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An Irreducible BRST Approach of Topological Yang-Mills Theory

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Abstract. The topological Yang-Mills theory is quantized in an irreducible manner, at both Lagrangian and Hamiltonian levels. Our procedure resides in replacing the starting reducible generating set, respectively the reducible first-class constraints with some irreducible ones. The ghosts of ghosts will no longer appear. Some cohomological aspects are briefly discussed.

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1 Introduction

It is widely known by now that the BRST formalism stands for the strongest quantization method for gauge theories. This approach has been developed in both Lagrangian [1]-[5] and Hamiltonian versions [5]-[9], for irreducible as well as for reducible theories. In the irreducible case the ghosts can be viewed as one-forms dual to the vector fields associated to the gauge transformations. In the opposite situation, where the generating set is reducible, the above geometrical interpretation fails as the vector fields form no longer a basis. Moreover, in this case it is necessary to introduce ghosts with ghost numbers greater than one. This objects, traditionally named ghosts of ghosts, accommodate the reducibility relations to the cohomology of the exterior derivative along the gauge orbits, while their antifields kill the non-trivial co-cycles in the homology of the Koszul-Tate operator at higher resolution degrees. An interesting model for testing the reducible BRST machinery appears to be the

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topological Yang-Mills theory [10]-[11] due to the field dependence of both gauge generators and reducibility functions. From the mathematical point of view, this model is interesting due to its connection [11] to Donaldson theory [12].

This model has not been quantized until now in an irreducible BRST fashion. This will be done here. In consequence, there will be no need for ghosts of ghosts. Our method consists in: *i)* the introduction of some fields with the help of which we replace the reducible generating set by an irreducible one at the Lagrangian level, respectively the reducible first-class constraints by some irreducible ones in the Hamiltonian procedure, and *ii)* the BRST quantization of the resulting irreducible systems. We note that the idea of transforming some reducible first-class constraints into some irreducible ones is exposed in [5].

The paper is structured in six sections. Section 2 exposes in brief the classical analysis of the model under study. In Section 3 we are dealing with the irreducible Lagrangian BRST treatment, while Section 4 covers the irreducible Hamiltonian procedure. In Section 5 we give some cohomological explanations related to our mechanism. Section 6 closes the paper with some conclusions.

2 The classical analysis of the topological Yang-Mills theory

Our starting point is given by the Lagrangian action

$$S_0^L\left[A_\mu^a\right]=-\frac{1}{4}\int d^4x\varepsilon_{\mu\nu\lambda\rho}F^{a\mu\nu}F_a^{\lambda\rho}, \tag{2.1}$$

where the field strength is defined by

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a-f^a{}_{bc}A_\mu^b A_\nu^c, \tag{2.2}$$

and $\varepsilon_{\mu\nu\lambda\rho}$ denotes the completely four-dimensional antisymmetric symbol. The group indices a, b, c, etc. are raised and lowered with the Killing metric. Action (2.1) is invariant under the gauge transformations

$$\delta_\varepsilon A_\mu^a=\left(D_\mu\right)^a{}_b\varepsilon^b+\varepsilon_\mu^a, \tag{2.3}$$

with

$$\left(D_\mu\right)^a{}_b(x)=\delta^a{}_b\partial_\mu^x+f^a{}_{bc}A_\mu^c(x), \tag{2.4}$$

where $\partial_\mu^x=\partial/\partial x^\mu$. For subsequent purposes, we make the notations

$$Z^{\alpha_0}{}_{\alpha_1}=\left(\left(D_\mu\right)^a{}_b(x)\,\delta^4(x-y)\ ,\ \delta^a{}_c\delta^\nu{}_\mu\delta^4(x-y)\right), \tag{2.5}$$

such that $\alpha_0=(a,\mu,x)$ and $\alpha_1=((b,y),(c,\nu,y))$. The gauge generators (2.5) are first-stage reducible, with the reducibility functions

$$Z^{\alpha_1}{}_{\alpha_2}=\begin{pmatrix}\delta^b{}_d\delta^4(y-z)\\ -\left(D_\nu\right)^c{}_d(y)\,\delta^4(y-z)\end{pmatrix}, \tag{2.6}$$

where $\alpha_2 = (d, z)$. We mention that the reducibility relations

$$Z^{\alpha_0}_{\ \alpha_1} Z^{\alpha_1}_{\ \alpha_2} = 0, \tag{2.7}$$

hold off-shell. In (2.7), we employed the De Witt condensed notation, i.e., the summation over α_1 implies also the integration over y.

Next, we pass to the canonical analysis. The definition of the canonical momenta leads to the primary constraints

$$G^a \equiv \pi_0^a = 0, \tag{2.8}$$

$$\chi_i^a \equiv \pi_i^a + \varepsilon_{0ijk} F^{ajk} = 0, \tag{2.9}$$

respectively to the canonical Hamiltonian

$$H = -\int d^3x A_a^0 \left(D^i\right)^a_{\ b} \pi_i^b. \tag{2.10}$$

The consistency of the above constraints implies the secondary ones of the form

$$F^a \equiv -\left(D^i\right)^a_{\ b} \pi_i^b = 0. \tag{2.11}$$

There are no further constraints, the previous ones being first-class. The gauge algebra reads

$$\left[G^a, G^b\right] = 0,\ \left[G^a, \chi_i^b\right] = 0,\ \left[G^a, F^b\right] = 0,\ \left[\chi_i^a, \chi_j^b\right] = 0, \tag{2.12}$$

$$[\chi_i^a, F_b] = f^a_{\ bc} \chi_i^c,\ [F_a, F_b] = f^c_{\ ab} F_c. \tag{2.13}$$

In addition, constraints (2.9) and (2.11) are first-stage reducible, the reducibility relations being expressed by

$$\left(D^i\right)^a_{\ b} \chi_i^b + \delta^a_{\ b} F^b = 0, \tag{2.14}$$

and taking place throughout the phase-space. For further convenience, we denote the reducible constraints, respectively the reducibility functions by

$$G_{a_0} = \left(\chi_i^a(x)\ ,\ F^b(x)\right), \tag{2.15}$$

$$Z^{a_0}_{\ a_1} = \begin{pmatrix} \left(D^i\right)^c_{\ a}(x)\,\delta^3(\vec{x} - \vec{y}) \\ \delta^c_{\ b}\delta^3(\vec{x} - \vec{y}) \end{pmatrix}, \tag{2.16}$$

with $a_0 = ((a, i, \vec{x}), (b, \vec{x}))$ and $a_1 = (c, \vec{y})$. This completes the classical analysis of our model.

3 The irreducible Lagrangian BRST quantization

In this section, we quantize the topological Yang-Mills theory along the irreducible antifield BRST prescriptions. In this respect, we transform the reducible gauge transformations (2.3) into some irreducible one by means of enlarging the original field spectrum by some scalar fields. These new fields are introduced such that the physical content of the model to remain unaltered. Subsequently, we quantize the inferred irreducible system within the antifield BRST background.

The conversion to an irreducible system goes as follows. To every reducibility relation (2.7), we associate a new field $\varphi^{\alpha_2} = \varphi^a$ with the gauge transformation

$$\delta_\varepsilon \varphi^{\alpha_2} = A_{\alpha_1}^{\alpha_2} \varepsilon^{\alpha_1}, \tag{3.1}$$

where

$$\varepsilon^{\alpha_1} = \begin{pmatrix} \varepsilon^b \\ \varepsilon_\nu^b \end{pmatrix}, \tag{3.2}$$

and $A_{\alpha_1}^{\alpha_2}$ a matrix with the property that $A_{\alpha_1}^{\alpha_2} Z^{\alpha_1}_{\beta_2}$ is invertible. We take $A_{\alpha_1}^{\alpha_2}$ ($\alpha_2 = (b, x)$, $\alpha_1 = ((a, y), (a, \nu, y))$) under the form

$$A_{\alpha_1}^{\alpha_2} = \left(M_b{}^a (x, y) \ , \ (N_\nu)_b{}^a (x, y) \right), \tag{3.3}$$

with unknown $M_b{}^a$ and $(N_\nu)_b{}^a$ and demand that

$$A_{\alpha_1}^{\alpha_2} Z^{\alpha_1}_{\beta_2} = -\partial^\nu \partial_\nu \delta^{\alpha_2}_{\beta_2}. \tag{3.4}$$

It is clear that the right hand-side of (3.4), which in fact reads $\delta^d{}_b \partial_x^\nu \partial_\nu^x \delta^4 (x - z)$, is invertible. On behalf of (2.6) and (3.3-3.4), we find after simple computation

$$M_b{}^a (x, y) = -f^a{}_{bc} A^{\nu c} (y) \partial_\nu^y \delta^4 (x - y) , \ (N_\nu)_b{}^a (x, y) = -\delta^a{}_b \partial_\nu^y \delta^4 (x - y) . \tag{3.5}$$

In this way, the gauge transformations of the new fields, (3.1), become

$$\delta_\varepsilon \varphi^a = f^a{}_{bc} \partial_\nu \left(A^{\nu c} \varepsilon^b \right) + \partial^\nu \varepsilon_\nu^a. \tag{3.6}$$

Next, we consider the theory described by the action

$$S_0^L \left[A_\mu^a, \varphi^a \right] = S_0^L \left[A_\mu^a \right], \tag{3.7}$$

subject to the gauge transformations (2.3) and (3.6). The new gauge transformations are now irreducible in virtue of (3.4). The irreducible gauge theory is physically equivalent to the reducible one because both theories display the same gauge invariant functions (classical observables). This can be seen as follows. Let $f (A^{\alpha_0}, \varphi^{\alpha_2})$ be a gauge invariant function with respect to (2.3) and (3.6). Then, we have that

$$\frac{\delta f}{\delta A^{\alpha_0}} Z^{\alpha_0}_{\alpha_1} + \frac{\delta f}{\delta \varphi^{\alpha_2}} A_{\alpha_1}^{\alpha_2} = 0. \tag{3.8}$$

Multiplying the last relations by $Z^{\alpha_1}_{\beta_2}$ and summing over α_1 (in the De Witt sense), we derive

$$\frac{\delta f}{\delta \varphi^{\alpha_2}} A_{\alpha_1}^{\alpha_2} Z^{\alpha_1}_{\beta_2} = 0. \tag{3.9}$$

As $A_{\alpha_1}^{\ \alpha_2} Z^{\alpha_1}_{\ \beta_2}$ is by construction invertible, it results that

$$\frac{\delta f}{\delta \varphi^{\alpha_2}} = 0. \tag{3.10}$$

Replacing (3.10) back in (3.8), we find

$$\frac{\delta f}{\delta A^{\alpha_0}} Z^{\alpha_0}_{\ \alpha_1} = 0. \tag{3.11}$$

The last equations state clearly that if $f\left(A^{\alpha_0}, \varphi^{\alpha_2}\right)$ is a gauge invariant function for the irreducible gauge theory, then it is gauge invariant also for the reducible one. Conversely, if $f\left(A^{\alpha_0}\right)$ is a gauge invariant function for the reducible system, then it remains so for the irreducible one because (3.10) hold. In conclusion, the zeroth order cohomological groups of the BRST operator, s, in both reducible and irreducible cases coincide. This indicates that the path integrals respectively associated to the reducible and irreducible situations describe the same theory. It is obvious that the observables in the case of topological Yang-Mills theory are constant as the number of physical degrees of freedom is equal to zero. However, the above proof is more general and instructive when approaching theories with a non-vanishing number of physical degrees of freedom.

Let us pass to the antifield BRST quantization of the irreducible theory. The minimal antifield and ghost spectra are expressed by

$$\left(A_a^{*\mu}, \varphi_a^*, \eta_a^*, \eta_a^{*\mu}\right), \tag{3.12}$$

$$\left(\eta^a, \eta_\mu^a\right), \tag{3.13}$$

with

$$\varepsilon\left(A_a^{*\mu}\right) = \varepsilon\left(\varphi_a^*\right) = 1,\ gh\left(A_a^{*\mu}\right) = gh\left(\varphi_a^*\right) = -1, \tag{3.14}$$

$$\varepsilon\left(\eta_a^*\right) = \varepsilon\left(\eta_a^{*\mu}\right) = 0,\ gh\left(\eta_a^*\right) = gh\left(\eta_a^{*\mu}\right) = -2, \tag{3.15}$$

$$\varepsilon\left(\eta^a\right) = \varepsilon\left(\eta_\mu^a\right) = 1,\ gh\left(\eta^a\right) = gh\left(\eta_\mu^a\right) = 1, \tag{3.16}$$

ε and gh denoting the Grassmann parity, respectively the ghost number. The non-minimal solution of the master equation is given by

$$S = \int d^4x \left(-\frac{1}{4}\varepsilon_{\mu\nu\lambda\rho} F^{a\mu\nu} F_a^{\lambda\rho} + A_a^{*\mu}\left(\left(D_\mu\right)^a_{\ b}\eta^b + \eta_\mu^a\right) + \right.$$

$$\left. \varphi_a^*\left(f^a_{\ bc}\partial_\nu\left(A^{\nu c}\eta^b\right) + \partial^\nu \eta_\nu^a\right) + \cdots + \bar{\eta}_a^* B^a + \bar{\eta}_a^{*\mu} B_\mu^a\right), \tag{3.17}$$

where $\cdots$ signify terms of antighost number greater than one, while $\left(\bar{\eta}^a, \bar{\eta}_\mu^a, B^a, B_\mu^a\right)$ together with their corresponding antifields form the non-minimal sector. We choose the gauge-fixing fermion

$$\psi = \int d^4x \left(\bar{\eta}_a\left(\partial^\mu A_\mu^a + \varphi^a\right) + \bar{\eta}_a^\mu A_\mu^a\right). \tag{3.18}$$

Eliminating in the standard way the antifields from (3.17) with the help of (3.18), we get the gauge-fixed action

$$S_\psi = \int d^4x \left(-\frac{1}{4}\varepsilon_{\mu\nu\lambda\rho} F^{a\mu\nu} F_a^{\lambda\rho} + \left(-\partial^\mu \bar{\eta}_a + \bar{\eta}_a^\mu\right)\left(\left(D_\mu\right)^a_{\ b}\eta^b + \eta_\mu^a\right) + \right.$$

$$\bar{\eta}_a \left(f^a_{\ bc}\partial_\nu \left(A^{\nu c}\eta^b\right) + \partial^\nu \eta^a_\nu\right) + \left(\partial^\mu A^a_\mu + \varphi^a\right) B_a + A^a_\mu B^\mu_a\right). \tag{3.19}$$

It is remarkable that (3.19) has no residual invariances as the term $\bar{\eta}_a \left(f^a_{\ bc}\partial_\nu \left(A^{\nu c}\eta^b\right) + \partial^\nu \eta^a_\nu\right)$ simultaneously freezes the supplementary freedom of the ghosts and antighosts implied by $\left(-\partial^\mu \bar{\eta}_a + \bar{\eta}^\mu_a\right)\left(\left(D_\mu\right)^a_{\ b}\eta^b + \eta^a_\mu\right)$. The corresponding path integral leads, after integration over B^μ_a, A^a_μ , B_a, φ^a, to

$$Z^L_\psi = \int \mathcal{D}\eta^a_\mu \mathcal{D}\bar{\eta}^\mu_a \mathcal{D}\eta^a \mathcal{D}\bar{\eta}_a \exp i\tilde{S}_\psi, \tag{3.20}$$

where

$$\tilde{S}_\psi = \int d^4x \left(\left(-\partial^\mu \bar{\eta}_a + \bar{\eta}^\mu_a\right)\left(\partial_\mu \eta^a + \eta^a_\mu\right) + \bar{\eta}_a \partial^\nu \eta^a_\nu\right). \tag{3.21}$$

The path integral (3.20) furnishes a finite number if one integrates in it also over the ghosts. Thus, our irreducible Lagrangian BRST treatment was proved to be consistent without using the ghosts of ghosts.

4 The irreducible Hamiltonian BRST approach

Here, we specialize to the irreducible Hamiltonian BRST approach, using on the one hand the antifield BRST method with respect to the extended action of the irreducible theory, and on the other, the standard Hamiltonian BRST device with respect to an irreducible set of first-class constraints to be derived below.

We start from the canonical analysis of action (3.7), which outputs the first-class constraints (2.8–2.9), (2.11) and

$$\Pi_a = 0, \tag{4.1}$$

where $\Pi_a \equiv \Pi_{a_1}$ stand for the canonical momenta associated with φ^a. The first-class constraints (2.8–2.9), (2.11) and (4.1) are reducible, the reducibility relations being expressed by (2.14). Initially, we transform the reducible constraints into some irreducible ones in agreement with the suggestion given in [5] (chapter 10, exercise 10.12). The above reducible first-class constraints are equivalent to the irreducible ones (2.8) and

$$\gamma_{a_0} \equiv G_{a_0} + A_{a_0}^{\ \ b_1}\Pi_{b_1} = 0, \tag{4.2}$$

where G_{a_0} is expressed by (2.15), and $A_{a_0}^{\ \ b_1}$ $(b_1 = (a, \vec{x}),\ a_0 = ((b, i, \vec{y}), (b, \vec{y})))$ stands for a matrix chosen such that $Z^{a_0}_{\ \ a_1} A_{a_0}^{\ \ b_1}$ is invertible. Obviously, when (2.9), (2.11) and (4.1) hold, (4.2) also hold. Conversely, when (4.2) hold, (2.9), (2.11) and (4.1) hold. This can be seen as follows. Applying $Z^{a_0}_{\ \ a_1}$ on (4.2), we find

$$Z^{a_0}_{\ \ a_1} A_{a_0}^{\ \ b_1}\Pi_{b_1} = 0, \tag{4.3}$$

which implies precisely (4.1) (because $Z^{a_0}_{\ \ a_1} A_{a_0}^{\ \ b_1}$ is invertible). When (4.1) hold, the first-class constraints (4.2) reduce to (2.9) and (2.11). This establishes the equivalence between the above reducible and irreducible first-class constraints. Requesting

$$A_{a_0}^{\ \ b_1} Z^{a_0}_{\ \ a_1} = -\partial_i \partial^i \delta^{b_1}_{\ \ a_1}, \tag{4.4}$$

we get

$$A_{a_0}^{\ \ b_1} = \left(\delta^b{}_a \partial_i^y \delta^3\left(\vec{x}-\vec{y}\right) \ , \ -f^b{}_{ac} A_i^c\left(y\right) \partial_y^i \delta^3\left(\vec{x}-\vec{y}\right)\right). \tag{4.5}$$

The right hand-side of (4.4) must be understood in the same way as the similar hand-side of (3.4). Inserting (4.5) in (4.2), we arrive at the concrete form of (4.2), namely

$$\gamma_{a_0} \equiv \left(\gamma_i^a \ , \ \gamma^a\right) = 0, \tag{4.6}$$

where

$$\gamma_i^a \equiv \chi_i^a + \partial_i \Pi^a = 0, \tag{4.7}$$

$$\gamma^a \equiv F^a - f^a{}_{bc} A_i^c \partial^i \Pi^b = 0. \tag{4.8}$$

At this moment, the theory possessing the constraints (2.8), (4.7-4.8) can be quantized in the irreducible BRST framework. Accordingly the subsequent development, it is necessary to add the supplementary purely gauge pairs (Φ_1^a, Π_{1a}), (Φ_2^a, Π_{2a}), together with the primary constraints

$$\Pi_{1a} = 0, \tag{4.9}$$

such that their consistencies implies the secondary ones

$$\Pi_{2a} = 0. \tag{4.10}$$

It is clear that (4.9-4.10) are first-class. Adding to any first-class constraint a combination of first-class constraints results also in a first-class constraint. In this way, we can replace (4.9) with the first-class constraints

$$\sigma_a \equiv \Pi_a + \Pi_{1a} = 0, \tag{4.11}$$

because Π_a is a combination of first-class constraints, namely

$$\Pi_a = \frac{1}{\Box}\left(\left(D^i\right)^a{}_b \gamma_i^b + \gamma^a\right). \tag{4.12}$$

The constraint set (2.8), (4.7-4.8), (4.10-4.11) is first-class and also irreducible. The necessity of introducing the pairs (Φ_1^a, Π_{1a}) and (Φ_2^a, Π_{2a}) has a technical nature, and will be explained later. In this way, the number of physical degrees of freedom of the irreducible theory is equal with the starting one. The first-class Hamiltonian of the irreducible theory can be taken under the form

$$H' = \int d^3x \left(A_a^0 \gamma^a + \varphi^a \Pi_{2a} - \Phi_{2a}\left(\left(D^i\right)^a{}_b \gamma_i^b + \gamma^a\right)\right), \tag{4.13}$$

such that the new gauge algebra becomes

$$\left[G^a, G^b\right] = 0, \ \left[G^a, \gamma_i^b\right] = 0, \ \left[G^a, \gamma^b\right] = 0, \ \left[G^a, \sigma_b\right] = \left[G^a, \Pi_{2b}\right] = 0, \tag{4.14}$$

$$\left[\gamma_i^a, \gamma_j^b\right] = 0, \ \left[\gamma_i^a, \gamma_b\right] = f^a{}_{bc}\gamma_i^c, \ \left[\gamma_i^a, \sigma_b\right] = \left[\gamma_i^a, \Pi_{2b}\right] = 0, \tag{4.15}$$

$$\left[\gamma_a, \gamma_b\right] = -f^c{}_{ab}\left(D^i\right)_{cd}\gamma_i^d, \ \left[\gamma_a, \sigma_b\right] = \left[\gamma_a, \Pi_{2b}\right] = 0, \ \left[\sigma_a, \sigma_b\right] = 0, \tag{4.16}$$

$$\left[\sigma_a, \Pi_{2b}\right] = \left[\Pi_{2a}, \Pi_{2b}\right] = 0, \tag{4.17}$$

$$[H', G^a] = \gamma^a,\ [H', \gamma_i^a] = -f^a{}_{bc} A_0^b \gamma_i^c + \partial_i \Pi_2^a, \tag{4.18}$$

$$[H', \gamma_a] = f^c{}_{ab} A_0^b \left(D^i\right)_{cd} \gamma_i^d + f^c{}_{ab} A_i^b \partial^i \Pi_{2c},\ [H', \sigma_a] = \Pi_{2a}, \tag{4.19}$$

$$[H', \Pi_2^a] = -\left(\left(D^i\right)^a{}_b \gamma_i^b + \gamma^a\right). \tag{4.20}$$

With the above gauge algebra at hand, we can write down the gauge transformations of the extended action

$$S_0^E = \int d^4x \left(\dot{A}_i^a \pi_a^i + \dot{A}_0^a \pi_a^0 + \dot{\varphi}^a \Pi_a + \dot{\Phi}_1^a \Pi_{1a} + \dot{\Phi}_2^a \Pi_{2a} - \mathcal{H}' - \right.$$

$$\left. u^a G_a - u_i^a \gamma_a^i - \nu^a \sigma_a - v^a \gamma_a - \mu^a \Pi_{2a}\right). \tag{4.21}$$

It is simply to see that the Lagrangian action corresponding to (4.21) is precisely (3.7). This follows by passing to the total action associated with (4.21) (taking $\mu^a = v^a = 0$), and subsequently eliminating the momenta and the remaining multipliers on their equations of motion. In order to derive the gauge invariances of (4.21), we associate to the constraint functions the following gauge parameters

$$G^a \to \varepsilon_1^a,\ \sigma^a \to \theta_1^a,\ \gamma_i^a \to \varepsilon_i^a,\ \gamma^a \to \varepsilon_2^a,\ \Pi_2^a \to \theta_2^a. \tag{4.22}$$

Then, the gauge transformations of (4.21) result as

$$\delta A_0^a = \varepsilon_1^a,\ \delta A_i^a = \left(D_i\right)^a{}_b \varepsilon_2^b + \varepsilon_i^a,\ \delta\Phi_1^a = \theta_1^a,\ \delta\Phi_2^a = \theta_2^a,\ \delta\pi_0^a = 0, \tag{4.23}$$

$$\delta\varphi^a = \theta_1^a - \partial^i \varepsilon_i^a - f^a{}_{bc} \partial^i \left(A_i^c \varepsilon_2^b\right),\ \delta\Pi_a = \delta\Pi_{1a} = \delta\Pi_{2a} = 0, \tag{4.24}$$

$$\delta\pi_i^a = -2\varepsilon_{0ijk} \left(D^j\right)^a{}_b \varepsilon^{kb} + f^a{}_{bc} \varepsilon_2^b \left(\pi_i^c + \partial_i \Pi^c\right),\ \delta u^a = \dot{\varepsilon}_1^a,\ \delta\nu^a = \dot{\theta}_1^a, \tag{4.25}$$

$$\delta u_i^a = \dot{\varepsilon}_i^a + f^a{}_{bc} \left(\left(A_0^c + v^c\right) \varepsilon_i^b + u_i^c \varepsilon_2^b + \varepsilon_2^b \left(D_i\right)^c{}_d A_0^d + A_0^c \left(D_i\right)^b{}_d \varepsilon_2^d\right) -$$

$$f^a{}_{bc} \left(\varepsilon_2^b \left(D_i\right)^c{}_d v^d + v^c \left(D_i\right)^b{}_d \varepsilon_2^d\right) - \left(D_i\right)^a{}_b \theta_2^b, \tag{4.26}$$

$$\delta v^a = \dot{\varepsilon}_2^a + \theta_2^a - \varepsilon_1^a,\ \delta\mu^a = \dot{\theta}_2^a + \partial^i \varepsilon_i^a + f^a{}_{bc} \partial^i \left(A_i^c \varepsilon_2^b\right) - \theta_1^a. \tag{4.27}$$

Now, we can state that the purely gauge pairs (Φ_1^a, Π_{1a}) and (Φ_2^a, Π_{2a}) have been introduced in order to obtain a gauge algebra of the form (4.14–4.20) that further yields some appropriate terms which, in turn, simplifies the quantization procedure.

Now, we pass to the antifield BRST quantization of the extended action (4.21). The minimal antifield and ghost spectra are expressed by

$$\left(A_a^{*0}, A_a^{*i}, \Phi_{1a}^*, \Phi_{2a}^*, \varphi_a^*\right), \tag{4.28}$$

$$\left(\pi_0^{*a}, \pi_{ka}^*, \Pi_1^{*a}, \Pi_2^{*a}, \Pi^{*a}\right), \tag{4.29}$$

$$\left(u_a^*, \nu_a^*, u_a^{*i}, v_a^*, \mu_a^*\right), \tag{4.30}$$

$$\left(\eta_{1a}^*, \eta_{2a}^*, \eta_a^{*i}, C_{1a}^*, C_{2a}^*\right), \tag{4.31}$$

$$\left(\eta_1^a, \eta_2^a, \eta_i^a, C_1^a, C_2^a\right). \tag{4.32}$$

The antifields (4.28-4.30) have Grassmann parities one and ghost numbers minus one, the antifields (4.31) possess Grassmann parities zero and ghost numbers minus two, while the ghosts (4.32) are of Grassmann parity one and ghost number one. The non-minimal solution of the master equation reads

$$\begin{aligned} S^E = S_0^E + \int d^4x \Big(& A_a^{*0}\eta_1^a + A_a^{*i}\left(\left(D_i\right)^a{}_b\,\eta_2^b + \eta_i^a\right) + \Phi_{1a}^* C_1^a + \Phi_{2a}^* C_2^a + \\ & \varphi_a^*\left(C_1^a - \partial^i\eta_i^a - f^a{}_{bc}\partial^i\left(A_i^c\eta_2^b\right)\right) + u_a^*\dot{\eta}_1^a + \nu_a^*\,\dot{C}_1^a + \\ & \pi_a^{*i}\left(-2\varepsilon_{0ijk}\left(D^j\right)^a{}_b\,\eta_k^b + f^a{}_{bc}\eta_2^b\left(\pi_i^c + \partial_i\Pi^c\right)\right) + \\ & u_a^{*i}\left(\dot{\eta}_i^a + f^a{}_{bc}\left(A_0^c + v^c\right)\eta_i^b + u_i^c\eta_2^b + \eta_2^b\left(D_i\right)^c{}_d\,A_0^d + A_0^c\left(D_i\right)^b{}_d\,\eta_2^b - \right. \\ & \left. f^a{}_{bc}\left(\eta_2^b\left(D_i\right)^c{}_d\,v^d + v^c\left(D_i\right)^b{}_d\,\eta_2^b\right) - \left(D_i\right)^a{}_b\,C_2^b\right) + v_a^*\left(\dot{\eta}_2^a + C_2^a - \eta_1^a\right) + \\ & \mu_a^*\left(\dot{C}_2^a + \partial^i\eta_i^a + f^a{}_{bc}\partial^i\left(A_i^c\eta_2^b\right) - C_1^a\right) + B_\mu^a\bar{\eta}_a^{*\mu} + \cdots \Big), \end{aligned} \tag{4.33}$$

where $\cdots$ signify other terms of antighost number greater that one, which are not important in the context of the further discussion, while $\left(B_\mu^a, \bar{\eta}_\mu^a\right)$ together with their associated antifields form the non-minimal sector. We pick up the gauge-fixing fermion

$$\psi' = \int d^4x \left(u_a^*\left(\partial^i A_i^a + \Phi_1^a\right) + \bar{\eta}_\mu^a A_a^\mu\right). \tag{4.34}$$

With the help of (4.34), we eliminate all the antifields from (4.33) excepting (u_a^*, v_a^*, μ_a^*) which are maintained in favor of their corresponding fields. Switching now to the path integral attached to the gauge-fixed action yielded from (4.33) and further integrating over all the variables except u_a^*, $\bar{\eta}_\mu^a$, η_2^a, C_2^a and η_i^a, we reach the expression

$$Z_{\psi'} = \int \mathcal{D}u_a^*\mathcal{D}\eta_\mu^a\mathcal{D}\bar{\eta}_\mu^a\mathcal{D}\eta_2^a \exp iS', \tag{4.35}$$

where

$$S' = \int d^4x \left(\left(-\partial^\mu u_a^* + \bar{\eta}_a^\mu\right)\left(\partial_\mu\eta_2^a + \eta_\mu^a\right) + u_a^*\partial^\mu\eta_\mu^a\right), \tag{4.36}$$

with $\eta_\mu^a = (C_2^a, \eta_i^a)$. Relations (4.35-4.36) are nothing but (3.20-3.21) modulo the identifications

$$u_a^* \leftrightarrow \bar{\eta}_a. \tag{4.37}$$

In conclusion, we succeeded in replacing the original reducible constraints with some irreducible ones such that the equivalence between the Lagrangian and Hamiltonian treatments is manifest.

In the sequel, we treat the irreducible standard Hamiltonian BRST approach of the model under study. We begin with the irreducible set of first-class constraints (2.8), (4.7-4.8), (4.10-4.11), the first-class Hamiltonian (4.13), and also with the gauge algebra (4.14-4.20). Although the first-order structure functions are field dependent, one can prove that all the higher-order structure functions vanish identically. The minimal ghost and antighost spectra respectively contains the fields

$$G^a \rightarrow \left(\eta_1^a, \mathcal{P}_{1a}\right),\ \sigma^a \rightarrow \left(C_1^a, P_{1a}\right),\ \gamma_i^a \rightarrow \left(\eta_i^a, \mathcal{P}_a^i\right), \tag{4.38}$$

$$\gamma^a \rightarrow (\eta_2^a, \mathcal{P}_{2a}), \ \Pi_2^a \rightarrow (\mathcal{C}_2^a, P_{2a}), \tag{4.39}$$

all the new fields being fermionic, with the ghosts having ghost number one, and the antighosts minus one.

The BRST charge will consequently take the form

$$\Omega = \int d^3x \left(\eta_1^a G_a + \mathcal{C}_1^a \sigma_a + \eta_i^a \gamma_a^i + \eta_2^a \gamma_a + \mathcal{C}_2^a \Pi_{2a} - \frac{1}{2} f^b{}_{ac} \eta_2^a \eta_2^c (D_i)_b{}^d \mathcal{P}_d^i + f^b{}_{ac} \eta_2^a \eta_i^c \mathcal{P}_b^i \right). \tag{4.40}$$

The BRST invariant extension of (4.13) is given by

$$H_B = H' + \int d^3x \left(\eta_1^a \mathcal{P}_{2a} + \eta_a^i \left(-f^a{}_{bc} A_0^b \mathcal{P}_i^c + \partial_i P_2^a\right) + \mathcal{C}_1^a P_{2a} + f^c{}_{ab} \eta_2^a \left(A_0^b (D_i)_c{}^d \mathcal{P}_d^i + A^{ib} \partial_i P_{2c}\right) - \mathcal{C}_2^a \left((D_i)_a{}^b \mathcal{P}_b^i + \mathcal{P}_{2a}\right)\right). \tag{4.41}$$

In order to fix the gauge, we introduce no non-minimal sector, and choose the gauge-fixing fermion

$$K = \int d^3x \left(\mathcal{P}_{1a} \left(\partial^i A_i^a + \Phi_1^a\right) - \mathcal{P}_a^i \left(\dot{A}_i^a + A_i^a - \partial_i A_0^a\right) - P_{1a} \left(A_0^a + \dot{\Phi}_1^a\right)\right). \tag{4.42}$$

Using the formula

$$[K, \Omega] = \int d^3x \left(-\pi_a^0 \left(\partial^i A_i^a + \Phi_1^a\right) + (\Pi_a + \Pi_{1a}) \left(A_0^a + \dot{\Phi}_1^a\right) + \left(\pi_i^a + \varepsilon_{0ijk} F^{jka} + \partial_i \Pi^a\right) \left(\dot{A}_a^i + A_a^i - \partial^i A_a^0\right) + \mathcal{P}_{1a} \mathcal{C}_1^a - P_{1a} \eta_1^a + \dot{P}_{1a} \mathcal{C}_1^a - \left(\partial^i \mathcal{P}_{1a} - \dot{\mathcal{P}}_a^i + \mathcal{P}_a^i\right) \left(\eta_i^a + (D_i)^a{}_b \eta_2^b\right) - \left(\partial_i \mathcal{P}_a^i\right) \eta_1^a\right), \tag{4.43}$$

we find, after simple computation, the Hamiltonian path integral

$$Z_K = \int D\mathcal{P}_{1a} D\mathcal{P}_a^\mu D\eta_2^a D\eta_\mu^a \exp i\bar{S}, \tag{4.44}$$

where

$$\bar{S} = \int d^4x \left(\left(\partial^\mu \mathcal{P}_{1a} + \mathcal{P}_a^\mu\right) \left(\partial_\mu \eta_2^a + \eta_\mu^a\right) - \mathcal{P}_{1a} \partial^\mu \eta_\mu^a\right), \tag{4.45}$$

with the notations

$$\mathcal{P}_a^\mu = \left(P_{1a}, \mathcal{P}_a^i\right), \ \eta_\mu^a = (\mathcal{C}_2^a, \eta_i^a). \tag{4.46}$$

The result expressed by (4.44-4.45) is identical with (4.35-4.36) modulo the identifications

$$\mathcal{P}_{1a} \leftrightarrow -u_a^*, \ \mathcal{P}_a^\mu \leftrightarrow \bar{\eta}_a^\mu. \tag{4.47}$$

Thus, we derived the same path integral as in the previous treatments without introducing ghosts of ghosts. We mention that we used the general results from [13] in order to establish the manifest equivalence between the irreducible Lagrangian and Hamiltonian path integrals derived earlier.

5 Some cohomological arguments

Here, we clarify some aspects linked to the cohomological significance of our procedure. It is well-known that the BRST symmetry, s, is constructed with the help of two nilpotent derivatives, namely the Koszul-Tate operator, δ_K, and the exterior derivative along the gauge orbits, D. Accordingly our discussion from Section 3, the observables from the reducible and irreducible cases coincide, i.e., the zeroth order cohomological groups of D modulo δ_K are the same. In agreement with standard cohomological BRST arguments, all the higher order homological groups of δ_K vanish. However, an obscure point in our irreducible method can be caused by the acyclicity of δ_K [14]-[16]. More precisely, defining the action of δ_K on the fields by

$$\delta_K A_\mu^a = 0,\ \delta_K \varphi^a = 0, \tag{5.1}$$

and on their corresponding antifields by

$$\delta_K A_a^{*\mu} = -\frac{\delta S_0^L}{\delta A_\mu^a} \equiv 0,\ \delta_K \varphi_a^* = -\frac{\delta S_0^L}{\delta \varphi^a} \equiv 0, \tag{5.2}$$

we reach the conclusion that there might exist non-trivial co-cycles of the type

$$M^{ab\cdots c\, de\cdots m}_{\quad\ \mu\nu\cdots\rho} \varphi_a^* \varphi_b^* \cdots \varphi_c^* A_d^{*\mu} A_e^{*\nu} \cdots A_m^{*\rho}, \tag{5.3}$$

in the homology of δ_K. The coefficients $M^{ab\cdots c\, de\cdots m}_{\quad\ \mu\nu\cdots\rho}$ from (5.3) are some functions of A_μ^a and φ^a. Using the definition of δ_K on the antifields $(\eta_a^*, \eta_a^{*\mu})$, namely

$$\delta_K \eta_a^* = -\left(D_\mu\right)^b_{\ a} A_b^{*\mu} + f^b_{\ ac}\left(\partial^\mu \varphi_b^*\right) A_\mu^c, \tag{5.4}$$

$$\delta_K \eta_a^{*\mu} = A_a^{*\mu} - \partial^\mu \varphi_a^*, \tag{5.5}$$

we infer, after some simple computation, and also taking into account (3.4), that

$$\varphi_a^* = \delta_K\left(-\frac{1}{\Box}\left(\left(D_\mu\right)^b_{\ a} \eta_b^{*\mu} + \eta_a^*\right)\right). \tag{5.6}$$

Replacing (5.6) in (5.5), we get that $A_a^{*\mu}$ are also δ_K-exact

$$A_a^{*\mu} = \delta_K\left(\eta_a^{*\mu} - \frac{\partial^\mu}{\Box}\left(\left(D_\mu\right)^b_{\ a} \eta_b^{*\mu} + \eta_a^*\right)\right). \tag{5.7}$$

The last two formulas state clearly that the polynomials (5.3) are δ_K-exact, hence killed in the homology of δ_K. This ensures the acyclicity of the Koszul-Tate operator [17] in our irreducible BRST formalism.

6 Conclusion

In this paper, we quantized the topological Yang-Mills theory accordingly an irreducible BRST manner. Our procedure mainly relies on substituting the original reducible generating set or first-class constraints with some irreducible ones and further quantizing the resulting irreducible gauge theory along the BRST Lagrangian and Hamiltonian lines. The replacement process implies the introduction of some scalar fields and the modification of the initial gauge transformations, respectively of the first-class constraints into some irreducible ones. In consequence, there is no need either for ghosts of ghosts, or their antifields. The acyclicity of the Koszul-Tate operator is made manifest in the context of our treatment.

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[17] In our analysis, we called δ_K the Koszul-Tate operator, although we deal with the irreducible case. As it is well-known, paper [16] is concerning with the reducible case, but we shall traditionally maintain the full name for δ_K.