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Pull-backs and Product Tests

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be test spaces. We study the test space $B(\mathcal{A}, \mathcal{B})$ consisting of graphs of bijections $f : E \to F$ between tests $E \in \mathcal{A}$ and $F \in \mathcal{B}$. Elements of $B(\mathcal{A}, \mathcal{B})$ may be interpreted as *products*, in something like the sense of Piron, of tests in $\mathcal{A}$ and $\mathcal{B}$.

Introduction

In a long series of papers (cf [2], [3], [4] and references therein), D. J. Foulis and the late C.H. Randall developed a straightforward but versatile generalized probability theory based on what are now usually called *test spaces.* In brief: A test space $\mathcal{A}$ is simply a non-empty collection of discrete sets $E, F, ...$, each thought of as the outcome-set for some measurement or *test*. When $\mathcal{A}$ contains only one test, one recovers (discrete) classical probability theory; when it consists of the set of maximal orthonormal bases of a Hilbert space, one recovers quantum probability theory.

This note concerns the following construction: If $\mathcal{A}$ and $\mathcal{B}$ are test spaces, let $B(\mathcal{A}, \mathcal{B})$ denote the set of bijections $f : E \to F$ between tests $E \in \mathcal{A}$ and $F \in \mathcal{B}$. Identifying each such a bijection with its graph, $B(\mathcal{A}, \mathcal{B})$ may be regarded as a test space in its own right.

We propose to interpret $B(\mathcal{A}, \mathcal{B})$ as the test space consisting of *products*, in something close to the sense of Piron [8] and Aerts [1], of tests $E \in \mathcal{A}$ and $F \in \mathcal{B}$. The construction is also of interest on purely mathematical grounds. On the one hand, it preserves various standard regularity conditions on $\mathcal{A}$ and $\mathcal{B}$; on the other hand, as soon as $\mathcal{A}$ and $\mathcal{B}$ contain tests with more than two outcomes, the structure of $B(\mathcal{A}, \mathcal{B})$ becomes quite rich, *even if* $\mathcal{A}$ *and* $\mathcal{B}$ *are classical.* Moreover, for certain categories of "uniform" test spaces, $B(\mathcal{A}, \mathcal{B})$ is effective as the direct product of $\mathcal{A}$ and $\mathcal{B}$.

In section 1, we discuss our construction in general terms. In section 2, we discuss the stability of various regularity conditions on $\mathcal{A}$ and $\mathcal{B}$ under passage to $B(\mathcal{A},\mathcal{B})$. In particular, we show that if $\mathcal{A}$ and $\mathcal{B}$ are algebraic, then $B(\mathcal{A},\mathcal{B})$ is algebraic as well. In section 3, we characterize the logic of $B(\mathcal{A},\mathcal{B})$ in the case that $\mathcal{A}$ and $\mathcal{B}$ are algebraic.

1 Questions, Products and Pull-Backs

As explained above, a **test space**[1] is a non-empty set $\mathcal{A}$ of non-empty sets $E, F, \ldots$. Elements of $\mathcal{A}$ are called **tests** and elements of $X := \bigcup \mathcal{A}$ are called **outcomes**. The intended interpretation is that each test $E \in \mathcal{A}$ is an exhaustive set of mutually exclusive outcomes, as, for instance, the set of outcomes of some experiment. Borrowing terminology from classical probability theory, we refer to any subset of any test $E \in \mathcal{A}$ as an **event** of $\mathcal{A}$. We write $\mathcal{E}(\mathcal{A})$ for the set of all events of $\mathcal{A}$.

Test spaces provide the foundation for a very natural – and conceptually uncomplicated – generalization of elementary probability theory having both classical measure-theoretic and quantum-mechanical probability as special cases. It is worth a moment to give a sketch of this. One defines a **state** on a test space $\mathcal{A}$ to be a map $\omega : X \to [0,1]$ such that $\omega(x) \geq 0$ for each $x \in X$ and $\sum_{x \in E} \omega(x) = 1$ for each test $E \in \mathcal{A}$. In other words, a state is a real-valued function on the set of outcomes that restricts to a probability weight on each test.

Note that if $\mathcal{A}$ consists of but a single test – i.e., if $\mathcal{A} = \{E\}$ — then a state is simply a discrete probability distribution and we recover discrete classical probability theory. In this case, we call $\mathcal{A}$ a *classical* test space. One can also consider the test space consisting of all countable partitions of a measurable space by measurable sets; this may be called a *Kolmogorov test space*. A *quantum test space* (or *frame manual*) is the set $\mathcal{A}$ of all orthonormal bases of a Hilbert space $\mathbf{H}$. The outcomes of $\mathcal{A}$ are the unit vectors of $\mathbf{H}$. Gleason's theorem [5] allows us to identify the states ω on $\mathcal{A}$ with density operators W on $\mathbf{H}$ via the prescription $\omega(x) = \langle Wx, x \rangle$ (where x is a unit vector of $\mathbf{H}$).

We may wish to attach numerical or other labels to the outcomes of a test. This motivates the following terminology:

1.1 Definition: Given a set V, we define a V**-valued question** on a test space $\mathcal{A}$ to be a bijection[2] $\alpha : E \to V$, where E is a test belonging to $\mathcal{A}$. The question is *posed* by executing the test E; its *answer* is the value $\alpha(x) \in V$ corresponding to the secured outcomes $x \in E$.

Note that if $V = \{yes, no\}$, this corresponds to the notion of a question as defined in the work of Piron [8].

If $\alpha : E \to V$ and $\beta : F \to V$ are two V-valued questions, it is very natural to form their

[1] called also a *manual* or *generalized sample space* in the older literature

[2] the condition that α be bijective is benign: If not, replace V by the range of α and E, by the partition $\{\alpha^{-1}(x) | x \in V\}$.

pull-back — that is, the canonical bijection $\alpha \cdot \beta : E \times_V F \to V$ where

$$E \times_V F \;=\; \{\ (x, y) \in E \times F \mid \alpha(x) = \beta(y)\ \}.$$

$$\begin{array}{ccc} E \times_V F & \longrightarrow & F \\ \big\downarrow & & \big\downarrow \beta \\ E & \underset{\alpha}{\longrightarrow} & V \end{array}$$

More generally, given an arbitrary collection $\{\alpha_i\}_{i \in I}$ of V-valued questions $\alpha_i : E_i \to V$, one can construct $E = \{\ x \in \Pi_{i \in I} E_i \mid \alpha_i(x_i) = \alpha_j(x_j)\ \forall i, j \in I\ \}$ and set $\Pi_i \alpha_i(x) = \Pi_j \alpha_j$ for any $j \in I$. (Indeed, by iterating this construction and taking a suitable direct limit, one can construct a test space that is in some sense closed under the formation of products of V-valued questions. We shall not pursue this here.)

We may interpret $E \times_V F$ as a test, as follows: One of the tests E or F is selected. If the outcome of the selected test is, say, $x \in E$, then the outcome of $E \times_V F$ is the unique pair $(x, y) \in E \times_V F$ having x as its first component. Similarly, if the secured outcome is $y \in F$, the outcome of $E \times_V F$ is the unique pair (x, y) with y as its second component. (Note that this in effect *erases* any record of *which* of the tests E and F was in fact selected.) To pose the question $\alpha \cdot \beta$, one executes $E \times_V F$. Upon securing, say, (x, y), one records the value $\alpha(x) = \beta(y)$ as answer.

As the reader familiar with [8] will have recognized, this construction is analogous to the notion of a product of yes-no questions as defined by Piron:

> If $\{\alpha_i\}$ is a family of questions, we denote by $\Pi_i \alpha_i$ the question defined in the following manner: One measures an arbitrary one of the α_i and attributes to $\Pi_i \alpha_i$ the answer thus obtained. ([8], p. 20).

This notion makes equal sense for V-valued questions generally, and we believe our construction adequately captures it in a precise way.

The balance of this paper is devoted to a discussion of the test space consisting of tests $E \times_V F$ arising from the formation of products of V-valued questions. This turns out to have a surprisingly rich structure. Before carrying on, it will be helpful to reformulate the definition of $E \times_V F$ in a manner not depending explicitly upon the questions α and β. To this end, notice that $E \times_V F$ is simply the graph of the bijection $\beta^{-1} \circ \alpha : E \to F$. Conversely, given any pair of tests $E, F \in \mathcal{A}$ and any bijection $f : E \to F$, we may understand f as a test corresponding to a product of V-valued questions defined on E and F, respectively. (To execute the test represented by f, one chooses E or F, executes it, and records the pair $(x, f(x))$ or $(f^{-1}(y), y)$ according as $x \in E$ or $y \in F$ is secured.)

1.2 Definition: For two sets E and F, we denote by $B(E, F)$ the set of (graphs of) bijections $f : E \to F$, abbreviating $B(E, E)$ to $B(E)$. For any two test-spaces $\mathcal{A}$, $\mathcal{B}$, we denote by $B(\mathcal{A}, \mathcal{B})$ the collection of sets $B(E, F)$ with $E \in \mathcal{A}$ and $F \in \mathcal{B}$. We abbreviate $B(\mathcal{A}, \mathcal{A})$ to $B(\mathcal{A})$.

Of course, $B(\mathcal{A}, \mathcal{B})$ may be empty. On the other hand, as $B(E, E) \neq \emptyset$, $\mathcal{B}(\mathcal{A})$ is always rather large. Indeed, if $\mathcal{A}$ is a totally finite test space having k operations each with n outcomes, $B(\mathcal{A})$ has $k^2 n!$ operations (each with n outcomes). There is a natural embedding of $\mathcal{A}$ in $B(\mathcal{A})$, namely, the diagonal map $X \to X \times X$ given by $x \mapsto (x, x)$. This maps each test $E \in \mathcal{A}$ to the corresponding identity function Id_E.

In general, the set of outcomes of $B(\mathcal{A}, \mathcal{B})$ will be smaller than $X \times Y$ (since, e.g., there may be outcomes in the former that belong only to tests with n outcomes, and outcomes of the latter belonging only to k-outcome tests with $k \neq n$). In any case, if (x, y) and (u, v) belong to $\bigcup B(\mathcal{A}, \mathcal{B})$, we have $(x, y) \perp (u, v) \Rightarrow x \perp u \;\&\; y \perp v$.

We now consider some examples.

1.3 Example: Suppose $\mathcal{A}$ is a collection of pair-wise disjoint two-element sets. Then

$$B(\mathcal{A}) = \{ \{(x, u), (y, v)\} \mid x \perp y, u \perp v \},$$

likewise a collection of pairwise-disjoint two- element sets. Notice that $B(\mathcal{A})$ is naturally isomorphic to the set of pairs $\{(\{x, u\}, \{y, v\}) | x \perp y, u \perp v\}$, which is the model for the manual of product questions given by Foulis, Piron and Randall in [4].

Once we admit test spaces having operations with more than two outcomes, the structure of $B(\mathcal{A})$ becomes quite involved. This is nicely illustrated even by the simplest example:

1.4 Example: Consider the hypergraph $\mathcal{A} = \{E\}$ consisting of a single three-outcome experiment $E = \{x, y, z\}$. Then $B(\mathcal{A}) = B(E)$ is isomorphic to the three-by-three "window" manual:

$$\begin{array}{ccccc}
(x,x) & --- & (y,z) & --- & (z,y) \\
| & & | & & | \\
(y,y) & --- & (z,x) & --- & (x,z) \\
| & & | & & | \\
(z,z) & --- & (x,y) & --- & (y,x)
\end{array}$$

As $B(E)$ contains four-loops but no three-loops, its logic is an orthomodular poset, but not an orthomodular lattice ([7]). The state-space of $B(E)$ is in effect the convex set of doubly stochastic 3×3 matrices.

1.5 Example: Consider a Hilbert space $\mathbf{H}$ (of any dimension, over any field) and let $\mathcal{A}$ be the associated quantum test space, i.e., the set of all (un-ordered) orthonormal bases of $\mathbf{H}$. Every bijection $f : E \to F$ between two bases $E, F \in \mathcal{A}$ extends uniquely to a unitary operator on $\mathbf{H}$. If U is such an operator, its graph is a closed subspace of $\mathbf{H} \times \mathbf{H}$, and hence a Hilbert space in its own right. An orthonormal basis for U is simply the graph of $U|_E$ for some $E \in \mathcal{A}$. Hence, $B(\mathcal{A})$ is just the union over all unitaries U, of the frame manuals of the corresponding subspaces $U \leq \mathbf{H} \times \mathbf{H}$. (It is interesting to note that the set of graphs of unitaries on $\mathbf{H}$ constitutes a *partial Hilbert space* in the sense of Gudder [6].)

We now consider a restricted class of test spaces for which the construction $\mathcal{A}, \mathcal{B} \mapsto B(\mathcal{A}, \mathcal{B})$ behaves in a particularly satisfactory manner.

1.6 Definition: Let κ be any cardinal. A test space $\mathcal{A}$ is κ-uniform iff every test $E \in \mathcal{A}$ has cardinality κ.

If $\mathcal{A}$ and $\mathcal{B}$ are both κ-uniform, then $Z = X \times Y$ and, in this case, $(x, y) \perp (u, v)$ iff $x \perp a$ and $y \perp b$. The class of κ-uniform test spaces is large enough to include both classical test spaces $\mathcal{A} = \{E\}$ with $\#(E) = \kappa$ and also the frame manual of any Hilbert space of dimension κ. Notice also that if $\mathcal{A}$ and $\mathcal{B}$ are both κ-uniform, then so also is $B(\mathcal{A}, \mathcal{B})$. In fact, as we shall now see, $B(\mathcal{A}, \mathcal{B})$ serves as the direct product of uniform test spaces, provided we define our morphisms correctly.

1.7 Definition: By a **uniform map** between two test spaces $\mathcal{A}$ and $\mathcal{B}$ with outcome-sets X and Y, respectively, we mean a function $\phi : X \to Y$ such that $\phi(\mathcal{A}) \subseteq \mathcal{B}$ and $x_1 \perp x_2 \Rightarrow \phi(x_1) \perp \phi(x_2)$ for all $x_i \in X$. (In the language of [4]: a uniform map is a *positive, outcome-preserving interpretation.*)

Note that if ϕ is a uniform map, then ϕ is locally bijective, in that for every $E \in \mathcal{A}$, $\phi_{|E} : E \to \phi(E) \in \mathcal{B}$ is a bijection.

1.8 Theorem: *Let $\mathcal{A}$ and $\mathcal{B}$ be κ-uniform. Then $B(\mathcal{A}, \mathcal{B})$ is the direct product of $\mathcal{A}$ and $\mathcal{B}$ in the category of uniform test spaces and uniform maps.*

Proof: Let $\mathcal{A}$ and $\mathcal{B}$ be κ-uniform test spaces with $\bigcup \mathcal{A} = X$ and $\bigcup \mathcal{B} = Y$. Note that $B(\mathcal{A}, \mathcal{B})$ is again κ-uniform, and that $\bigcup B(\mathcal{A}, \mathcal{B}) = X \times Y$. Let π_1 and π_2 be the projections of $X \times Y$ onto X and Y, respectively. If $(x, y) \perp (u, v) \in \bigcup B(\mathcal{A}, \mathcal{B})$, then $x \perp y$ and $u \perp v$, so $\pi_i(x, y) \perp \pi_i(u, v)$ for $i = 1, 2$. If $f \in B(\mathcal{A}, \mathcal{B})$, then $\pi_1(f) = \mathrm{dom}\,(f) \in \mathcal{A}$; similarly, $\pi_2(f) = \mathrm{ran}\,(f) \in \mathcal{B}$. Thus, both projections are uniform maps. It now suffices to show that if $\mathcal{C}$ is a κ-uniform test space with $\bigcup \mathcal{C} = Z$. and $\phi : Z \to X$ and $\psi : Z \to Y$ are uniform maps, then $\phi \times \psi : Z \to (X \times Y)$ is an uniform map. If $z \perp w$, then $\phi(z) \perp \phi(w)$ and $\psi(z) \perp \psi(w)$; hence, $(\phi \times \psi)(z) \perp (\phi \times \psi)(w)$. Now suppose $E \in \mathcal{C}$. We must show that $(\phi \times \psi)(E)$ belongs to $B(\mathcal{A}, \mathcal{B})$. Because $\phi|_E$ is a bijection, we have

$$(\phi \times \psi)(E) \;=\; \{\, (\phi(z), \psi(z)) \mid z \in E \,\} \;=\; \{\, (x, \psi(\phi^{-1}(x)) \mid x \in \phi(E) \,\}.$$

That is, $(\phi \times \psi)(E) = \psi \circ (\phi|_E)^{-1} : \phi(E) \to \psi(F)$. Since $\psi|_F$ is bijective, this last belongs to $B(\mathcal{A}, \mathcal{B})$.

2 The Structure of $B(\mathcal{A}, \mathcal{B})$

In this section, we establish (Theorems 2.2, 2.3 and 2.5) that passage from $\mathcal{A}$ and $\mathcal{B}$ to $B(\mathcal{A}, \mathcal{B})$ preserves each of three standard conditions often imposed on test spaces: That of being algebraic, that of being coherent (though here we need an additional uniformity assumption), and that of being regular.

Throughout this section, let $\mathcal{A}$ and $\mathcal{B}$ be test spaces with outcome-sets X and Y, respectively. As noted above, the outcome-set of $B(\mathcal{A}, \mathcal{B})$ is in general a *proper* (possibly empty) subset of $Z \subseteq X \times Y$. An event for $B(\mathcal{A}, \mathcal{B})$ is any subset of the graph of a bijection $f : E \to F$ with $E \in \mathcal{A}$ and $F \in \mathcal{B}$. Evidently, any such subset is the graph of a bijection between two events $A \subseteq E$ and $B \subseteq F$. Thus,

$$\mathcal{E}(B(\mathcal{A}, \mathcal{B})) \subseteq B(\mathcal{E}(\mathcal{A}), \mathcal{E}(\mathcal{B})).$$

Again, the inclusion is generally proper – indeed, it is easy to see we have identity iff $\mathcal{A}$ and $\mathcal{B}$ are n-uniform for some *finite* n.

Events A and B of a test space $\mathcal{A}$ are said to be *complementary* – the short-hand is $A \,\mathrm{c}\, B$ – iff $A \cap B = \emptyset$ and $A \cup B \in \mathcal{A}$. If A and B are both complementary to a common third event, one says that A and B are *perspective*, writing $A \sim B$. A test space is a *algebraic* (in the older literature, a *manual*) iff, given any events A, B and C, $A \sim B$ and $B \,\mathrm{c}\, C$ imply $A \,\mathrm{c}\, C$.

2.1 Lemma: *Let $f : A \to A'$ and $g : B \to B'$ be bijections belonging to $\mathcal{E}(B(\mathcal{A}, \mathcal{B}))$. Then*
(1) $f \,\mathrm{c}\, g$ iff $A \,\mathrm{c}\, B$ and $A' \,\mathrm{c}\, B'$.
(2) $f \sim g$ iff $A \sim B$ and $A' \sim B'$.

Proof: Note that (2) is an immediate consequence of (1). To establish (1), suppose $A \,\mathrm{c}\, B$ and $A' \,\mathrm{c}\, B'$. Then $f \cap g = \emptyset$ and $f \cup g \in B(A \cup B, A' \cup B') \subseteq B(\mathcal{A}, \mathcal{B})$; thus, $f \,\mathrm{c}\, g$. Conversely, if $f \,\mathrm{c}\, g$, then $f \cap g = \emptyset$ and $f \cup g \in B(E, F)$ for some $E \in \mathcal{A}, F \in \mathcal{B}$. But then $A \cup B = E \in \mathcal{A}$ and, as $f \cup g$ is again a bijection, we must have $A \cap B = \emptyset$ – whence, $A \,\mathrm{c}\, B$. Also, $A' \cup B' = f(A) \cup g(B) = F \in \mathcal{B}$, and, again because $f \cup g$ is a bijection, $A' \cap B' = \emptyset$, so $A' \,\mathrm{c}\, B'$.

2.2 Theorem: *If $\mathcal{A}$ and $\mathcal{B}$ are algebraic, then $B(\mathcal{A}, \mathcal{B})$ is likewise algebraic. If the test space $B(\mathcal{A})$ is algebraic, then $\mathcal{A}$ is algebraic.*

Proof: Suppose that $f : A \to A'$, $g : B \to B'$ in $\mathcal{E}(B(\mathcal{A}, \mathcal{B}))$ with $f \sim g$ and $g \,\mathrm{c}\, h : C \to C'$. By Lemma 1, $A \sim B \,\mathrm{c}\, C$ and $A' \sim B' \,\mathrm{c}\, C'$. If $\mathcal{A}$ and $\mathcal{B}$ are algebraic, it follows that $A \,\mathrm{c}\, C$ and $A' \,\mathrm{c}\, C'$. But then $f \,\mathrm{c}\, h$ by Lemma 2.1. Thus, $B(\mathcal{A}, \mathcal{B})$ is algebraic. if $B(\mathcal{A})$ is algebraic and $A \,\mathrm{c}\, C \,\mathrm{c}\, B \,\mathrm{c}\, D$ in $\mathcal{E}(\mathcal{A})$, then $\mathrm{Id}_A \sim \mathrm{Id}_B \,\mathrm{c}\, \mathrm{Id}_D$, hence, $\mathrm{Id}_A \,\mathrm{c}\, \mathrm{Id}_D$, whence, $\mathrm{Id}_A \cup \mathrm{Id}_D = \mathrm{Id}_{A \cup D}$ belongs to $B(\mathcal{A})$ – whence, $A \,\mathrm{c}\, D$, and it follows that $\mathcal{A}$ is algebraic.

A test space $\mathcal{A}$ is **coherent** [3, 4] iff for all events A and B of $\mathcal{A}$, $A \subseteq B^{\perp} \Rightarrow A \perp B$.

2.3 Theorem: *Let $\mathcal{A}$ and $\mathcal{B}$ be coherent and κ-uniform. Then $B(\mathcal{A}, \mathcal{B})$ is also coherent.*

Proof: Suppose $f, g \in \mathcal{E}(B(\mathcal{A}, \mathcal{B}))$ with $f : A \to A'$ and $g : B \to B'$. Suppose $f \subseteq g^{\perp}$. Then for every $x \in A$, $(x, f(x)) \perp (y, g(y))$ for every $y \in B$; hence, $x \in B^{\perp}$ and (since g is surjective), $f(x) \in B'^{\perp}$. Thus, $A \subseteq B^{\perp}$ and (since f is surjective) $A' \subseteq B'^{\perp}$. Since $\mathcal{A}$ and $\mathcal{B}$ are coherent, $A \perp B$ and $A' \perp B'$. Thus, $f \cap g = \emptyset$ and $f \cup g \in B(\mathcal{E}(\mathcal{A}))$. Since $\mathcal{A}$ and $\mathcal{B}$ are n-uniform, $f \perp g$. Thus, $B(\mathcal{A}, \mathcal{B})$ is coherent.

A *support* of a test space $\mathcal{A}$ is a set $S \subseteq X = \bigcup \mathcal{A}$ such that for all $E, F \in \mathcal{A}$,

$$E \cap S \subseteq F \Rightarrow F \cap S \subseteq E.$$

The usual heuristic is that S is the set of outcomes that are *possible* in some state of affairs. By way of example, if ω is a (probabilistic) state on $\mathcal{A}$, then $S_\omega = \{x \in X | \omega(x) > 0\}$ is a support of $\mathcal{A}$. Notice that X is a support, since test spaces are irredundant. It is straight-forward that the union of any collection of supports is a support; hence, the set of all supports of $\mathcal{A}$ is a complete lattice under set inclusion. More details and motivation will be found in [4].

Let $\bigcup B(\mathcal{A}, \mathcal{B}) = Z \subseteq X \times Y$. Suppose S and T are supports of $\mathcal{A}$. Then we define

$$S \odot T \;:=\; [X \times T \;\cup\; S \times Y] \cap Z.$$

2.4 Lemma: *If S and T are supports of $\mathcal{A}$ and $\mathcal{B}$, respectively, then $S \odot T$ is a support of $B(\mathcal{A}, \mathcal{B})$.*

Proof: Suppose $f : E \to E'$ and $g : F \to F'$ are operations in $B(\mathcal{A}, \mathcal{B})$, and that

$$f \cap (S \odot T) = \{(x, f(x)) | x \in E \cap S \text{ or } f(x) \in E' \cap T\} \subseteq g.$$

Then $E \cap S \subseteq F = \mathrm{dom}\,(g)$ and $E' \cap T \subseteq F' = \mathrm{ran}\,(g)$, whence, as S and T are supports, $E \cap S = F \cap S$ and $E' \cap T = F' \cap T$. Moreover, $f|_{E \cap S} = g|_{F \cap S}$ and $f^{-1}|_{E' \cap T} = g^{-1}|_{F' \cap T}$. Hence, $g \cap (S \odot T) = f \cap (S \odot T)$. Thus, $S \odot T$ is a support of $B(\mathcal{A})$.

Remark: If μ is a state on $\mathcal{A}$, then $\mu \circ \pi_1$ is a state on $B(\mathcal{A}, \mathcal{B})$ (provided that the latter test space exists). Hence, given a state μ on $\mathcal{A}$ and a state ν on $\mathcal{B}$, we may form a state

$$\mu \odot \nu := \frac{1}{2}(\mu \circ \pi_1 + \nu \circ \pi_2)$$

on $B(\mathcal{A}, \mathcal{B})$. It is easily checked that $S_{\mu \odot \nu} = S_\mu \odot S_\nu$.

A test space $\mathcal{A}$ is **regular** iff, for every $x \in X = \bigcup \mathcal{A}$, $X \setminus x^\perp$ is a support of $\mathcal{A}$ [4]. We have:

2.5 Theorem: *If $\mathcal{A}$ and $\mathcal{B}$ are regular, so is $B(\mathcal{A}, \mathcal{B})$.*

Proof: For a typical outcome $(x, y) \in Z$, we have

$$Z \setminus (x, y)^\perp = [(X \setminus x^\perp) \times Y \;\cup\; X \times (Y \setminus y^\perp)] \cap Y = (X \setminus x^\perp) \odot (X \setminus y^\perp).$$

Since $\mathcal{A}$ and $\mathcal{B}$ are regular, this last is a support by Lemma 3. Hence, $B(\mathcal{A}, \mathcal{B})$ is regular.

Let us adopt the following notation: If S is a support of a test-space $\mathcal{A}$ and $\alpha : E \to V$ is a V-valued observable, then we write $\{\alpha \in A\}$ for the collection of all supports of $\mathcal{A}$ such that $\alpha(S \cap E) \subseteq A$. That is: $\{\alpha \in A\}$ is the set of all supports making the event $\alpha^{-1}(A)$ certain to occur if the test E is made.

2.6 Lemma: *Let α and β be V-valued questions and $A \subseteq V$. Then*

$$\{\alpha \cdot \beta \in A\} = \{\alpha \in A\} \odot \{\beta \in A\}.$$

Proof: Suppose $\alpha : E \to V$ and $\beta : F \to V$. Let $f = \beta^{-1}\alpha = \{(x,y) \in E \times F | \alpha(x) = \beta(y)\}$. Then

$$(S \odot T) \cap f = \{(x,y)|\alpha(x) = \beta(y) \;\&x \in E \cap S \text{ or } y \in F \cap T\}.$$

Hence, $S \odot T \cap f \subseteq (\alpha \cdot \beta)^{-1}(A)$ iff $\alpha(S \cap E) \subseteq A$ and $\beta(T \cap F) \subseteq A$.

As a special case of the foregoing, note that $\alpha \cdot \beta$ is certain to take a value in $A \subseteq V$ in a state of affairs represented by $S \odot S$ iff both α and β are certain to lie in A in the state of affairs represented by S.

3 The Logic of $B(\mathcal{A}, \mathcal{B})$

If $\mathcal{A}$ is algebraic, the relation $\sim$ of perspectivity is an equivalence relation on the set of events of $\mathcal{A}$. The set of equivalence classes of events is the **logic** of $\mathcal{A}$, here denoted by $L(\mathcal{A})$. The equivalence class $p(A) := \{B \in \mathcal{E}(\mathcal{A}) | B \sim A\}$ of an event A is called the *operational proposition* corresponding to A. As is well-known, $L(\mathcal{A})$ can be organized into an orthoalgebra via the partial binary operation $p(A) \oplus p(B) \;:=\; p(A \cup B)$, (well)-defined for pairs of events A, B with $A \perp B$. (For details, see [2] and [3], or [4].)

If $\mathcal{A}$ and $\mathcal{B}$ are algebraic, then $B(\mathcal{A}, \mathcal{B})$ is also algebraic, by Theorem 2.2. In this section, we characterize $\Pi(B(\mathcal{A}, \mathcal{B}))$ in terms of $\Pi(\mathcal{A})$ and $\Pi(\mathcal{B})$ for a large class of algebraic test spaces.

3.1 Definition: Events $A \in \mathcal{E}(\mathcal{A})$ and $B \in \mathcal{E}(\mathcal{B})$ are **comparable** iff there exists a bijection $f \in \mathcal{E}(B(\mathcal{A}, \mathcal{B}))$ with $f : A \to B$.

Note that if $\mathcal{A}$ is κ-uniform, then any two proper events A and B of a given cardinality are comparable.

Let A and B be comparable events. By Lemma 2.1, the proposition $p(f)$ corresponding to any (hence, all) bijections $f : A \to B$ consists exactly of the union of the sets $B(C, D)$ of bijections between C and D with $C \sim A$ and $D \sim B$. Thus, the proposition $p(f)$ is completely determined by the pair $p(A)$ and $p(B)$. Let us write $p(A, B)$ for this proposition.

3.2 Lemma: *Let $A, B \in \mathcal{E}(\mathcal{A})$ and $C, D \in \mathcal{E}(\mathcal{B})$ with A and C comparable and B and D comparable. If $p(A, B) \perp p(C, D)$, then $A \perp C$, $B \perp D$, $A \cup C$ and $B \cup C$ are comparable, and $p(A, B) \oplus p(C, D) = p(A \cup C, B \cup D)$.*

Proof: If $p(A, B) \perp p(C, D)$ then for every bijection $f : A \to B$ and every bijection $g : C \to D$, $f \cap g = \emptyset$ and $f \cup g : A \cup C \to B \cup D$ belongs to $\mathcal{E}(B(\mathcal{A}))$ – whence, $A \perp C$, $B \perp D$, and $p(A \cup C, B \cup D) = p(f \cup g) = p(f) \oplus p(g) = p(A, B) \oplus p(C, D)$.

Note that $A \perp C, B \perp D$ need not imply that $p(A, B) \perp p(B, D)$ unless $\mathcal{A}$ is uniform.

If $\mathcal{A}$ is uniform, then any two perspective events have the same cardinality. Hence, we may define a map $\rho : \Pi(\mathcal{A}) \to \kappa$ (where κ is the cardinality of a test in $\mathcal{A}$) by

$$\rho(p(A)) = \#(A).$$

We call $\rho(p)$ the **rank** of the proposition $p \in \Pi(\mathcal{A})$. Note also that if $p \perp q$ then $\rho(p \oplus q) = \rho(p) + \rho(q)$ for all $p, q \in \Pi(\mathcal{A})$. The proof of the following is straightforward:

3.3 Theorem: *Let $\mathcal{A}$ and $\mathcal{B}$ be κ-uniform test spaces with logics L and M, respectively. Let*

$$L \times_\rho M = \{ (p,q) \in L \times M \mid \rho(p) = \rho(q) \}.$$

For all $(p,q), (u,v) \in L \times_\rho M$, write $(p,q) \perp (u,v)$ iff $p \perp u$ and $q \perp v$, and, if this is the case, set $(p,q) \oplus (u,v) := (p \oplus q, u \oplus v)$. Then $(L \times_\rho M, \perp, \oplus)$ is an orthoalgebra, and there is a canonical isomorphism $L \times_\rho M \to \Pi(B(\mathcal{A},\mathcal{B}))$ given by $p(A,B) \mapsto (p(A), p(B))$.

3.4 Definition: Call an algebraic test space $\mathcal{A}$ **saturated** iff for every $A \in \mathcal{E}(\mathcal{A})$ there is some $x_A \in X = \bigcup \mathcal{A}$ with $\{x_A\} \sim A$.

By way of example, if $\mathcal{A}$ is any manual, the manual $\mathcal{A}^\#$ of partitions of $\mathcal{A}$-operations by $\mathcal{A}$-events is saturated, with $\{\bigcup A\} \sim A$ for any subset A of such a partition.

3.5 Lemma: *Let $\mathcal{A}$ and $\mathcal{B}$ be saturated. Then every bijection between proper events of $\mathcal{A}$ and $\mathcal{B}$ can be extended to an element of $B(\mathcal{A},\mathcal{B})$.*

Proof: If A and B are proper events of the same cardinality with $A \subseteq E \in \mathcal{A}$ and $B \subseteq F$ in $\mathcal{B}$, then there exist outcomes x and y with $\{x\} \sim E \setminus A$ and $\{y\} \sim F \setminus A$. Since $\mathcal{A}$ and $\mathcal{B}$ are algebraic, $\{x\} \cup A$ and $\{y\} \cup B$ are tests in $\mathcal{A}$ and $\mathcal{B}$, respectively, to which we may extend any given bijection $f : A \to B$ by setting $f(x) = y$.

3.6 Definition: Let L and M be two orthoalgebras. Let

$$L * M := \{ (p,q) \in L \times M \mid p = 0 \Leftrightarrow q = 0 \ \&\ p = 1 \Leftrightarrow q = 1. \}.$$

For (p,q) and (r,s) in $L * M$, set $(p,q) \perp (r,s)$ iff $p \perp r$, $q \perp s$, and $(p \oplus q, r \oplus s) \in L * M$. If this is the case, define $(p,q) \oplus (r,s) := (p \oplus r, q \oplus s)$.

It is easily verified that $(L * M, \perp, \oplus, (1,1))$ is an orthoalgebra in which the orthocomplement of an element (p,q) is given by $(p,q)' = (p', q')$.

3.7 Proposition: *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be saturated algebraic test spaces. Then*

$$L(B(\mathcal{A}_1, \mathcal{A}_2)) \simeq L(\mathcal{A}_1) * L(\mathcal{A}_2).$$

Proof: Let $L = L(B(\mathcal{A}_1, \mathcal{A}_2))$ and $L_i = L(\mathcal{A}_i)$, $i = 1, 2$. The two coordinate projections $\pi_i : B(\mathcal{A}_1, \mathcal{A}_2) \to \mathcal{A}_i$ introduced in the proof of Theorem 1.7 lift to orthoalgebra homomorphisms $L \to L_i$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are saturated, these are surjections, by Lemma 3.5 above. Hence, we have a natural map $\phi : L \to L_1 \times L_2$ given by $\phi(p) = (\pi_1(p), \pi_2(p))$ for all $p \in L$. If $\phi(p) = (1, q) \in L_1 \times L_2$, then $p = p(E, B)$ for some $E \in \mathcal{A}_1$ and some event $B \in \mathcal{A}_2$

with $q = p(B)$. In order for A and B to be comparable, there must exist a bijection $f \in B(\mathcal{A}_1, \mathcal{A}_2)$ with $B = f(A)$. But then $B \in \mathcal{A}_2$, whence, $q = 1$. Similarly, if $p = 0$, $q = 0$ in order to preserve comparability. On the other hand, if $p \in L(\mathcal{A}_1)$, $q \in L(\mathcal{A}_2)$, and neither p nor q is 0 or 1, then, since each manual is saturated, we may choose outcomes $x \in X_1 = \bigcup \mathcal{A}_1$ and $y \in X_2 = \bigcup \mathcal{A}_2$ with $p = p(x)$ and $q = p(y)$. Likewise, $p' = p(x')$ and $q' = p(y')$ for some outcomes $x' \in X_1$ and $y' \in X_2$. Thus $\{x, x'\}$ and $\{y, y'\}$ are two-element tests in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, whence, $f = \{(x, y), (x', y')\}$ belongs to $B(\mathcal{A}_1, \mathcal{A}_2)$. Thus, $\pi(p(f)) = \pi(p(x, y)) = (p, q)$ is defined. The image of ϕ is therefore precisely $L_1 * L_2$. It remains to see that ϕ is an faithful (hence, injective) orthoalgebra homomorphism. But this follows from Lemma 3.2.

A **partition of unity** in an orthoalgebra L is a finite set $E = \{p_1, ..., p_n\} \subseteq L \setminus \{0\}$ such that $p_1 \oplus \cdots \oplus p_n = 1$. The collection $\mathcal{A}_L$ of all such partitions of unity is easily seen to be a saturated manual, the logic of which is canonically isomorphic to L.

3.8 Corollary: *For any orthoalgebras L and M, $L * M \simeq L(B(\mathcal{A}_L, \mathcal{A}_M))$.*

Call two test spaces $\mathcal{A}$ and $\mathcal{B}$ *uniformly compatible* iff every bijection between events of $\mathcal{A}$ and $\mathcal{B}$ extends to an element of $B(\mathcal{A}, \mathcal{B})$. (By way of example: Any two saturated algebraic test spaces, or any two uniform test spaces). The following generalization of Theorem 3.7 is straightforward. We omit the proof.

3.9 Proposition: *Let $\mathcal{A}$ and $\mathcal{B}$ be uniformly compatible test spaces. There is a canonical embedding of $L(B(\mathcal{A}, \mathcal{B}))$ into $L(\mathcal{A}) * L(\mathcal{B})$ given by*

$$(p(A, B)) \mapsto (p(A), p(B))$$

for compatible events A and B.

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