and $A_\ell$ for $A(E+\ell\hbar\omega)$, and similarly for $B$. Also, we set $c_\ell \equiv \sqrt{2m(E+\ell\hbar\omega-\mu_0)}$)

$$\begin{aligned} I_\ell(E) &= \frac{1}{(c+c_\ell)^2}\Big\{4ABA_\ell B_\ell \cos 2(c+c_\ell)r \\ &+ (A^2+B^2)\left[A_\ell^2 e^{2i(c+c_\ell)r} + B_\ell^2 e^{-2i(c+c_\ell)r}\right]\Big\} \\ &+ \frac{1}{(c-c_\ell)^2}\Big\{4ABA_\ell B_\ell \cos 2(c-c_\ell)r \\ &+ (A^2+B^2)\left[A_\ell^2 e^{-2i(c-c_\ell)r} + B_\ell^2 e^{2i(c-c_\ell)r}\right]\Big\} \\ &+ \frac{4}{c^2-c_\ell^2}\Big\{\left[(A^2+B^2)A_\ell B_\ell + (A_\ell^2+B_\ell^2)AB\right]\cos 2cr \\ &- (A^2+B^2)A_\ell B_\ell \cos 2c_\ell r - AB\left[A_\ell^2 e^{2ic_\ell r} + B_\ell^2 e^{-2ic_\ell r}\right] \\ &+ i\frac{c_\ell \sin 2cr}{c}(A_\ell^2 - B_\ell^2)AB + i c_\ell r (A^2+B^2)(A_\ell^2-B_\ell^2)\Big\} \\ &- 2\frac{c^2+c_\ell^2}{(c^2-c_\ell^2)^2}\left[(A^2+B^2)(A_\ell^2+B_\ell^2) + 4ABA_\ell B_\ell\right]. \end{aligned} \tag{106}$$

Note that the denominators in (106) never vanish since the sum in (105) runs only over $\ell \neq 0$ as $\lambda_0 = 0$ by definition.

## 5.2 The oscillating position barrier

We consider the potential (84) with $v_0(x) = \mu_0 \chi_{[-r,r]}(x)$ a square barrier of a given arbitrary coupling $\mu_0$ and $\int_0^{2\pi/\omega} dt\, a(t) = 0$. Then, we have the Taylor expansion (in the sense of distributions)

$$v_0(x-a(t)) = \mu_0\,\chi_{[-r,r]}(x) - \mu_0\,a(t)\,[\delta(x+r)-\delta(x-r)] + \mu_0\,\frac{a^2(t)}{2}\,[\delta'(x+r)-\delta'(x-r)] + O(a^3), \tag{107}$$

where $\delta'(x)$ denotes the derivative of the Dirac distribution, defined by

$$\int dx\,\delta'(x\pm r)f(x) = -f'(\pm r), \tag{108}$$

for sufficiently smooth functions $f(x)$.

### 5.2.1 Sidebands

For the first-order correction to the sidebands amplitudes, we need to consider only the linear approximation for the displaced potential (107). For the case of transmission, we have $(n \neq 0)$

$$\begin{aligned}\langle s+n|T(\varepsilon)|s\rangle &= -2\pi i\mu_0\,a_n\langle E+n\hbar\omega,+|\Omega_+^{0\,\dagger}(\delta_r-\delta_{-r})\Omega_-^0|E,+\rangle + O(a^2)\\ &= -i\frac{\mu_0\,a_n}{\hbar^2}\frac{m}{\sqrt{kk_n}}\int dx\,\psi^-(E+n\hbar\omega,x)\psi^+(E,x)\,[\delta(x-r)-\delta(x+r)] + O(a^2)\\ &= -i\frac{\mu_0\,a_n}{\hbar^2}\frac{m}{\sqrt{kk_n}}\Big[\psi^-(E+n\hbar\omega,r)\psi^+(E,r)-\psi^-(E+n\hbar\omega,-r)\psi^+(E,-r)\Big]\\ &+ O(a^2).\end{aligned} \tag{109}$$

Using the asymptotic forms (90) and (91), we obtain

$$\begin{aligned}\langle s+n|T(\varepsilon)|s\rangle &= i\frac{\mu_0\,a_n}{\hbar^2}\frac{m}{\sqrt{kk_n}}\Big\{T(E+n\hbar\omega)e^{i(k_n-k)r} - T(E)e^{-i(k_n-k)r}\\ &+ e^{i(k_n+k)r}\,[T(E+n\hbar\omega)R(E)-T(E)R(E+n\hbar\omega)]\Big\}\\ &+ O(a^2),\end{aligned} \tag{110}$$

where the transmission and reflexion amplitudes in (110) are given by (104). For the case of reflexion, a similar calculation yields

$$\begin{aligned}\langle s+n|R(\varepsilon)|s\rangle &= i\frac{\mu_0\,a_n}{\hbar^2}\frac{m}{\sqrt{kk_n}}\Big\{R(E+n\hbar\omega)e^{i(k_n-k)r} + R(E)e^{-i(k_n-k)r}\\ &- e^{i(k_n+k)r}\Big[T(E+n\hbar\omega)T(E)-R(E+n\hbar\omega)R(E)-e^{-2i(k_n+k)r}\Big]\Big\}\\ &+ O(a^2).\end{aligned} \tag{111}$$

### 5.2.2 Elastic channel

As for the case of the modulated barrier, to calculate the first non vanishing correction to the elastic channel one has to resort to a second order perturbation expansion. Using (107) and the definition (108) for the derivative of the Dirac distribution, one finds after a long but straightforward calculation (we consider here only the case of transmission)

$$\langle s|T(\varepsilon)|s\rangle = T(E) - \sum_{\ell} |a_\ell|^2 \frac{\mu_0\, m^2}{k k_\ell} J_\ell(E) + O(a^3), \tag{112}$$

where the function $J_\ell(E)$ is given by (we set $R \equiv R(E)$, $R_\ell \equiv R(E + \ell\hbar\omega)$, and same for the transmission amplitude)

$$\begin{aligned} J_\ell(E) &= e^{2i(k+k_\ell)r}\left[2TRR_\ell - T_\ell(T^2 + R^2)\right] \\ &+ 2e^{2ikr}TR + 2e^{2ik_\ell r}(TR_\ell - RT_\ell) \\ &- e^{-2i(k-k_\ell)r}T_\ell + 2T. \end{aligned} \tag{113}$$

# 6 Graphical solutions

We present in this Section, for the modulated and oscillating position barriers, some graphs of transmission probabilities, sidebands and time delays, as a function of the frequency $\omega$ of the external field. We consider only the case of an incoming energy in the tunnelling regime i.e., $E < \mu_0$, and of cosinusoidal time-dependences $\lambda(t) = \mu_1 \cos\omega t$ and $a(t) = \beta\cos\omega t$. According to (87)-(89), to the leading order in the perturbation the transmission probability (36) can be written as the sum of three channels

$$\mathcal{P}_{\rm tr}(E) = \mathcal{P}^0_{\rm tr}(E) + \mathcal{P}^1_{\rm tr}(E) + \mathcal{P}^{-1}_{\rm tr}(E) + \cdots, \tag{114}$$

where $\mathcal{P}^m_{\rm tr}(E)$ are the sidebands intensities defined in (38).

More specifically, we shall consider the following model parameters: $E = 5$, $\mu_0 = 8$, $r = 2$, $\mu_1 = 1$, $\beta = 1/8$, $m = 1/2$ and $\hbar = 1$.

## 6.1 The modulated barrier

In Figure 1, we have plotted the transmission probability (114), calculated according to formulae (97) and (105), and the associated time-delay (59). The dashed vertical line gives the inverse of the Büttiker-Landauer "traversal" time $\tau_{\rm BL} = 2mr/\hbar|c|$ [2], and the full vertical lines the frequencies at which $E + \hbar\omega$ corresponds to an energy-resonance of the static unperturbed barrier. The figure shows that, as the modulation frequency increases from the adiabatic ($\omega = 0$) to the high frequency regime, the transmission probability increases drastically to reach resonances corresponding to maximums of the transmission time-delay. The resonances correspond rather well to the condition for the coincidence of the sideband energies $E + \hbar\omega$ with the resonance energies of the unperturbed static barrier. On the other hand, at the Büttiker-Landauer frequency $\omega_{\rm BL} = 1/\tau_{\rm BL}$ there is apparently no particular sign of crossover (we shall come back to this point in the next Section). Notice that time-delay (curve B) has been scaled by a factor of $10^{-2}$ in the graph.

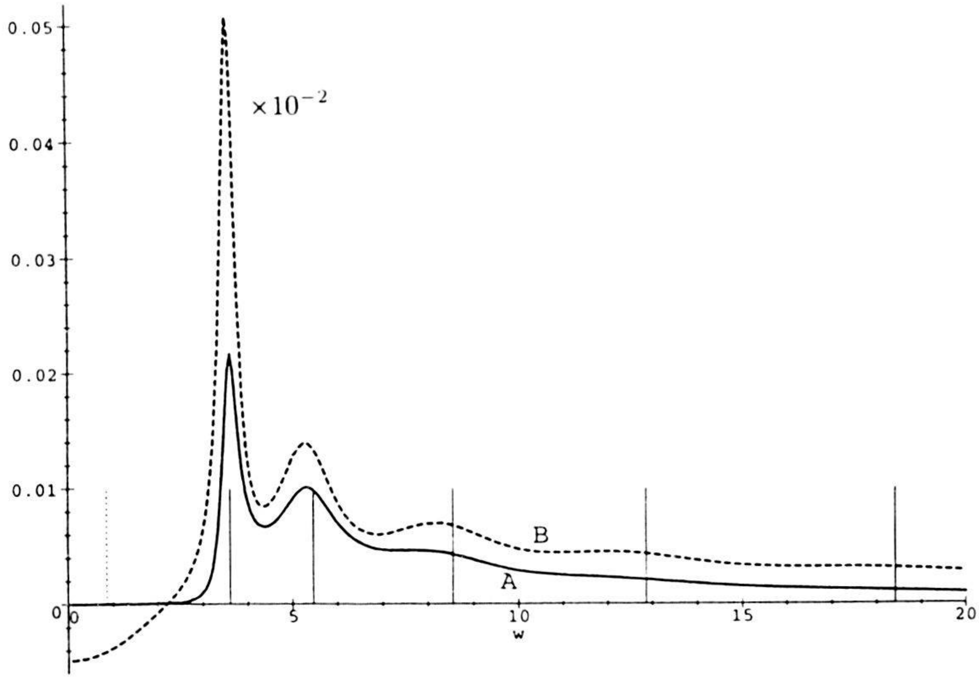


**Figure 1**: Transmission probability $\mathcal{P}_{tr}(E)$ (full line A) and transmission time delay $\tau^{tr}(E)$ (dashed line B), as a function of frequency $\omega$. Time delay is scaled by a factor $10^{-2}$. The vertical dotted line corresponds to $\omega_{BL} = 1/\tau_{BL}$. The other full vertical lines correspond to frequencies such that $E + \hbar\omega$ is equal to a resonance energy of the static transmission probability.

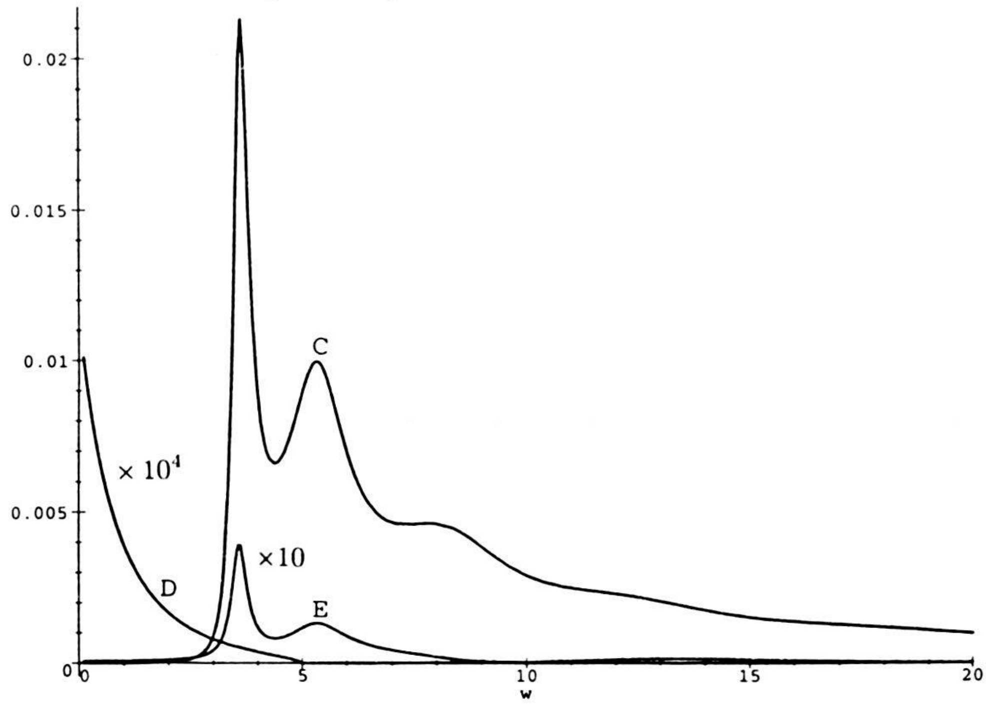


**Figure 2**: Absorption, emission and elastic contributions to the total transmission probability. Curve (C) corresponds to $\mathcal{P}^{1}_{tr}(E)$ (transmission with absorption of one quanta $\hbar\omega$), curve (D) corresponds to $\mathcal{P}^{-1}_{tr}(E)$ (transmission with emission of one quanta $\hbar\omega$), and curve (E) corresponds to the transmission probability $\mathcal{P}^{0}_{tr}(E)$ with no transfer of energy of the particle to the field (elastic channel). The emission and elastic contributions are scaled by factors of $10^4$ and 10, respectively.

In Figure 2, the three different contributions (114) to the full transmission probability are plotted separately. The Figure shows that the process is dominated by the absorption channel, corresponding to a transfer of one quanta of energy $\hbar\omega$ from the external field to the particle. For questions of clarity, the emission and elastic probabilities have been scaled in the graph by factors of $10^4$ and 10, respectively. Note that the emission probability is identically zero for $\hbar\omega > E$, since there is no contribution to the scattering for outgoing negative energies (corresponding to the quasi-bound exponentially decaying states described in Section 3).

## 6.2 The oscillating position barrier

Figures 3 and 4 are the same as Figures 1 and 2, but for the case of the oscillating position barrier. It is worth noting that curves in Figures 3 and 4 are very similar (at least qualitatively) to curves in Figures 1 and 2, so that the same remarks hold. This also shows that the two models, even though very different from the classical point of view [5], show similarities at the quantum level.

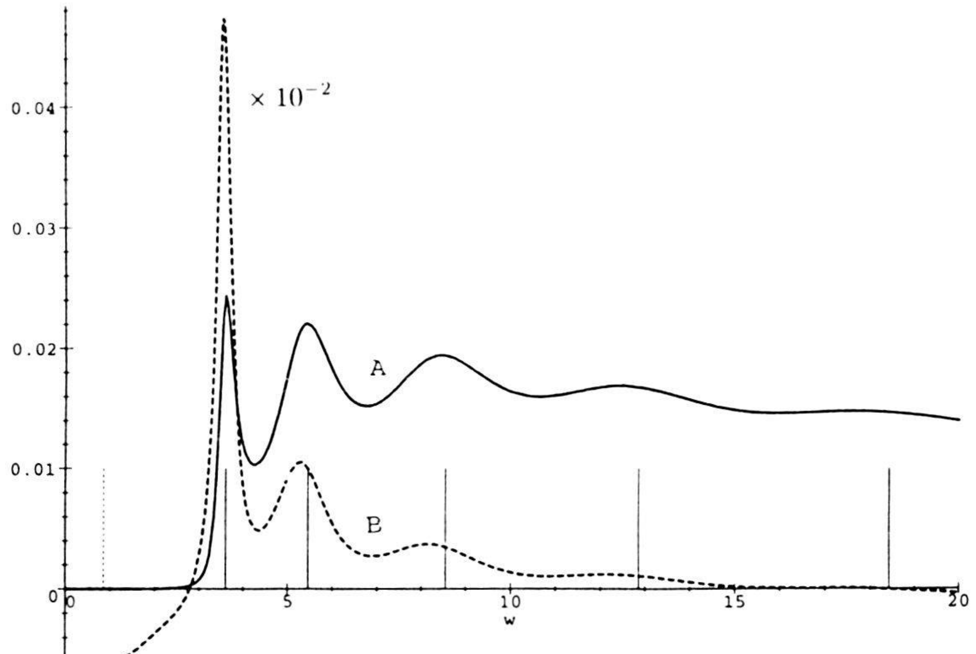


**Figure 3**: Transmission probability $\mathcal{P}_{tr}(E)$ (full line A) and transmission time delay $\tau^{tr}(E)$ (dashed line B), as a function of frequency $\omega$. Time delay is scaled by a factor of $10^{-2}$. The vertical dotted line corresponds to $\omega_{BL} = 1/\tau_{BL}$. The other full vertical lines correspond to frequencies such that $E + \hbar\omega$ is equal to an energy-resonance of the static transmission probability.

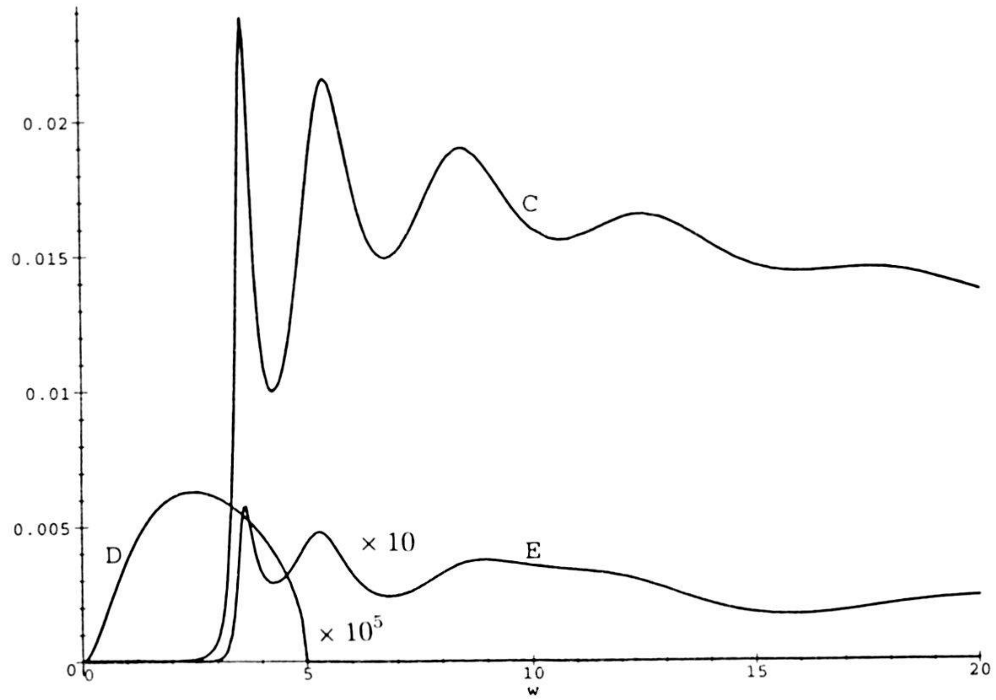


**Figure 4**: Absorption, emission and elastic contributions to the total transmission probability. Curve (C) corresponds to $\mathcal{P}^{1}_{tr}(E)$ (transmission with absorption of one quanta $\hbar\omega$), curve (D) corresponds to $\mathcal{P}^{-1}_{tr}(E)$ (transmission with emission of one quanta $\hbar\omega$), and curve (E) corresponds to the transmission probability $\mathcal{P}^{0}_{tr}(E)$ with no transfer of energy of the particle to the field (elastic channel). The emission and elastic contributions are scaled by factors of $10^5$ and 10, respectively.

# 7 Concluding remarks

In the first part of the paper, we have described the basic formalism and concepts of the one-dimensional scattering problem by time-periodic perturbations. Although the description of the formalism is far from being complete (scattering from time-dependent potentials is a very rich subject, both from the mathematical and from the physical point of view, which is still under intense investigation), we have given a simple and self-contained presentation of the problem. Since such a presentation is hardly found in the literature, we hope that this work will also serve as a useful didactical purpose.

A generalization of the Eisenbud-Wigner time-delay formula was derived in Section 3, by a stationary phase argument. The same formula can also be obtained using the more transparent concept of sojourn time [25]. For this, and similarly to the case of $N$-body scattering [26], one needs to define a free reference time which is symmetric with respect to the incoming and outgoing asymptotic states. We plan to come back to this problem in future work.

A generalization of the Born series was presented in Section 4. Although the series was derived in the one-dimensional context, it is straightforward to check that the final result (79),(82) will also hold for an arbitrary number of dimensions.

The Born series was applied in Section 5 to two specific models (the barrier of modulated height and the barrier with oscillating position), and in Section 6 a graphical discussion of the obtained perturbation formulae was presented for the case of transmission in the tunnelling regime. The case of an incoming energy which is above the barrier and the case of reflexion have not been discussed, in order to not increase dramatically the length of the paper. Also, we have not discussed the complementary case of perturbation of a transparent barrier i.e., a barrier having a (static) transmission probability which is close to one. In that case, one finds that, contrary to the case of the opaque barrier, the process is essentially dominated by its elastic channel contribution.

To conclude, let us spend a few words on the recent controversy over tunnelling times, where the intriguing results of Büttiker and Landauer have played a central role [2]. The original proposition of these authors was as follows: if the period $T = 2\pi/\omega$ of the external modulation is long compared to the time during which the particle interacts with the barrier, then the particle sees an effective static barrier during its traversal. On the other hand, at modulation frequencies high compared to the reciprocal traversal time, the particle sees many cycles of the oscillation and can emit or absorb modulation quanta (in particular, by absorbing quanta it will tunnel more easily through the barrier). As the modulation frequency is varied, there is a crossover frequency $\omega_{\mathrm{BL}}$ between these two different regimes, and the inverse of this frequency gives the traversal time of the particle (or at least a magnitude of it). In [2], the time $\tau_{\mathrm{BL}} = 2mr/\hbar|c|$ was found for the case of the opaque rectangular barrier.

The above reasoning assumes implicitly the existence in quantum mechanics of a concept of traversal time, and the existence of a crossover frequency. Concerning the first assumption, it has now been noted by many authors (see [27]-[28], and the references cited therein) that contrary to the case of classical mechanics, a traversal time cannot be uniquely defined in quantum mechanics, because of the uncertainty principle (at least, not within the standard theory of measurement). Conversely, it can also be argued that the time proposed by Büttiker and Landauer should be referred to as an interaction time, in the sense that it estimates the time of interaction of the particle with the additional dynamical degree of freedom of the barrier. In this way, one avoids the controversy over interpretation, and simply asks if there is a characteristic frequency separating the pure adiabatic regime with the regime where inelastic effects start to become relevant in the scattering process.

According to Figures 1 and 3, the Büttiker-Landauer frequency $\omega_{\mathrm{BL}}$ is apparently unrelated to any particular crossover regime, the only critical frequency emerging from the Figures being the one for which the energy $E + \hbar\omega$ of the first sideband becomes equal to the first resonance energy of the static barrier (this was the conclusion of the numerical study [21]). However, in Figure 5, we have plotted the transmission probabilities of Figures 1 and 3, magnifying the adiabatic region going from $\omega = 0$ to $\omega = 5$. These curves show that the Büttiker-landauer frequency does indeed correctly estimate the critical region where the transmission probability starts to deviate from its adiabatic (static) limit, in the sense that it correctly separates the pure adiabatic regime with the regime where inelastic effects start to become relevant in the scattering process.

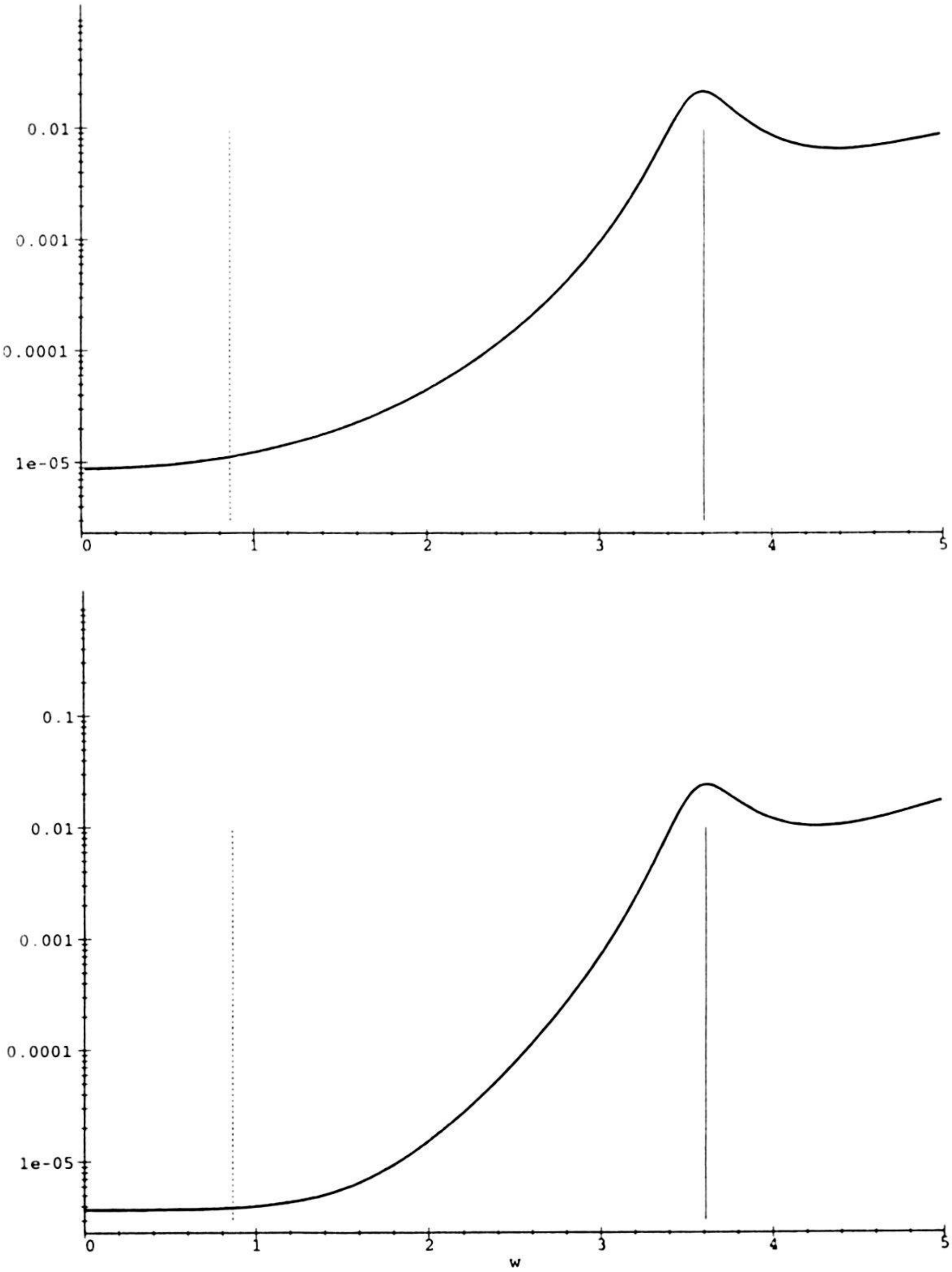


**Figure 5**: Total transmission probability for the modulated barrier (up curve) and for the oscillating position barrier (down curve), as a function of frequency $\omega$. The vertical dotted line corresponds to $\omega_{\rm BL} = 1/\tau_{\rm BL}$. The full vertical line corresponds to the frequency such that $E + \hbar\omega$ is equal to the first resonance energy of the static transmission probability.

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