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ON GENERALIZED DARBOUX TRANSFORMATIONS AND SYMMETRIES OF SCHRÖDINGER EQUATIONS

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Abstract

Some generalizations of Darboux transformations have already been proposed and related. We extend these considerations to their most general forms including all the preceding approaches and get (very easily) the maximal set of symmetries subtended by the physical applications corresponding to exactly solvable potentials. The multidimensional matrix formulation of supersymmetric quantum mechanics is particularly well-adapted to such generalized Darboux transformations, so that maximal invariance superalgebras come out very naturally.

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1 Introduction

Physical applications characterized by exactly solvable stationary or nonstationary potentials are very interesting parts of (nonrelativistic) quantum mechanics which have to be exploited as far as possible. In particular, chains of exactly solvable potentials have already been constructed and enlightened through Darboux transformations [1] also in the recent contexts of supersymmetric [2, 3] and parasupersymmetric [4, 5] quantum mechanics leading in particular to specific remarkable invariance Lie superalgebras and parasuperalgebras.

Through the nonstationary developments, we want to take here advantages of the study of Schrödinger equations and associated multidimensional matrix Hamiltonians. Such a context can be presented in the following way. Let us consider two nonstationary Schrödinger equations in a one-dimensional space, i.e.

$$i\, \partial_t\, \Psi(x,t) = H_0\, \Psi(x,t)\ ,\ H_0 \equiv -\partial_x^2 + V_0(x,t) \tag{1}$$

and

$$i\, \partial_t\, \varphi(x,t) = H_1\, \varphi(x,t)\ ,\ H_1 \equiv -\partial_x^2 + V_1(x,t), \tag{2}$$

where ∂_t and ∂_x evidently refer to partial derivatives with respect to time and space and where V_0 and V_1 are potential energies defining a first example of a chain of exactly solvable potentials. In fact, the Darboux transformation solves the following problem : if we assume that the solutions of eq.(1) are known for a fixed V_0(x,t) , we can derive a (set of) potential(s) V_1(x,t) , so that the solutions φ(x,t) of eq.(2) could be obtained through the relation

$$\varphi\ (x,t) = L\ \Psi(x,t) \tag{3}$$

where L is the so-called Darboux operator. As quoted very clearly by Matveev and Salle [1], this gives to L and V_1 the following forms in terms of a particular solution - let us call it u(x,t) - of eq. (1)

$$L = \partial_x - (ln\ u)_x \tag{4}$$

and

$$V_1 = V_0 - 2\ (ln\ u)_{xx}, \tag{5}$$

so that the solutions of eq. (2) take the forms

$$\varphi(x,t) = \Psi_x(x,t) - (ln\ u(x,t))_x\ \Psi(x,t). \tag{6}$$

Knowing that the inverse problem is also well-defined, it is evident that we have an example of the study of 2-dimensional matrix Hamiltonians already considered in the literature [6] and directly connected to *supersymmetric* quantum mechanical developments [2]. The important role being played by the relation (3) and its explicit operator (4), let us notice that the latter can be written in the form

$$L \equiv L_1(x,t)\ \partial_x + L_0(x,t) \tag{7}$$

with

$$L_1(x,t) = 1\ ,\ L_0(x,t) = -(ln\ u(x,t))_x \tag{8}$$

and that it can be generalized, after Bagrov and Samsonov [3], in the form (7) but with

$$L_1(x,t) = exp\left[2\int Im(ln\ u)_{xx}\ dt\right], \tag{9}$$

$$L_0(x,t) = -(ln\ u(x,t))_x\ L_1(x,t). \tag{10}$$

In that context we then get the second potential in the form

$$V_1(x,t) = V_0(x,t) - 2(ln\ \mid u \mid)_{xx}. \tag{11}$$

Here we want to extend once more such a generalization and exploit the idea in order to characterize different physical applications by structures of invariance giving all the possible (super)symmetries of the corresponding contexts.

The contents are distributed as follows. In *Section 2* we summarize some results already quoted in Matveev and Salle [1] and generalized by Bagrov and Samsonov [3] but also extend the latter on the basis of very simple arguments. In *Section 3* we exploit our proposal and construct even and odd symmetry operators leading in the supersymmetric context(s) to typical invariance superalgebras. *Section 4* is devoted to remarks and conclusions as well as to the generalization to an arbitrary order N in the derivatives, nowadays a purely mathematical but interesting context.

2 Towards generalized Darboux transformations

Different steps have already been proposed (see Section 1) in order to generalize effective Darboux transformations. Besides some extensions to time-dependent potentials (and associated nonstationary Schrödinger equations) and some developments in super- and parasuper- quantum mechanics, there are also other possible generalizations with respect, for example, to the one proposed by Bagrov and Samsonov [3]. Let us first notice that, through eqs. (8) and (9-10) , we immediatly see that Bagrov-Samsonov's developments are more general than Matveev-Salle's ones : they correspond to each other only when

$$Im(ln\ u)_{xx} = 0. \tag{12}$$

Such a remark says that there is no reason at all to choose $(ln\ u)_{xx}$ as being a real quantity. Secondly, here we want to stress the fact that, moreover, there is no reason at all to limit ourselves to real functions $L_1(x,t)$ in eq.(9) defining the operator $L \equiv (7)$. Such a further simple remark will lead us to new results and consequences which have to be exploited.

For example, in the supersymmetric context dealing with a 2-dimensional matrix (super)Hamiltonian H

$$H = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} \tag{13}$$

and a supercharge Q defined by

$$Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & L^\dagger \\ 0 & 0 \end{pmatrix}, \tag{14}$$

possible choices of complex functions $L_1(x,t)$ open a more general discussion coming from the condition

$$L\ (i\ \partial_t - H_0)\ \Psi(x,t) = (i\ \partial_t - H_1)\ L\ \Psi(x,t) \tag{15}$$

which corresponds to ask for a Darboux operator $L \equiv (7)$ acting on the solutions $\Psi(x,t)$ of eq.(1) and transforming $\Psi(x,t)$ into $\varphi(x,t)$ according to eq.(3). Let us also point out that the condition (15) corresponds to the conservation requirement

$$[i\,\partial_t - H\;,\;Q] = 0. \tag{16}$$

This leads to the following system of equations on $L_0(x,t)$ and $L_1(x,t)$ introduced in the Darboux operator (7) :

$$\begin{aligned}
&L_1(x,t) = L_1(t),\\
&L_0(x,t) = L_0(t) - L_1(t)\;(ln\;u)_x\; - \frac{i}{2}\,\frac{dL_1(t)}{dt}x, \qquad (17)\\
&L_1(t)\;V_{0\,,\,x} - i\frac{dL_1(t)}{dt}(ln\;u)_x + \frac{1}{2}\frac{d^2L_1(t)}{dt^2}x - L_1(t)(ln\;u)_{xxx} + i\frac{dL_0(t)}{dt}\\
&+2\left(L_0(t) - L_1(t)(ln\;u)_x - \frac{i}{2}\frac{dL_1(t)}{dt}x\right)\;(ln\;u)_{xx} - iL_1(t)\;(ln\;u)_{xt} = 0.
\end{aligned}$$

Let us notice that the last condition can be directly exploited : by integrating on the space variable and taking care of u as a particular solution of eq.(1), we get

$$\frac{1}{4}\frac{d^2L_1(t)}{dt^2}\;x^2 + i\frac{dL_0(t)}{dt}\;x - i\frac{dL_1(t)}{dt}\;(ln\;u)_x\;x + 2L_0(t)\;(ln\;u)_x = f(t). \tag{18}$$

Such a condition implies that we will obtain more general results with respect to the Bagrov-Samsonov ones if and only if we choose

$$(ln\;u)_x = a\;x + b + \frac{c}{x}\;,\;a \neq 0, \tag{19}$$

where a, b, c are arbitrary time-dependent functions included in the corresponding Schrödinger solutions

$$u(x,t) = d\;x^c\;e^{\,\frac{1}{2}\,ax^2+bx}\;, d \neq 0. \tag{20}$$

By taking care once again of

$$i\;u_t = -u_{xx} + V_0\;u, \tag{21}$$

the required solutions (20) lead to families of exactly solvable potentials of the following form

$$V_0(x,t) = A(t)\;x^2 + B(t)\;x + C(t) + D(t)\;ln\;x + E(t)\;x^{-2} + F(t)\;x^{-1} \tag{22}$$

where

$$
\begin{aligned}
A(t) &= a^2 + \frac{i}{2}\, a_t, \quad & B(t) &= 2ab + i\, b_t, \\
C(t) &= a + b^2 + 2ac + i\, d^{-1} d_t, \quad & D(t) &= i\, c_t, \\
E(t) &= c\,(c-1), \quad & F(t) &= 2bc.
\end{aligned}
\tag{23}
$$

In the following section, we plan to consider a few examples entering into such categories but here let us propose a further extension of these considerations by including *second order* derivatives in the Darboux operator. In fact, we now propose to generalize the formula (7) as follows

$$
L \equiv L_2(x,t)\, \partial_x^2 + L_1(x,t)\, \partial_x + L_0(x,t), \tag{24}
$$

so that all the preceding study corresponds to $L_2 = 0$. Let us immediatly point out that our proposal once again contains other approaches [1, 3, 7, 8] but none of these does consider the possible complexification of the L_2-function. We also notice another generalization [9] similar to the one introduced in eq.(24) up to the important differences that the polynomial expression of the Eleonsky - Korolev operator L connects different eigenfunctions of the *same* Hamiltonian (while in our approach *two* Hamiltonians are connected) and is independent of time.

Coming back now on the condition corresponding to eq. (15) but with the expression (24) for the Darboux operator, we get a new system which reads :

$$
\begin{aligned}
&L_2(x,t) = L_2(t), \\
&L_1(x,t) = L_1(t) - \frac{i}{2}\,\frac{dL_2(t)}{dt}\, x - (ln\, W(u_1,u_2))_x\, L_2(t), \\
&L_0(x,t) = L_0(t) - \frac{1}{8}\,\frac{d^2 L_2(t)}{dt^2}\, x^2 - \frac{i}{2}\,\frac{dL_1(t)}{dt}\, x - L_2(t)\, V_0(x,t) \\
&\quad +(ln\, W(u_1,u_2))_x \left(\frac{i}{2}\,\frac{dL_2(t)}{dt}\, x - L_1(t)\right) + \frac{1}{2}\, L_2(t)\, (ln\, W(u_1,u_2))_x^2 \\
&\quad +\frac{1}{2}\,(ln\, W(u_1,u_2))_{xx}\, L_2(t) + \frac{i}{2}\,(ln\, W(u_1,u_2))_t\, L_2(t), \\
&L_0(x,t)_{xx} + i\, L_0(x,t)_t + 2L_0(x,t)\,(ln\, W(u_1,u_2))_{xx} + L_1(x,t)\, V_0(x,t)_x \\
&\quad +L_2(x,t)\, V_0(x,t)_{xx} = 0.
\end{aligned}
\tag{25}
$$

The above expressions are evidently given in terms of the usual Wronskian determinant

$$W(u_1, u_2) = u_1\, u_{2x} - u_{1x}\, u_2, \tag{26}$$

where u_1 and u_2 are two particular solutions of eq. (1) while we have added a further Schrödinger equation (with respect to eqs. (1) and (2)) quoted in the form

$$i\, \partial_t\, \chi(x,t) = H_2\, \chi(x,t), \quad H_2 \equiv -\partial_x^2 + V_2(x,t), \quad \chi(x,t) = L\, \psi(x,t). \tag{27}$$

From Matveev-Salle's contribution [1], we get through eqs. (1) and (27) that

$$\begin{aligned}
&V_2(x,t) = V_0(x,t) - 2(ln\, W(u_1,u_2))_{xx}, \\
&L_2(x,t) = 1, \quad L_1(x,t) = -(ln\, W(u_1,u_2))_x, \\
&L_0(x,t) = \frac{1}{2}\,(ln\, W(u_1,u_2))_{xx} + \frac{i}{2}\,(ln\, W(u_1,u_2))_t \,+ \frac{1}{2}\,(ln\, W(u_1,u_2))_x^2 \\
&\qquad -V_0(x,t),
\end{aligned} \tag{28}$$

while, from Bagrov-Samsonov results [3], we obtain

$$\begin{aligned}
&V_2(x,t) = V_0(x,t) - 2\,(ln \mid W(u_1,u_2) \mid)_{xx}, \\
&L_2(x,t) = exp\left[2\,\int Im\,(ln \mid W(u_1,u_2) \mid)_{xx} dt\right], \\
&L_1(x,t) = -L_2(x,t)\,\,(ln \mid W(u_1,u_2) \mid)_x, \\
&L_0(x,t) = L_2(x,t)\left[\frac{1}{2}\,\,(ln \mid W(u_1,u_2) \mid)_{xx} + \frac{i}{2}\,(ln \mid W(u_1,u_2) \mid)_t \right.\\
&\qquad \left. +L_1(x,t) + \frac{1}{2}\,(ln \mid W(u_1,u_2) \mid)_x^2 - V_0(x,t)\,\right].
\end{aligned} \tag{29}$$

These expressions immediately show the more general character of Bagrov-Samsonov's developments which can once again be extended through our proposal by complexifying the L_2 -function. The interest of our generalization is illustrated in the following section on specific applications.

3 Some physical applications and their (super)symmetries

Let us call N the maximal order of derivatives included in the Darboux operator we are constructing in each physical context and consider N = 1 or

2 before giving some extensions in the last section.

A. a) The N = 1 - free case

It corresponds to the choice $V_0 = 0$ in eq. (1) and to its implications in eqs. (22) and (23). We immediately notice that, with $a \neq 0$, a particular solution of eq. (1) is given with

$$a = i\,(2t)^{-1}, \quad b = t^{-1}, \quad c = 0, \quad d = t^{-\frac{1}{2}}\, exp(-i\,t^{-1}) \tag{30}$$

by

$$u(x,t) = t^{-\frac{1}{2}}\, exp\,(-i\,t^{-1})\, exp\,\left[\frac{i\,x^2}{4\,t} + \frac{x}{t}\right]. \tag{31}$$

This implies $V_1 = 0$ and the system reduces to

$$L_1(x,t) = L_1(t), \quad L_0(x,t) = -\frac{i}{2}\frac{dL_1(t)}{dt}\,x + L_0(t), \tag{32}$$
$$\frac{1}{2}\,\frac{d^2L_1(t)}{dt^2}\,x + i\,\frac{dL_0(t)}{dt} = 0,$$

so that

$$L_1(t) = c_1 + c_2\,t, \quad L_0(x,t) = -\frac{i}{2}c_2\,x + c_3$$

and, consequently,

$$L(x,t) = (c_1 + c_2\,t)\,\partial_x - \frac{i}{2}\,c_2\,x + c_3. \tag{33}$$

Due to the three arbitrary parameters c_1, c_2 and c_3 now included in the Darboux operator appearing in the 2-dimensional matrix formulation (see eq. (14)), we can construct six *odd* operators defined as follows :

$$Y_1 \equiv \partial_x\,\sigma_-, \quad Y_1^\dagger \equiv -\partial_x\,\sigma_+, \quad Y_2 \equiv (t\,\partial_x - \frac{i}{2}\,x)\,\sigma_-, \quad Y_2^\dagger \equiv (-t\,\partial_x + \frac{i}{2}\,x)\,\sigma_+, \tag{34}$$

and

$$Y_3 \equiv \sigma_-, \quad Y_3^\dagger \equiv \sigma_+,$$

where $\sigma_\pm$ refer to the usual linear combinations of Pauli matrices. The corresponding supersymmetric context (with $V_0 = V_1 = 0$) associated with the formulation [(13),(14)] is here characterized by the *expected* Lie superalgebra sqm(2) with the structure relations

$$\{Q, Q^\dagger\} = H, \quad Q^2 = Q^{\dagger\,2} = 0, \quad [Q, H] = [Q^\dagger, H] = 0, \tag{35}$$

where we have identified

$$Q \equiv Y_1, \quad Q^\dagger \equiv Y_1^\dagger, \quad H \equiv -\partial_x^2. \tag{36}$$

Moreover this context admits a maximal invariance superalgebra readily obtained by anticommuting the odd generators (34) and, in that way, by constructing seven *even* operators which, besides the unit matrix, are given by

$$\begin{gathered} X_1 \equiv \partial_x, \quad X_2 \equiv t\,\partial_x - \frac{i}{2}\,x, \quad X_3 \equiv -\partial_x^2 = H, \\ X_4 \equiv -t\,\partial_x^2 + \frac{i}{2}\,x\,\partial_x + \frac{i}{4}\,(1-\sigma_3), \quad X_5 \equiv -t^2\,\partial_x^2 + i\,x\,t\,\partial_x + \frac{i}{2}\,t + \frac{1}{4}\,x^2, \\ X_6 \equiv \sigma_3. \end{gathered} \tag{37}$$

It is easy to confirm that we get a closed structure which is the semi-direct sum of the orthosymplectic superalgebra $osp(2 \mid 2)$ and the superHeisenberg one $sh(2 \mid 2)$, an expected result in connection with the maximal invariance superalgebra for the (isomorphic) 1-dimensional harmonic superoscillator [10] that we will also recover in the following, such results being nothing else than the superextension of Niederer's results [11].

A. b) The N = 2 - free case

Let us start with $V_0 = 0$ and with the particular solution $u_1(x,t) \equiv$ (31) supplemented by another one given as

$$u_2(x,t) = t^{-1}\,(x - 2i)\,u_1(x,t) \tag{38}$$

leading to the Wronskian determinant (26)

$$W(u_1, u_2) = t^{-2}\,exp\left[-\frac{2i}{t}\right]\,exp\left[\frac{ix^2}{2t} + \frac{2x}{t}\right] \tag{39}$$

and to its absolute value

$$| W(u_1, u_2) |= t^{-2} \, exp \left[\frac{2x}{t}\right]. \tag{40}$$

Once again, the second potential V_2 is equal to zero and we are led to a second order Darboux operator depending on six arbitrary parameters, i.e.

$$L(x,t) = (c_1 + c_2 \, t + \frac{1}{2} \, c_3 \, t^2) \, \partial_x^2 + \left[c_4 + c_5 \, t - \frac{i}{2} \, (c_2 + c_3 \, t) \, x\right] \, \partial_x$$
$$- \frac{1}{8} \, c_3 \, x^2 - \frac{i}{2} \, x \, c_5 - \frac{i}{4} \, c_3 \, t + c_6. \tag{41}$$

This ensures the appearance of twelve *odd* generators and eight *even* ones. Let us only point out that, in particular, we have

$$Y_1 \equiv \partial_x^2 \, \sigma_-, \qquad Y_1^\dagger \equiv \partial_x^2 \, \sigma_+ \tag{42}$$

as odd "charges" leading to

$$\{Y_1, Y_1^\dagger\} = \partial_x^4 = H^2 \tag{43}$$

and showing that we detect here a *deformed* superalgebra sqm (2) inside the corresponding maximal invariance superalgebra subtending this context.

B. a) The N = 1 - harmonic (super)oscillator

By letting the usual angular frequency equal to unity, the potential evidently is

$$V_0 = x^2 \tag{44}$$

in the 1-dimensional space context. Here we choose as an example

$$a = -1, \;\; b = 0 = c, \;\; d = exp\,(-it) \tag{45}$$

and get the particular

$$u(x,t) = exp\,(-it) \; exp\left(-\frac{1}{2}\, x^2\right) \tag{46}$$

leading to

$$V_1(x,t) = V_1(x) = x^2 + 2 \tag{47}$$

and to the system (17) easily exploited. We get

$$L_1(x,t) = c_1 \, exp\,[-4it\,] + c_2, \qquad (48)$$
$$L_0(x,t) = (c_2 - c_1 \, exp\,[-4it\,])\, x + c_3 \, exp\,[-2it\,].$$

Here we can deal again with six odd generators which lead to the realization

$$Y_1 \equiv exp\,[-4it\,]\,(\partial_x - x)\,\sigma_-, \quad Y_1^\dagger \equiv exp\,[4it\,]\,(-\partial_x - x)\,\sigma_+,$$
$$Y_2 \equiv (\partial_x + x)\,\sigma_-, \qquad Y_2^\dagger \equiv (-\partial_x + x)\,\sigma_+, \qquad (49)$$
$$Y_3 \equiv exp\,[-2it\,]\,\sigma_-, \qquad Y_3^\dagger \equiv exp\,[2it\,]\,\sigma_+.$$

It is easy to get the other six even operators leading once again to the semi-direct sum $osp\ (2 \mid 2)$ with $sh\ (2 \mid 2)$ as expected [10] as it was recovered in the free case. For convenience, let us also mention these six even generators (besides the unit matrix)

$$X_1 \equiv exp\,[2it\,]\,(\partial_x + x), \quad X_2 = X_1^\dagger \equiv exp\,[-2it\,]\,(-\partial_x + x),$$
$$X_3 \equiv exp\,[4it\,]\,(-\partial_x^2 - x^2 - 1 - 2\,x\,\partial_x), \qquad (50)$$
$$X_4 = X_3^\dagger \equiv exp\,[-4it\,]\,(-\partial_x^2 - x^2 + 1 + 2\,x\,\partial_x),$$
$$X_5 \equiv -\partial_x^2 + x^2 + \sigma_3, \quad X_6 \equiv -\partial_x^2 + x^2 - \sigma_3.$$

B. b) The N = 2 - harmonic (super)oscillator

With $u_1(x,t)$ given by eq. (46) we consider

$$u_2(x,t) = 2\, exp\,[-3it\,]\left(exp\left[-\frac{1}{2}\,x^2\right]\right)\, x \qquad (51)$$

as a second solution of eq. (1) with the potential (45). We then get

$$W(u_1, u_2) = 2\, exp\,[-4it\,]\, exp(-x^2) \qquad (52)$$

and the second potential becomes

$$V_2(x,t) = V_2(x) = x^2 + 4. \qquad (53)$$

The Darboux operator is now dependent on six arbitrary parameters, so that this context shows twelve odd and eight even generators, a result isomorphic to that deduced from (42). Let us point out here (as in (49) and (50)) the combinations

$$A^{\pm} \equiv \mp\, \partial_x + x \tag{54}$$

playing the role of annihilation and creation operators for characterizing superpartners in the 2-dimensional matrix representation. Here again we obtain a deformation of the superalgebra sqm (2) characterized by the relations

$$\begin{aligned} &\{\, Y_1, Y_1^{\dagger} \,\} = (H-1)\,(H-3), \quad Y_1^2 = Y_1^{\dagger^2} = 0, \\ &[\, H, Y_1 \,] = \left[\, H, Y_1^{\dagger} \,\right] = 0, \end{aligned} \tag{55}$$

and

$$Y_1 \equiv (A^-)^2\, \sigma_-, \quad H = \begin{pmatrix} A^+A^- + 1 & 0 \\ 0 & A^-A^+ + 3 \end{pmatrix}. \tag{56}$$

Let us insist on the fact that the above deformation of sqm (2) is of *second* order in the superhamiltonian.

C. a) The N = 1 - Calogero context

It is characterized by the so-called Calogero potential

$$V_0(x) = x^2 + \frac{\lambda}{x^2} + \mu, \quad (\lambda, \mu = constants) \tag{57}$$

and corresponds to

$$a = -1, \;\; b = 0, \;\; c = \frac{1}{2}(1 \pm \sqrt{1+4\lambda}), \;\; d = exp\,[-it(1+2c+\mu)] \tag{58}$$

suggesting the particular solution

$$u(x,t) = exp\,\left[-it(\sqrt{1+4\lambda} + 2 + \mu)\right]\; x^{\frac{1}{2}\sqrt{1+4\lambda}+\frac{1}{2}}\; exp\left(-\frac{x^2}{2}\right). \tag{59}$$

This implies in the second equation a potential

$$V_1(x) = x^2 + \frac{\lambda + \sqrt{1+4\lambda} + 1}{x^2} + 2 + \mu \tag{60}$$

and a Darboux operator depending on two arbitrary parameters. Effectively we get

$$L_0(x,t) = [c_2 - c_1\ exp\ (-4it)]x - [c_1\ exp(-4it) + c_2]\ \left[\sqrt{1+4\lambda} + \frac{1}{2}\right]\frac{1}{x} \quad (61)$$

and

$$L_1(x,t) = c_1\ exp\ (-4it) + c_2. \quad (62)$$

We thus point out here four *odd* generators

$$\begin{aligned} Y_1 &\equiv exp\ (-4it)\ \left[\partial_x - \left(\sqrt{1+4\lambda} + \frac{1}{2}\right)\frac{1}{x} - x\right]\ \sigma_-, \\ Y_1^\dagger &\equiv exp\ (4it)\ \left[-\partial_x - \left(\sqrt{1+4\lambda} + \frac{1}{2}\right)\frac{1}{x} - x\right]\ \sigma_+, \\ Y_2 &\equiv \left[\partial_x - \left(\sqrt{1+4\lambda} + \frac{1}{2}\right)\frac{1}{x} + x\right]\ \sigma_-, \\ Y_2^\dagger &\equiv \left[-\partial_x - \left(\sqrt{1+4\lambda} + \frac{1}{2}\right)\frac{1}{x} + x\right]\ \sigma_+ \end{aligned} \quad (63)$$

and four *even* generators which can be determined by anticommuting the odd ones. This leads once again to a closed superstructure.

C. b) The N = 2 - Calogero context

By calling $u_1(x,t)$ the first particular solution as the one given in eq. (59), let us add a second one in the form

$$u_2(x,t) = exp\ \left[-it(\sqrt{1+4\lambda} + 6 + \mu)\right]\ x^{\frac{1}{2}\sqrt{1+4\lambda}+\frac{1}{2}}\ exp\ \left(-\frac{x^2}{2}\right) \left[1 + \frac{1}{2}\sqrt{1+4\lambda} - x^2\right], \quad (64)$$

so that, through the Wronskian determinant, we get

$$V_2(x) = x^2 + \frac{\lambda + 2\sqrt{1+4\lambda} + 4}{x^2} + 4 + \mu. \quad (65)$$

The Darboux operator takes the form

$$L(x,t) = [c_1 + c_2\ exp\ (-4it) + c_3\ exp\ (-8it)]\ \partial_x^2 +$$

$$+[2x(c_1 - c_3\, exp\,(-8it)) - (\sqrt{1+4\lambda}+2)\,\frac{1}{x}$$
$$(c_1 + c_2\, exp\,(-4it) + c_3\, exp\,(-8it))]\,\partial_x$$
$$+\frac{1}{2}\,(3\sqrt{1+4\lambda}+2\lambda+3)\,(c_1 + c_2\, exp\,(-4it) + c_3\, exp\,(-8it))\frac{1}{x^2}$$
$$+(c_1 - c_2\, exp\,(-4it) + c_3\, exp\,(-8it))x^2$$
$$+(\sqrt{1+4\lambda}+1)\,(c_3\, exp\,(-8it) - c_1) \tag{66}$$

and leads to six odd operators. Let us only mention one of them called Y_1 and its hermitean conjugate $Y_1^\dagger$ which are given by

$$Y_1 = \Big[\partial_x^2 + (2x - (\sqrt{1+4\lambda}+2)\,\frac{1}{x})\partial_x + \frac{1}{2}(3\sqrt{1+4\lambda}+2\lambda+3)\,\frac{1}{x^2}$$
$$+x^2 - (\sqrt{1+4\lambda}+1)\Big]\;\sigma_-,$$
$$(67)$$
$$Y_1^\dagger = \Big[\partial_x^2 - (2x - (\sqrt{1+4\lambda}+2)\,\frac{1}{x})\partial_x + \frac{1}{2}(\sqrt{1+4\lambda}+2\lambda-1)\,\frac{1}{x^2}$$
$$+x^2 - (\sqrt{1+4\lambda}+3)\Big]\;\sigma_+.$$

They lead to a deformed sqm (2)-superalgebra characterized by the following anticommutation relation

$$\{Y_1\,,\,Y_1^\dagger\} = H^2 - (2\,\sqrt{1+4\lambda}+2\mu+8)\,H + 8\,\sqrt{1+4\lambda}$$
$$+2\mu\,\sqrt{1+4\lambda}+8\mu+\mu^2+4\lambda+13 \tag{68}$$

which is once again at most of the second order in the superhamiltonian

$$H = \begin{pmatrix} H_0 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{69}$$

D. a) The N = 1 - Coulomb context

It is given by the potential

$$V_0(x) = -\frac{1}{x} + \frac{\ell(\ell+1)}{x^2} \tag{70}$$

with usual units for the mass and the charge appearing in the discussion. Let us also recall that $\ell(\ell+1)$ are the eigenvalues of the square of the orbital angular momentum operator entering into these considerations.

The particular solution issued from simple Laguerre polynomials [12] can be chosen with $a = 0$ in the form

$$u_\ell(x,t) = exp\left(\frac{it}{4(\ell+1)^2}\right)\, exp\left(-\frac{x}{2(\ell+1)}\right)\, x^{\ell+1}, \tag{71}$$

so that it leads to the second potential

$$V_1(x) = -\frac{1}{x} + \frac{(\ell+1)\,(\ell+2)}{x^2}. \tag{72}$$

The corresponding Darboux operator becomes

$$L(x,t) = L(x) = c_1\, \partial_x + c_1\left(-\frac{\ell+1}{x} + \frac{1}{2(\ell+1)}\right) \tag{73}$$

and depends only on one arbitrary constant. We have thus only two odd (super)charges defined as follows :

$$Q \equiv \left(\partial_x - \frac{\ell+1}{x} + \frac{1}{2(\ell+1)}\right)\sigma_-, \quad Q^\dagger \equiv \left(-\partial_x - \frac{\ell+1}{x} + \frac{1}{2(\ell+1)}\right)\sigma_+, \tag{74}$$

leading to the sqm (2)-superalgebra (35) as expected but with

$$H = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} + \frac{1}{4(\ell+1)^2}\, I. \tag{75}$$

D. b) The N = 2 - Coulomb context

Besides the first solution $u_{1,\ell}(x,t)$ given for arbitrary ℓ 's by (71), we consider as a second particular solution the following one

$$u_{2,\ell}(x,t) = exp\left(\frac{it}{4\,(\ell+2)^2}\right)\, exp\left(-\frac{x}{2\,(\ell+2)}\right)\left[2(\ell+1)\, x^{\ell+1} - \frac{x^{\ell+2}}{\ell+2}\right]. \tag{76}$$

We are led to a second potential given by

$$V_2(x) = -\frac{1}{x} + \frac{(\ell+2)\,(\ell+3)}{x^2} \tag{77}$$

and to a Darboux transformation characterized by

$$L(x,t) = L(x) = c_1\,\partial_x^2 + c_1\left(\frac{2\ell+3}{2\,(\ell+1)\,(\ell+2)} - \frac{2\ell+3}{x}\right)\partial_x$$
$$+c_1\left(\frac{(\ell+1)\,(\ell+3)}{x^2} - \frac{2\ell^2+6\ell+5}{2\,(\ell+1)\,(\ell+2)\,x} + \frac{1}{4\,(\ell+1)\,(\ell+2)}\right) \tag{78}$$

showing that here again we only get two possible *odd* supercharges. They are such that

$$\{Q, Q^\dagger\} = H^2 + \frac{2\ell^2+6\ell+5}{4\,(\ell+1)^2\,(\ell+2)^2}H + \frac{1}{16\,(\ell+1)^2\,(\ell+2)^2}. \tag{79}$$

Let us close this section and ask for some general remarks and conclusions rather than by considering more and more contexts like those referring to Morse, Pösch -Teller, ..., potentials belonging to the category of solvable problems.

4 Remarks and conclusions

We have learned, through very simple arguments, that it is possible to generalize the Bagrov-Samsonov results concerning the Darboux operator, such generalizations depending on the N = 1 or 2-values as discussed in Section 3. Such an order N implying specific Darboux transformations, let us add a few results which could have some interests in the future. Let us indeed look at the generalization of our ideas to arbitrary values of N and let us propose the Darboux operator

$$L(x,t) = \sum_{j=0}^{N} L_j(x,t)\,\partial_x^j \tag{80}$$

admitting complex values for the function $L_N(x,t)$. For 2-dimensional matrix formulations, the corresponding condition (15) with L given by (80) leads to the following system where W is once again the Wronskian determinant corresponding to N particular solutions of eq.(1) including the potential $V_0(x,t)$. For each fixed value of N, we get systems of (N+2) conditions of the forms :

$$\begin{aligned}
&L_{N,x} = 0,\\
&iL_{N,t} + 2\, L_{N-1,x} + 2\,(ln\; W(u_1,u_2,...,u_N))_{xx}\, L_N = 0,\\
&iL_{N-1,t} + 2L_{N-2,x} - L_{N-1,xx} + 2(ln\; W)_{xx}\, L_{N-1} + NL_N V_{0,x} = 0, \quad (81)\\
&iL_{\ell,t} + 2L_{\ell-1,x} - L_{\ell,xx} + 2(ln\; W)_{xx} L_\ell + (\ell+1)L_{\ell+1}V_{0,x}\\
&+\frac{1}{2}(\ell+1)(\ell+2)L_{\ell+2}V_{0,xx} + \;...\; + \frac{N!}{(N-\ell)!\;\ell!}L_N(\partial_x^{N-\ell}V_0) = 0,\\
&\ell = 0,1,...,N-2.
\end{aligned}$$

As an illustration, let us come back on the harmonic oscillator context but with N arbitrary. We can choose N solutions $u_j(x,t)$ such that

$$i\partial_t\, u_j(x,t) = H_0\, u_j(x,t), \quad \forall j = 1,2,...,N, \tag{82}$$

expressed in terms of Hermite polynomials [12] $H_N(x)$ as follows

$$u_j(x,t) = exp\left[-i(2j-1)t\right]\, exp\left[-\frac{1}{2}x^2\right]\, H_{j-1}(x), \tag{83}$$

so that

$$(ln\; W)_x = -Nx, \quad V_N(x) = V_0(x) + 2N = x^2 + 2N. \tag{84}$$

The system (81) then leads to (N + 1) (N + 2) *odd* symmetries given in the 2-dimensional context by

$$\sigma_\pm,\; A^\pm\,\sigma_\pm,\; A^\mp\,\sigma_\pm,\; A^\pm\, A^\pm\,\sigma_\pm,\; ...\;, (A^+)^k\,(A^-)^{n-k}\,\sigma_\pm, \tag{85}$$

with k = O, 1, ..., N and $A^\pm$ given by eq. (54). In this oscillator context, they lead to *deformed* superalgebras. In particular, it is always possible to define the two supercharges

$$Q \equiv (A^-)^N\,\sigma_-,\;\; Q^\dagger \equiv (A^+)^N\,\sigma_+, \tag{86}$$

which are such that

$$Q^2 = (Q^\dagger)^2 = 0 \tag{87}$$

and

$$\{Q, Q^\dagger\} = \prod_{j=0}^{N-1} (H - 2j - 1), \tag{88}$$

$$[H, Q] = [H, Q^\dagger] = 0 \tag{89}$$

with

$$H = \begin{pmatrix} A^- A^+ - 1 & 0 \\ 0 & A^- A^+ + 2N - 1 \end{pmatrix}. \tag{90}$$

We thus get a deformed sqm (2)-superalgebra always included in the maximal invariance superalgebra corresponding to this study for arbitrary N 's.

All these results show that, *on the one hand*, supersymmetric quantum mechanics and its (at least) 2-dimensional formulation are particularly well exploited through the above developments associated with such Darboux transformations. *On the other hand*, supersymmetries in quantum mechanics have already been determined in a systematic study [13] for arbitrary superpotentials simply related to usual potentials appearing in Schrödinger equations. This approach gives a classification of all solvable interactions and associates nontrivial invariance superalgebras with each context. It is thus evident that these two points of views are not independent. Indeed, we have already noticed in the four physical applications developed in Section 3 (i.e. the free case, the harmonic oscillator, the Calogero and Coulomb problems) that the results are strongly related to those contained in reference [13] . Then, we immediatly deduce that all the physical applications different with respect to the four preceding ones (i.e. Morse, Pösch-Teller, ... potentials) cannot be characterized by superstructures larger than the deformed sqm(2)-superalgebras we are always constructing here in each context.

References

[1] **V.B. MATVEEV and M.A. SALLE**, *Darboux Transformations and Solitons* , in *Series in Nonlinear Dynamics* (Springer, Berlin, 1991).

[2] **E.WITTEN**, Nucl.Phys. **B188**, 513 (1981) ;
A.A. ANDRIANOV, N.V. BORISOV and M.V. IOFFE, Phys.Lett. **A105**, 19 (1984) ;
**A.A. ANDRIANOV, N.V. BORISOV, M.I. EIDES and M.V.

IOFFE, Phys.Lett. **A109**, 143 (1985) ;
V.P. BEREZOVOY and A.I. PASHNEV, Z.Phys. **C51**, 525 (1991).

[3] **V.G. BAGROV and B.F. SAMSONOV**, Phys.Lett. **A210**, 60 (1996).

[4] **V.A. RUBAKOV and V.P. SPIRIDONOV**, Mod.Phys.Lett. **A3**, 1337 (1988) ;
J. BECKERS and N. DEBERGH, Nucl.Phys. **B340**, 767 (1990).

[5] **B.F. SAMSONOV**, Mod.Phys.Lett. **A11**, 2095 (1996) ;
N. DEBERGH, Phys.Lett. **A219**, 1 (1996).

[6] **A.A. ANDRIANOV, N.V. BORISOV and M.V. IOFFE**, Phys.Lett. **B181**, 141 (1984).

[7] **A.A. ANDRIANOV, M.V. IOFFE and V.P. SPIRIDONOV**, Phys.Lett. **A174**, 273 (1993);
A.A. ANDRIANOV, M.V. IOFFE and D.N. NISHNIANIDZE, Phys.Lett. **A201**, 103 (1995).

[8] **D.J. FERNANDEZ C.**, Int.J.Mod.Phys. **A12**, 171 (1997).

[9] **V.M. ELEONSKY and V.G. KOROLEV**, J.Phys. **A28**, 4973 (1995).

[10] **J. BECKERS and V. HUSSIN**, Phys.Lett. **A118**, 319 (1986).

[11] **U. NIEDERER**, Helv.Phys.Acta **45**, 802 (1972); **46**, 191 (1973).

[12] **W. MAGNUS, F. OBERHETTINGER and R.P. SONI**, *Formulas and Theorems for Special Functions of Mathematical Physics* (Springer, Berlin, 1966).

[13] **J. BECKERS, N. DEBERGH and A.G. NIKITIN**, J.Math.Phys. **33**, 152 (1992).