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Chiral Schwinger Model with the Faddeevian Regularization in the Light-Front Frame: Construction of the Gauge-Invariant Theory Through the Stueckelberg Term, Hamiltonian and BRST Formulations

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Abstract A chiral Schwinger model with the Faddeevian regularization à la Mitra is studied in the light-front frame. The front-form theory is found to be gauge-non-invariant. The Hamiltonian formulation of this gauge-non-invariant theory is first investigated and then the Stueckelberg term for this theory is constructed. Finally, the Hamiltonian and BRST formulations of the resulting gauge-invariant theory, obtained by the inclusion of the Stueckelberg term in the action of the above gauge-non-invariant theory, are investigated with some specific gauge choices.

1 Introduction

The Chiral Schwinger Model (CSM) in one-space one-time dimension has attracted very wide interest in the recent years [1-10] the model describes a massless Dirac field $\psi(x,t)$ in two dimensions with only one of its chiral components coupled to a $U(1)$ gauge field $A^{\mu}(x,t)$ [1]. Jackiw and Rajaraman [1], in particular have considered a gauge anomalous Chiral Schwinger model [1]. By studying the field equations and propagator obtained from the effective gauge field action, they concluded [1] that the theory was not gauge invariant, but was unitary and amenable to particle interpretation [1]. They also found that the vector gauge boson necessarily acquires a mass when consistency and unitarity are demanded.

The Jackiw-Rajaraman model [1-7] is found to admit exact solutions in positive metric Hilbert space, respecting unitarity, provided that the Jackiw-Rajaraman (J-R) regularization parameter a (introduced in Ref. [1] is restricted to the range $a \geq 1$ [1]. In fact, the model is seen to yield a sensible theory for a class of regularizations [1] and the spectrum of the theory is seen to depend on the regularization in a crucial way. The class of regularization that have been considered involve the dimensionless J-R regularization parameter a. For $a \geq 1$ the theory is sensible. The spectrum of the theory for $a > 1$ contains a massive photon in addition to a massless fermion and for $a = 1$ only a massless fermion.

Very recently, Mitra [10] has considered a new regularization which does not belong to the above class. With this regularization [10] the photon is once again massive and the massless fermion present in the theory has (unlike the J-R regularization) a chirality opposite to that entering the interaction with the electromagnetic field [10]. Further, this regularization, being in accordance with the Faddeev's picture [11] of anomalous gauge theories, has been called by Mitra [10] as the "Faddeevian regularization" [10,11].

It is important to mention here at this point that the nature of the matrix of the Poisson brackets of the constraints of the theory decides the nature of the set of constraints of the theory [12] and also as to whether the theory is gauge-invariant or not : so that if the matrix is singular , than the set of constraints of the theory is first-class and the theory is gauge-invariant (and if the matrix is null matrix and therefore also singular then the theory is a true or bonafide gauge-invariant theory) and if the matrix is non-singular than the set of constraints of the theory is second-class and the theory is gauge-non-invariant. Further, in the last case, if the matrix of the Poisson brackets of the constraints of the theory becomes non-singular because of the non-vanishing Poisson bracket of the Gauss law constraint of the theory with itself (-called Faddeev's anomaly [11,10]), so that the constraints become second-class and the theory becomes gauge-non-invariant or it looses gauge-invariance because of this Faddeev's anomaly [11,10] then the theory fits into the Faddeev's scenario [11,10]. In the Chiral Schwinger model with the Faddeevian regularization considered by Mitra [10], the Faddeevian mechanism works because the constraints of the theory become second class through the Faddeev's anomaly for the Gauss law constraint of the theory.

Mitra [10] has studied the Hamiltonian formulation of the above Chiral Schwinger model with the Faddeevian regularization in a recent paper [10,11] in the instant-form [13], where the instant-form theory is seen to be gauge-non-invariant possessing a set of three second-class constraints.

In the present work, we study the above theory in the light-front frame. The front-form theory is also found to be gauge-non-invariant possessing a set of three second-class constraints. The Hamiltonian formulation of this gauge-non-invariant front-form theory is first presented in Section 3A, and then the Stueckelberg term [14,9,7,15] for this theory is constructed. Finally, the Hamiltonian [12] and Becchi-Rouet-Stora and Tyutin (BRST) [16,15,17,9,7] formulations of the resulting theory, obtained by the inclusion of the Stueckelberg term in the action of the above gauge-non-invariant front-form theory, are investigated with some specific gauge choice in Sections 3C and 3D.

Further, in the usual Hamiltonian formulation of a gauge-invariant theory under some gauge-fixing conditions, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under the gauge fixing). To achieve the quantization of a gauge-invariant theory such that the gauge invariance of the theory is maintained even under gauge fixing, one goes to a more generalized procedure called the BRST formulation [16,15,17,9,7]. In the BRST formulation [16,15,17,9,7] of a gauge-invariant theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called the BRST symmetry. For this, one enlarges the Hilbert space of the gauge-invariant theory and replaces the notion of the gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators having different statistics. In view of this, one introduces new anti-commuting variables c and $\bar{c}$ called the Faddeev-Popov ghost and anti-ghost fields, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable b called the Nakanishi-Lautrup field [16,15,17,9,7]. In the BRST formulation, one thus embeds a gauge-invariant theory into a BRST invariant system, and quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator $\bar{Q}$ as well as with the anti-BRST charge operator $\vec{Q}$, the new symmetry of the quantum system (the BRST symmetry) that replaces the gauge invariance is maintained (even under the gauge-fixing) and hence projecting any state onto the sector of BRST and anti-BRST invariant states yields a theory which is isomorphic to the original gauge-invariant theory. The unitarity and consistency of the BRST-invariant theory described by the gauge-fixed quantum Lagrangian is guaranteed by the conservation and nilpotency of the BRST charge Q.

The plan of the paper is as follows. In Section 2, we briefly recapitulate the Chiral Schwinger model with the Faddeevian regularization [10] in the instant-form [13]. In Section 3, we consider this theory in the light-front frame. This front-form theory is also found to be gauge-non-invariant. In Section 3A, the Hamiltonian formulation of this gauge-non-invariant front-form theory is considered. The construction of the gauge-invariant theory and the calculation of the Stueckelberg term for this gauge-non-invariant front-form theory is considered in Section 3B. Finally, the Hamiltonian and BRST formulations of the gauge-invariant front-form theory (obtained by the inclusion of the Stueckelberg term) are studied in Sections 3C and 3D respectively, with some specific gauge choices.

2 The Instant-Form Theory : A Brief Recapitulation [10]

The Chiral Schwinger model with the Faddeevian regularization due to Mitra [10] in one-space one-time dimension in the instant-form [10,13] is described by the bosonized action [10] :

$$\tilde{S}^n = \int \tilde{\mathcal{L}}^n dx\ dt \tag{2.1a}$$

$$\tilde{\mathcal{L}}^n = (\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2) + e(\dot{\phi} + \phi')(A_0 - A_1) + \frac{1}{2}(\dot{A}_1 - A_0')^2 + \frac{1}{2}e^2(A_0 - A_1)^2 - 2e^2 A_1^2 \tag{2.1b}$$

$$g^{\mu\nu} = \text{diag}(+1 - 1) \tag{2.1c}$$

The overdots and primes denote time and space derivatives respectively. The first term in (2.1b) represents [10] a massless boson, which is equivalent to a massless fermion in two dimensions. The second term represents the chiral coupling of this fermion to the electromagnetic field A_μ. The third term is the kinetic energy term of the electromagnetic field. The last two terms involve only the electromagnetic field and may be regarded as a signature of the regularization [10]. Here the sum of the last two terms in (2.1b) (namely, $[\frac{1}{2}e^2(A_0 - A_1)^2 - 2e^2 A_1^2]$) has been chosen [10] as the mass-like term of the model [10], to be compared with the mass-term ($\frac{1}{2}ae^2 A_\mu A^\mu$) of the Jackiw-Rajaraman Chiral Schwinger model [1]; where a is the Jackiw-Rajaraman regularization parameter [1], and it has been called a different (namely, Faddeevian) regularization in Ref. [10].

The action (2.1) is seen to possess a set of three second-class constraints [10] :

$$\Omega_1 = \Pi_0 \approx 0 \tag{2.2a}$$

$$\Omega_2 = [E' + e(\Pi + \phi')] \approx 0 \tag{2.2b}$$

$$\Omega_3 = (A_0' + A_1') \approx 0 \ . \tag{2.2c}$$

Where Ω_1 is a primary constraints and Ω_2 and Ω_3 are secondary constraints [10]. Here, Π, Π_0 and $E(\equiv \Pi^1)$ are the momenta canonically conjugate respectively to ϕ, A_0 and A_1. The matrix of the Poisson brackets of the constraints Ω_i is seen to be non-singular, implying that the set of constraints Ω_i is second-class and that the theory is gauge-non-invariant (GNI) [10]. Using the Hamilton's equations of motion of the theory that preserve the constraints of the theory in the course of time, one can see that A_1 satisfies the Klein-Gordon equation [10] :

$$(\Box + 4e^2)A_1 = 0 \tag{2.3}$$

implying that the photon has a mass $2|e|$. Further, by defining a new field χ by [10] :

$$\chi = \phi + (\frac{1}{2}e)(\dot{A}_1 + A_1') \ ; \tag{2.4}$$

it is seen that χ satisfies [10]

$$\dot{\chi} + \chi' = 0 \tag{2.5}$$

implying that χ is a self-dual boon, and there by showing that the theory contains a chiral boson, which could also be thought of as a chiral fermion [10]. The fields ϕ, A_0 and A_1

could then be expressed in terms of the free massive scalar field A_1 and the free self-dual boson χ (or equivalently a chiral fermion) [10]. For further details of this theory, we refer to the work of Ref. [10].

The mass-like term of this model (i.e., the sum of the last two terms in (2.1b)) does *not* have the Lorentz invariance and therefore the theory (2.1) lacks manifest Lorentz covariance. However, the three Poincaré generators namely, the Hamiltonian operator $H_R(\equiv P_R^o)$, the field momentum operator $P_R(\equiv P_R^1)$; and the Lorentz boost operator $M_R(\equiv M_R^{10})$; all defined on the constraints hypersurface of the theory (i.e., under the constraints of the theory) are seen to satisfy the Poincaré algebra [10] :

$$[P_R^o, P_R^1] = 0 \ ; \ [M_R^{10}, P_R^o] = -iP^1 \ ; [M_R^{10}, P_R^1] = -iP^o \ . \qquad (2.6)$$

In view of this, the theory described by the action (2.1), despite the lack of manifest Lorentz covariance, is seen to be implicitly Lorentz-invariant [10]. In view of the Faddeev's arguments [11]. The gauge-non-invariant theory described by the action (2.1) would have more physical degrees of freedom than the gauge-invariant theories because no gauge-fixing conditions are required for quantizing the theory [10,11]. Also, the spectrum of this theory in this Faddeevian regularization is found [10] to contain a self-dual boson [10]. This is in contrast with the case of the Chiral Schwinger model with the Jackiw-Rajaraman regularization schemes [1].

3 The Theory in the Light-Front Frame

3A. Hamiltonian Formulation of the Gauge-Non-Invariant Theory

In the light-front frame approach one defines the coordinates [13] :

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^o \pm x^1)$$

and then writes all the quantities involved in the action in terms of $x^{\pm}$ instead of x^o and x^1. After doing this the instant-form action of the theory given by (2.1) [10] in the light-front frame [13] becomes :

$$S^N = \int \mathcal{L}^N dx^- dx^+ \qquad (3.1a)$$

$$\begin{aligned}\mathcal{L}^N := (\partial_+\phi)(\partial_-\phi) + 2eA^+(\partial_+\phi) + \frac{1}{2}(\partial_+A^+ - \partial_-A^-)^2 \\ - e^2(A^-)^2 + 2e^2A^+A^-\end{aligned} \qquad (3.1b)$$

$$A^{\pm} = \frac{1}{\sqrt{2}}(A_0 \mp A_1) \; ; \; \partial^{\pm} = \frac{1}{\sqrt{2}}(\partial_0 \mp \partial_1) \; ; \tag{3.1c}$$

$$g^{\mu\nu} := \text{diag}(+1, -1) \; ; \; \mu, \nu = 0, 1 \tag{3.1d}$$

the Greek indices μ, ν, appearing in the text represent Minkowsky indices and take on values 0 and 1. The light-cone canonical momenta obtained from $\mathcal{L}^N$ are

$$\Pi^+ = \frac{\partial \mathcal{L}^N}{\partial(\partial_+ A^-)} = 0 \tag{3.2a}$$

$$\Pi^- = \frac{\partial \mathcal{L}^N}{\partial(\partial_+ A^+)} = (\partial_+ A^+ - \partial_- A^-) \tag{3.2b}$$

$$\Pi = \frac{\partial \mathcal{L}^N}{\partial(\partial_+ \phi)} = (\partial_- \phi + 2eA^+) \tag{3.2c}$$

where Π^+, Π^- and Π are the momenta canonically conjugate respectively to A^-, A^+ and ϕ.

Equations (3.2a) and (3.2c) imply that $\mathcal{L}^N$ possesses two primary constraints :

$$\rho_1 = (\Pi^+) \approx 0 \tag{3.3a}$$

$$\rho_2 = (\Pi - \partial_- \phi - 2eA^+) \approx 0 \; . \tag{3.3b}$$

The canonical Hamiltonian density corresponding to $\mathcal{L}^N$ is :

$$\begin{aligned} \mathcal{H}_C^N &:= \Pi^+(\partial_+ A^-) + \Pi^-(\partial_+ A^+) + \Pi(\partial_+ \phi) - \mathcal{L}^N \\ &= \frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_- A^-) + e^2(A^-)^2 - 2e^2 A^+ A^- \; . \end{aligned} \tag{3.4}$$

After including the primary constraints ρ_1 and ρ_2 in the canonical Hamiltonian density $\mathcal{H}_C^N$ with the help of Lagrange multiplier fields w_1 and w_2, one can write the total Hamiltonian density of the theory $\mathcal{H}_T^N$ as :

$$\begin{aligned} \mathcal{H}_T^N &= \frac{1}{2}(\Pi^-)^2 + (\Pi^-(\partial_- A^-) + e^2(A^-)^2 - 2e^2 A^+ A^- + \Pi^+ w_1 \\ &+ (\Pi - \partial_\phi - 2eA^+) w_2 \; . \end{aligned} \tag{3.5}$$

The Hamilton's equations of motion of the theory obtained from the total Hamiltonian $H_T^N = \int dx^- \mathcal{H}_T^N$ are :

$$\partial_+ \phi = \frac{\partial H_T^N}{\partial \Pi} = w_2 \tag{3.6a}$$

$$-\partial_+\Pi = \frac{\partial H_T^N}{\partial \phi} = \partial_- w_2 \tag{3.6b}$$

$$\partial_+ A^- = \frac{\partial H_T^N}{\partial \Pi^+} = w_1 \tag{3.6c}$$

$$-\partial_+\Pi^- = \frac{\partial H_T^N}{\partial A^+} = (-2e^2 A^- - 2ew_2) \tag{3.6d}$$

$$\partial_+ A^+ = \frac{\partial H_T^N}{\partial \Pi^-} = (\Pi^- + \partial_- A^-) \tag{3.6e}$$

$$-\partial_+\Pi^+ = \frac{\partial H_T^N}{\partial A^-} = (-\partial_-\Pi^- + 2e^2 A^- - 2e^2 A^+) \tag{3.6f}$$

$$\partial_+ w_1 = \frac{\partial H_T^N}{\partial \Pi_{w_1}} = 0 \tag{3.6g}$$

$$-\partial_+\Pi_{w_1} = \frac{\partial H_T^N}{\partial w_1} = \Pi^+ \tag{3.6h}$$

$$\partial_+ w_2 = \frac{\partial H_T^N}{\partial \Pi_{w_2}} = 0 \tag{3.6i}$$

$$-\partial_+\Pi_{w_2} = \frac{\partial H_T^N}{\partial w_2} = (\Pi - \partial_-\phi - 2eA^+) \ . \tag{3.6j}$$

These are the equations of motion of the theory that preserve the constraints of the theory in the course of time. For Poisson bracket $\{\ ,\ \}_p$ of two functions A and B, we choose the convention :

$$\{A(x), B(y)\}_p := \int dz^- \sum_\alpha \left[\frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)} \right] \ . \tag{3.7}$$

Demanding that the primary constraint ρ_1 be preserved in the course of time, we obtain the secondary constraint

$$\rho_3 := \{\rho_1, \mathcal{H}_T^N\}_p = [\partial_-\Pi^- + 2e^2(A^- - A^+)] \approx 0 \ . \tag{3.8}$$

The preservation of ρ_3 for all time does not give rise to any further constraints. The preservation of ρ_2 for all time also does not yield any further constraints. The theory is thus seen to possess only three constraints ρ_1, ρ_2 and ρ_3. The matrix of the Poisson brackets of the constraint ρ_i is

$$
\begin{aligned}
S_{\alpha\beta}(w,z) &= \{\rho_\alpha(w)\ ,\ \rho_\beta(z)\}_p \\
&= \begin{bmatrix} 0 & 0 & 2e^2\delta(w^- - z^-) \\ 0 & -2\partial_-\delta(w^- - z^-) & 2e\partial_-\delta(w^- - z^-) \\ -2e^2\delta(w^- - z^-) & 2e\partial_-\delta(w^- - z^-) & -4e^2\partial_-\delta(w^- - z^-) \end{bmatrix} .
\end{aligned}
\tag{3.9}
$$

The matrix $S_{\alpha\beta}$ is seen to be non-singular implying that the set of constraints ρ_i is second-class and that the theory described by the action S^N (3.1) is a gauge-non-invariant theory. The reduced Hamiltonian of the theory $H_R^N = \int dx^- \mathcal{H}_R^N$, obtained from the total Hamiltonian H_T^N (3.5), after implementation of the constraints ρ_i is given by

$$H_R^N = \int dx^- [\frac{1}{2}(\Pi^-)^2 - e^2(A^-)^2] . \tag{3.10}$$

The Dirac bracket $\{\ ,\ \}_D$ of two functions A and B is defined as [12] :

$$
\begin{aligned}
\{A, B\}_D := \{A, B\}_P - \int dw^- dz^- \sum_{\alpha,\beta} [\{A, \Gamma_\alpha(w)\}_P \\
[\Delta^{-1}_{\alpha\beta}(w,z)]\{\Gamma_\beta(z), B\}_P] .
\end{aligned}
\tag{3.11}
$$

Where Γ_i are the constraints of the theory and $\Delta_{\alpha\beta}(w,z)[:= \{\Gamma_\alpha(w), \Gamma_\beta(z)\}_P]$ is the matrix of the Poisson brackets of the constraints Γ_i. The transition to quantum theory is made by the replacement of the Dirac brackets by the operator commutation relations according to :

$$\{A, B\}_D \to (-i)[A, B]\ ,\ \ i = \sqrt{-1} . \tag{3.12}$$

Finally, the nonvanishing equal light-cone-time ($x^+ = y^+$) commutators obtained for the gauge-non-invariant front-form theory $\mathcal{L}^N$ (3.1) are :

$$[\phi(x^-), \Pi(y^-)] = \frac{3}{2} i\delta(x^- - y^-) \tag{3.13a}$$

$$[\phi(x^-), \Pi^-(y^-)] = -\frac{1}{2} ie\epsilon(x^- - y^-) \tag{3.13b}$$

$$[A^-(x^-), \Pi(y^-)] = -\frac{1}{2e} i\partial_-\delta(x^- - y^-) \tag{3.13c}$$

$$[A^+(x^-), \Pi^-(y^-)] = i\delta(x^- - y^-) \tag{3.13d}$$

$$[\phi(x^-), \phi(y^-)] = -\frac{1}{4} i\epsilon(x^- - y^-) \tag{3.13e}$$

$$[\phi(x^-), A^-(y^-)] = \frac{1}{2e} i\delta(x^- - y^-) \tag{3.13f}$$

$$[A^-(x^-), A^-(y^-)] = -\frac{1}{2e^2} i\partial_-\delta(x^- - y^-) \tag{3.13g}$$

$$[A^-(x^-), A^+(y^-)] = -\frac{1}{2e^2} i\partial_-\delta(x^- - y^-) \tag{3.13h}$$

$$[\Pi(x^-), \Pi(y^-)] = -\frac{1}{2} i\partial_-\delta(x^- - y^-) \tag{3.13i}$$

$$[\Pi(x^-), \Pi^-(y^-)] = -ie\delta(x^- - y^-) \tag{3.13j}$$

$$[\Pi^-(x^-), \Pi^-(y^-)] = -ie^2\epsilon(x^- - y^-) \ . \tag{3.13k}$$

Here $\epsilon(x^- - y^-)$ is a step function defined as :

$$\epsilon(x^- - y^-) = \begin{cases} +1 \ , & (x^- - y^-) > 0 \\ -1 \ , & (x^- - y^-) < 0 \end{cases} \ . \tag{3.14}$$

3B. Construction of the Gauge-Invariant Theory : The Stueckelberg Term

In constructing a gauge-invariant model corresponding to $\mathcal{L}^N$ (3.1), we calculate the Stueckelberg term for $\mathcal{L}^N$. For this, we enlarge the Hilbert space of the theory described by $\mathcal{L}^N$, and introduce a new field θ, called the Stueckelberg field [14,15,9,7], through the following redefinition of fields ϕ and $A^{\pm}$ in the original Lagrangian density $\mathcal{L}^N$ (the motivation for which comes from the gauge transformations (3.27) of the expected gauge-invariant theory (3.16) :

$$\phi \to \Phi = \phi - \theta \ ; \ AA^{\pm} \to A^{\pm} = A^{\pm} + \partial_{\mp}\theta \tag{3.15}$$

the Stueckelberg field θ is a full Quantum field [14,15,9,7]. Performing the changes (3.15) in $\mathcal{L}^N$, we obtain the modified Lagrangian density as :

$$\mathcal{L}^I = \mathcal{L}^N + \mathcal{L}^S \tag{3.16a}$$

with

$$\begin{aligned} \mathcal{L}^S = &[(1 - 2e + 2e^2)(\partial_+\theta)(\partial_-\theta) - (1 - 2e)(\partial_+\phi)(\partial_-\theta) \\ &- (\partial_+\theta)(\partial_-\phi) + 2e(e-1)A^+(\partial_+\theta) - e^2(\partial_+\theta)^2 \\ &- 2e^2A^-(\partial_+\theta - \partial_-\theta)] \ . \end{aligned} \tag{3.16b}$$

Here $\mathcal{L}^S$ is the appropriate Stueckelberg term corresponding to $\mathcal{L}^N$. We shall see later that $\mathcal{L}^I$ describes a gauge-invariant theory possessing a set of three first-class constraints. In fact, we will be able to recover the physical content of the gauge-non-invariant theory described by $\mathcal{L}^N$ (3.1), under some special choice of gauge.

The Euler-Lagrange equations obtained from $\mathcal{L}^I$ are :

$$2\partial_+\partial_-\phi = \partial_-\partial_-\theta + (1-2e)\partial_+\partial_-\theta - 2e\partial_+A^+ \tag{3.17a}$$

$$eJ_+ := \partial_+(\partial_+A^+ - \partial_-A^-) = 2e^2A^- + 2e\partial_+\phi + 2e(e-1)\partial_+\theta \tag{3.17b}$$

$$eJ_- := \partial_-(\partial_+A^+ - \partial_-A^-) = 2e^2(A^- - A^+) + 2e^2(\partial_+\theta - \partial_-\theta) \tag{3.17c}$$

$$\begin{aligned}[2e^2\partial_+\partial_+\theta - (1-2e+2e^2)\partial_+\partial_-\theta + 2(1-e)\partial_+\partial_-\phi + 2e\partial_+A^+ \\ - 2e^2(\partial_+A^+ + \partial_-A^- - \partial_+A^-)] = 0\ .\end{aligned} \tag{3.17d}$$

The vector current (J^μ) has the divergence :

$$\begin{aligned}\partial_\nu J^\nu :&= \frac{1}{e}\partial_\nu(\partial_\mu F^{\mu\nu}) = \partial_+J_- + \partial_-J_+ \\ &= 2\partial_+\partial_-\phi + (2e^2-1)\partial_+\partial_-\theta + 2e\partial_+A^+ \\ &= 2e(e-1)\partial_+(\partial_-\theta) + \partial_-(\partial_-\theta)\end{aligned} \tag{3.18}$$

which is seen to vanish in the gauge $-(\partial_-\theta) = 0$ (which is a consequence of the unitary gauge choice $\partial^\mu\theta = 0$ to be considered later). This implies that the theory possesses at the classical level, a vector gauge symmetry under the gauge $\partial^\mu\theta = 0$. The divergence of the axial-vector current J_5^μ, at the same time, is (by virtue of the Euler-Lagrange equations (3.17)) non-zero ($\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}, \varepsilon^{01} = +1$) :

$$\partial_\mu J_5^\mu := \frac{1}{2}\varepsilon^{\mu\nu}F_{\mu\nu} = -(\partial_+A^+ - \partial_-A^-) \neq 0 \tag{3.19}$$

this further implies that under the gauge $\partial^\mu\theta = 0$ the theory $\mathcal{L}^I$ does not possess any vector-gauge anomaly. On the other hand, for the gauge-non-invariant theory $\mathcal{L}^N$, one has a nonvanishing divergence of vector-current (J^μ), implying that the latter theory is anomalous.

3C. Hamiltonian Formulation of the Gauge-Invariant Theory

The light-cone canonical momenta for the gauge-invariant theory described by $\mathcal{L}^I$ (with Π_θ being the light-cone momentum canonically conjugate to the Stueckelberg field θ) are :

$$\Pi^+ := \frac{\partial\mathcal{L}^I}{\partial(\partial_+A^-)} = 0 \tag{3.20a}$$

$$\Pi^- := \frac{\partial \mathcal{L}^I}{\partial(\partial_+ A^+)} = (\partial_+ A^+ - \partial_- A^-) \tag{3.20b}$$

$$\Pi := \frac{\partial \mathcal{L}^I}{\partial(\partial_+ \phi)} = \partial_- \phi + 2eA^+ - (1-2e)(\partial_- \theta) \tag{3.20c}$$

$$\Pi_\theta := \frac{\partial \mathcal{L}^I}{\partial(\partial_+ \theta)} = 2e^2(\partial_- \theta + A^+ - A^-) - 2e^2(\partial_+ \theta) - \Pi \ . \tag{3.20d}$$

Equations (3.20a) and (3.20c) imply that $\mathcal{L}^I$ possesses two primary constraints :

$$\psi_1 := \Pi^+ \approx 0 \tag{3.21a}$$

$$\psi_2 := [\Pi - \partial_- \phi - 2eA^+ + (1-2e)\partial_- \theta] \approx 0 \ . \tag{3.21b}$$

The canonical Hamiltonian density corresponding to $\mathcal{L}^I$ is :

$$\begin{aligned} \mathcal{H}_C^I &:= \Pi^+(\partial_+ A^-) + \Pi^-(\partial_+ A^+) + \Pi(\partial_+ \phi) + \Pi_\theta(\partial_+ \theta) - \mathcal{L}^I \\ &= \{\frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_- A^-) + e^2(A^-)^2 - 2e^2 A^+ A^- \\ &- 2e^2 A^-(\partial_- \theta) - \frac{1}{4e^2}[\Pi_\theta - (1-2e+2e^2)(\partial_- \theta) + \partial_- \phi \\ &- 2e(e-1)A^+ + 2e^2 A^-]^2\} \ . \end{aligned} \tag{3.22}$$

After including the primary constraints ψ_1 and ψ_2 in the canonical Hamiltonian density $\mathcal{H}_C^I$ with the help of Lagrange multiplier fields u and v, one can write the total Hamiltonian density of the theory $\mathcal{H}_T^I$ as :

$$\mathcal{H}_T^I = \mathcal{H}_C^I + \Pi^+ u + [\Pi - \partial_- \phi - 2eA^+ + (1-2e)\partial_- \theta]v \tag{3.23}$$

the Hamilton's equations of motion of the theory obtained from the total Hamiltonian $H_T^I = \int dx^- \mathcal{H}_T^I$, are :

$$\partial_+ \phi = \frac{\partial H_T^I}{\partial \Pi} = v \tag{3.24a}$$

$$\begin{aligned} -\partial_+ \Pi = \frac{\partial H_T^I}{\partial \phi} &= \frac{1}{2e^2}[\partial_- \Pi_\theta - (1-2e+2e^2)\partial_- \partial_- \theta + \partial_- \partial_- \phi \\ &- 2e(e-1)\partial_- A^+ + 2e^2 \partial_- A^-] + \partial_- v \end{aligned} \tag{3.24b}$$

$$\partial_+ A^- = \frac{\partial H_T^I}{\partial \Pi^+} = u \tag{3.24c}$$

$$-\partial_+\Pi^- = \frac{\partial H_T^I}{\partial A^+} = [\frac{2e(e-1)}{2e^2}\{\Pi_\theta - (1-2e+2e^2)\partial_-\theta + \partial_-\phi - 2e(e-1)A^+ + 2e^2A^-\} - 2e^2A^- - 2ev] \tag{3.24d}$$

$$\partial_+A^+ = \frac{\partial H_T^I}{\partial \Pi^-} = \Pi^- + \partial_-A^- \tag{3.24e}$$

$$-\partial_+\Pi^+ = \frac{\partial H_T^I}{\partial A^-} = [-\{\Pi_\theta - (1-2e+2e^2)\partial_-\theta + \partial_-\phi - 2e(e-1)A^+ + 2e^2A^-\} - \partial_-\Pi^- + 2e^2(A^- - A^+) - 2e^2\partial_-\theta] \tag{3.24f}$$

$$\partial_+\theta = \frac{\partial H_T^I}{\partial \Pi_\theta} = \frac{-1}{2e^2}[\Pi_\theta - (1-2e+2e^2)\partial_-\theta + \partial_-\phi - 2e(e-1)A^+ + 2e^2A^-] \tag{3.24g}$$

$$-\partial_+\Pi_\theta = \frac{\partial H_T^I}{\partial \theta} = [-\frac{1}{2e^2}(1-2e+2e^2)\partial_-\{-(1-2e+2e^2)\partial_-\theta + \Pi_\theta + \partial_-\phi - 2e(e-1)A^+ + 2e^2A^-\} + 2e^2\partial_-A^- - (1-2e)\partial_-v] \tag{3.24h}$$

$$\partial_+u = \frac{\partial H_T^I}{\partial \Pi_u} = 0 \tag{3.24i}$$

$$-\partial_+\Pi_u = \frac{\partial H_T^I}{\partial u} = \Pi^+ \tag{3.24j}$$

$$\partial_+v = \frac{\partial H_T^I}{\partial \Pi_v} = 0 \tag{3.24k}$$

$$-\partial_+\Pi_v = \frac{\partial H_T^I}{\partial v} = [\Pi - \partial_-\phi - 2eA^+ + (1-2e)\partial_-\theta]\ . \tag{3.24l}$$

These are the equations of motion of the theory that preserve the constraints of the theory in the course of time. Demanding that the primary constraint ψ_1 be preserved in the course of time, we obtain the secondary constraint

$$\psi_3 := \{\psi_1, \mathcal{H}_T^I\}_P = [\partial_-\Pi^- + \Pi_\theta - (1-2e)(\partial_-\theta) + \partial_-\phi + 2eA^+] \approx 0\ . \tag{3.25}$$

The preservation of ψ_3 for all time does not give rise to any further constraints. The preservation of ψ_2 for all time also does not yield any further constraints. The theory is thus seen to possess three constraints ψ_1, ψ_2 and ψ_3. The matrix of the Poisson brackets of the constraints ψ_i is :

$$\begin{aligned} T_{\alpha\beta}(w,z) &:= \{\psi_\alpha(w), \psi_\beta(z)\}_P \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\partial_-\delta(w^- - z^-) & 2\partial_-\delta(w^- - z^-) \\ 0 & 2\partial_-\delta(w^- - z^-) & -2\partial_-\delta(w^- - z^-) \end{bmatrix} . \end{aligned} \tag{3.26}$$

The matrix $T_{\alpha\beta}$ is clearly singular implying that the set of constraints ψ_i is first-class and that the theory described by $\mathcal{L}^I$ is a gauge-invariant theory. In fact, the Lagrangian density $\mathcal{L}^I$ is seen to be invariant under the time-dependent chiral gauge transformations :

$$\begin{aligned} &\delta A^+ = \partial_-\beta \, , \delta A^- = \partial_+\beta \, , \delta\phi = -\beta \, , \delta\theta = -\beta \, , \\ &\delta u = \partial_+\partial_+\beta \, , \delta v = -\partial_+\beta \end{aligned} \tag{3.27a}$$

$$\delta\Pi^+ = \delta\Pi^- = \delta\Pi = \delta\Pi_\theta = \delta\Pi_u = \delta\Pi_v = 0 \tag{3.27b}$$

where $\beta \equiv \beta(x^+, x^-)$ is an arbitrary function of its arguments. The reduced Hamiltonian of the theory $H_R^I = \int dx^- \mathcal{H}_R^I$, obtained from H_T^I, after the implementation of the constraints ψ_i is given by :

$$\begin{aligned} \mathcal{H}_R^I = \int dx^- [\frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_- A^-) + e^2(A^-)^2 - 2e^2 A^+ A^- - 2e^2 A^- \partial_-\theta \\ - \frac{1}{4e^2}(2e^2\partial_-\theta - \partial_-\Pi^- - 2e^2 A^+ + 2e^2 A^-)^2] . \end{aligned} \tag{3.28}$$

In order to quantize the theory using Dirac's procedure [12], one has to convert the set of first-class constraints of the theory into a set of second-class ones. This one can achieve by imposing arbitrarily, some additional constraints on the system in the form of gauge-fixing conditions or the gauge-constraints. For this, we go to a special gauge given by $\partial^\mu\theta = 0$ (or equivalently $\partial_+\theta = 0$ and $-\partial_-\theta = 0$), and accordingly choose the gauge-fixing conditions of the theory as [4,15,9] :

$$\varsigma_1 = -(\partial_-\theta) \approx 0 \tag{3.29a}$$

$$\varsigma_2 = [\Pi_\theta + \partial_-\phi - 2e(e-1)A^+ + 2e^2 A^-] \approx 0 . \tag{3.29b}$$

With the gauge-fixing conditions (3.29), the total set of constraints of the theory becomes

$$\xi_1 = \psi_1 = \Pi^+ \approx 0 \tag{3.30a}$$

$$\xi_2 = \psi_2 = [\Pi - (\partial_-\phi) - 2eA^+ + (1-2e)(\partial_-\theta)] \approx 0 \tag{3.30b}$$

$$\xi_3 = \psi_3 = [(\partial_-\Pi^-) + \Pi_\theta - (1-2e)(\partial_-\theta) + (\partial_-\phi) + 2eA^+] \approx 0 \tag{3.30c}$$

$$\xi_4 = \varsigma_1 = -(\partial_-\theta) \approx 0 \tag{3.30d}$$

$$\xi_5 = \varsigma_2 = [\Pi_\theta + \partial_-\phi - 2e(e-1)A^+ + 2e^2A^-] \approx 0\ . \tag{3.30e}$$

The matrix of the Poisson brackets of the constraints ξ_i, namely, $M_{\alpha\beta}(w,z) := \{\xi_\alpha(w), \xi_\beta(z)\}$ is then calculated. The nonvanishing matrix elements of the matrix $M_{\alpha\beta}(w,z)$ are :

$$M_{15} = -M_{51} = -2e^2\delta(w^- - z^-) \tag{3.31a}$$

$$M_{22} = M_{33} = -2\partial_-\delta(w^- - z^-) \tag{3.31b}$$

$$M_{23} = M_{32} = 2\partial_-\delta(w^- - z^-) \tag{3.31c}$$

$$M_{25} = M_{52} = 2(1-e)\partial_-\delta(w^- - z^-) \tag{3.31d}$$

$$M_{34} = M_{43} = -\partial_-\delta(w^- - z^-) \tag{3.31e}$$

$$M_{35} = M_{53} = (2e^2 - 1)\partial_-\delta(w^- - z^-) \tag{3.31f}$$

$$M_{45} = M_{54} = -\partial_-\delta(w^- - z^-)\ . \tag{3.31g}$$

The matrix $M_{\alpha\beta}$ is seen to be nonsingular and therefore its inverse exists. The nonvanishing elements of the inverse of the matrix $M_{\alpha\beta}$ (i.e., the elements of the matrix $(M^{-1})_{\alpha\beta}$ are :

$$(M^{-1})_{11} = \frac{-1}{2e^2}\partial_-\delta(w^- - z^-) \tag{3.32a}$$

$$(M^{-1})_{12} = -(M^{-1})_{21} = [\frac{-1}{2e}]\delta(w^- - z^-) \tag{3.32b}$$

$$(M^{-1})_{13} = -(M^{-1})_{31} = [\frac{-1}{2e^2}]\delta(w^- - z^-) \tag{3.32c}$$

$$(M^{-1})_{14} = -(M^{-1})_{41} = [\frac{(2e^2 - 2e + 1)}{2e^2}]\delta(w^- - z^-) \tag{3.32d}$$

$$(M^{-1})_{15} = -(M^{-1})_{51} = [\frac{1}{2e^2}]\delta(w^- - z^-) \tag{3.32e}$$

$$(M^{-1})_{22} = [\frac{-1}{4}]\epsilon(w^- - z^-) \tag{3.32f}$$

$$(M^{-1})_{24} = (M^{-1})_{42} = [\frac{-1}{2}]\epsilon(w^- - z^-) \tag{3.32g}$$

$$(M^{-1})_{34} = (M^{-1})_{43} = [\frac{-1}{2}]\epsilon(w^- - z^-) \tag{3.32h}$$

and

$$\int dz^- M(x,z) M^{-1}(z,y) = 1_{5\times 5}\delta(x^- - y^-) \ . \tag{3.33}$$

The nonvanishing equal light-cone time ($x^+ = y^+$) commutators of the gauge-invariant front-form described by $\mathcal{L}^I$ under the gauge (3.29) are finally obtained as :

$$[\phi(x^-), \Pi(y^-)] = \frac{3}{2} i\delta(x^- - y^-) \tag{3.34a}$$

$$[\phi(x^-), \Pi^-(y^-)] = -\frac{1}{2} ie\epsilon(x^- - y^-) \tag{3.34b}$$

$$[A^-(x^-), \Pi(y^-)] = -\frac{1}{2e}\partial_-\delta(x^- - y^-) \tag{3.34c}$$

$$[A^+(x^-), \Pi^-(y^-)] = i\delta(x^- - y^-) \tag{3.34d}$$

$$[\phi(x^-), \phi(y^-)] = -\frac{1}{4} i\epsilon(x^- - y^-) \tag{3.34e}$$

$$[\phi(x^-), A^-(y^-)] = \frac{1}{2e} i\delta(x^- - y^-) \tag{3.34f}$$

$$[A^-(x^-), A^-(y^-)] = -\frac{1}{2e^2} i\partial_-\delta(x^- - y^-) \tag{3.34g}$$

$$[A^-(x^-), A^+(y^-)] = -\frac{1}{2e^2} i\partial_-\delta(x^- - y^-) \tag{3.34h}$$

$$[\Pi(x^-), \Pi(y^-)] = -\frac{1}{2} i\partial_- \delta(x^- - y^-) \quad (3.34i)$$

$$[\Pi(x^-), \Pi^-(y^-)] = -ie\delta(x^- - y^-) \quad (3.34j)$$

$$[\Pi^-(x^-), \Pi^-(y^-)] = -ie^2 \epsilon(x^- - y^-) \quad (3.34k)$$

$$[\phi(x^-), \Pi_\theta(y^-)] = \frac{1}{2} i(1 + 2e)\delta(x^- - y^-) \quad (3.34l)$$

$$[A^-(x^-), \Pi_\theta(y^-)] = \frac{1}{2e} i\partial_- \delta(x^- - y^-) \quad (3.34m)$$

$$[A^+(x^-), \Pi_{theta}(y^-)] = -i\partial_- \delta(x^- - y^-) \quad (3.34n)$$

$$[\theta(x^-), \Pi_\theta(y^-)] = 2i\delta(x^- - y^-) \quad (3.34o)$$

$$[\Pi(x^-), \Pi_\theta(y^-)] = \frac{1}{2} i(1 - 2e)\partial_- \delta(x^- - y^-) \quad (3.34p)$$

$$[\Pi^-(x^-), \Pi_\theta(y^-)] = ie(e - 1)\delta(x^- - y^-) \quad (3.34q)$$

$$[\Pi_\theta(x^-), \Pi_\theta(y^-)] = -\frac{1}{2} i(2e - 1)^2 \partial_- \delta(x^- - y^-) \ . \quad (3.34r)$$

Following the sequence of reasoning offered in [4,15,9,7], it is easy to see that (3.34) together with $\mathcal{H}_C^I$ (3.22) under the gauge (3.29), reproduce precisely the quantum system described by $\mathcal{L}^N$ (3.1) [4,15,9.7]. The gauge (3.29) translates the gauge-invariant version of the theory described by $\mathcal{L}^I$ into the gauge-non-invariant one described by $\mathcal{L}^N$. A comparison of (3.34) and (3.13) reveals that (3.34a - 3.34k) coincide completely with (3.13) as they should. The additional commutators appearing in (3.34) (viz., (3.34l)-(3.34r)) express merely the dependence on θ and Π_θ. In fact, the physical Hilbert spaces of the two theories ($\mathcal{L}^I$ and $\mathcal{L}^N$) are the same. The addition of the Stueckelberg term ($\mathcal{L}^S$) to the theory (i.e., to $\mathcal{L}^N$) enlarges only the unphysical part of the full Hilbert space of the theory $\mathcal{L}^N$, without modifying the physical content of the theory. The Stueckelberg field θ itself, in fact, represents only an unphysical degree of freedom and correspondingly the physics of the theories with and without the Stueckelberg term remains the same [4,15.9,7].

For the later use (in the next section), for considering the BRST formulation of the gauge-invariant theory described by $\mathcal{L}^I$, we convert the total Hamiltonian density $\mathcal{H}_T^I$ into the first-order Lagrangian density

$$\begin{aligned}
\mathcal{L}_{IO}^I &= \Pi^+(\partial_+ A^-) + \Pi^-(\partial_+ A^+) + \Pi(\partial_+\phi) + \Pi_\theta(\partial_+\theta) \\
&+ \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) - \mathcal{H}_T^I \\
&= \{-\frac{1}{2}(\Pi^-)^2 + \Pi_\theta(\partial_+\theta) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) + \Pi^-(\partial_+ A^+) \\
&- e^2(A^-)^2 + 2e^2 A^+ A^- + 2e^2 A^-(\partial_-\theta) - [(-\partial_-\phi - 2eA^+) \\
&+ (1-2e)\partial_-\theta](\partial_+\phi) + \frac{1}{4e^2}[\Pi_\theta - (1 - 2e + 2e^2)(\partial_-\theta) \\
&+ \partial_-\phi - 2e(e-1)A^+ + 2e^2 A^-]^2\} \,.
\end{aligned} \tag{3.35}$$

In the above equation (3.35), the terms $\Pi^+(\partial_+ A^- - u)$ and $\Pi(\partial_+\phi - v)$ drop out in view of the Hamilton's equations (3.24c) and (3.24a).

3D. The BRST Formulation of the Gauge-Invariant Theory

3D1. The BRST Invariance

For the BRST formulation of the theory, we rewrite the gauge-invariant theory $\mathcal{L}^I$ as a quantum system which possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge-invariant model $\mathcal{L}^I$ and replace the notion of gauge transformation which shifts operators by c-number functions by a BRST transformation which mixes operators with Bose and Fermi statistics. We then introduce new anti-commuting variables c and $\bar{c}$ (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable b such that [14,15,9,7,17] :

$$\hat{\delta}\phi = \hat{\delta}\theta = -c \; ; \hat{\delta}A^+ = \partial_- c \; ; \hat{\delta}A^- = \partial_+ c \; ; \hat{\delta}u = \partial_+\partial_+ c \tag{3.36a}$$

$$\hat{\delta}v = -\partial_+ c \; ; \hat{\delta}\Pi = \hat{\delta}\Pi_\theta = \hat{\delta}\Pi^+ = \hat{\delta}\Pi^- = \hat{\delta}\Pi_u = \hat{\delta}\Pi_v = 0 \tag{3.36b}$$

$$\hat{\delta}c = 0 \; ; \hat{\delta}\bar{c} = b \; , \hat{\delta}b = 0 \tag{3.36c}$$

with the property $\hat{\delta}^2 = 0$. We could now define a BRST-invariant function of the dynamical variables to be a function $f(\phi, A^+, A^-, \theta, u, v, c, \bar{c}, b, \Pi, \Pi^+, \Pi^-, \Pi_\theta, \Pi_u, \Pi_v, \Pi_c, \Pi_{\bar{c}}, \Pi_b)$ such that $\hat{\delta}f = 0$.

3D2. Gauge-Fixing in the BRST Formalism

Performing gauge-fixing in the BRST formalism implies adding to the first-order Lagrangian density (3.35) a trivial BRST-invariant function [14,15,9,7,17]. We thus write the gauge-fixing quantum Lagrangian density (taking e.g. a trivial BRST-invariant function as follows [14,15,9,7,17] :

$$
\begin{aligned}
\mathcal{L}^I_{BRST} &:= \mathcal{L}^I_{IO} + \hat{\delta}[\bar{c}(\partial_+ A^- + \partial_- \theta + \frac{1}{2} b + A^+ - \phi] \\
&= -\frac{1}{2}(\Pi^-)^2 + \Pi_\theta(\partial_+\theta) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) - e^2(A^-)^2 + \Pi^-(\partial_+ A^+) \\
&+ 2e^2 A^+ A^- + 2e^2 A^-(\partial_-\theta) - [-\partial_-\phi - 2eA^+ \\
&+ (1-2e)(\partial_-\theta)](\partial_+\phi) \\
&+ \frac{1}{4e^2}[\Pi_\theta - (1-2e+2e^2)(\partial_-\theta) + \partial_-\phi - 2e(e-1)A^+ \\
&+ 2e^2 A^-]^2 + \hat{\delta}[\bar{c}(\partial_+ A^- + \partial_-\theta + \frac{1}{2} b + A^+ - \phi)]
\end{aligned}
\tag{3.37}
$$

the last term in the above equation (3.37) is the extra BRST-invariant gauge-fixing term. Using the definition of $\hat{\delta}$ we can rewrite $\mathcal{L}_{BRST}$ (with one integration by parts) :

$$
\begin{aligned}
\mathcal{L}_{BRST} &= -\frac{1}{2}(\Pi^-)^2 + \Pi_\theta(\partial_+\theta) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) - e^2(A^-)^2 + \Pi^-(\partial_+ A^+) \\
&+ 2e^2 A^+ A^- + 2e^2 A^-(\partial_-\theta) - [-\partial_-\phi - 2eA^+ \\
&+ (1-2e)\partial_-\theta](\partial_+\phi) + \frac{1}{4e^2}[\Pi_\theta - (1-2e+2e^2)(\partial_-\theta) \\
&+ \partial_-\phi - 2e(e-1)A^+ + 2e^2 A^-]^2 + \frac{1}{2} b^2 + b(\partial_+ A^- \\
&+ \partial_-\theta + A^+ - \phi) + (\partial_+\bar{c})(\partial_+ c) - \bar{c}c \,.
\end{aligned}
\tag{3.38}
$$

Proceeding classically, the Euler-Lagrange equation for b reads :

$$-b = (\partial_+ A^- + \partial_-\theta + A^+ - \phi) \tag{3.39}$$

the requirement $\hat{\delta} b = 0$ (cf. (3.36c)) then implies :

$$-\hat{\delta} b = [\hat{\delta}(\partial_+ A^-) + \hat{\delta}(\partial_-\theta) + \hat{\delta} A^+ - \hat{\delta}\phi] = 0 \tag{3.40}$$

which in turn implies

$$-\partial_+(\partial_+ c) = c \,. \tag{3.41}$$

The above equation is also an Euler-Lagrange equation obtained by the variation of $\mathcal{L}_{BRST}$ with respect to $\bar{c}$. We now define the bosonic momenta in the usual way so that :

$$\Pi^+ := \frac{\partial \mathcal{L}_{BRST}}{\partial(\partial_+ A^-)} = +b \tag{3.42}$$

the fermionic momenta are, however, defined using the directional derivatives such that [14,15,9,7,17] :

$$\Pi_c := \mathcal{L}_{BRST} \frac{\partial^{\leftarrow}}{\partial(\partial_+ c)} = \partial_+ \bar{c} \; ; \Pi_{\bar{c}} := \frac{\vec{\partial}}{\partial(\partial_+ \bar{c})} \mathcal{L}_{BRST} = \partial_+ c \tag{3.43}$$

implying that the variable canonically conjugate to c is $(\partial_+ \bar{c})$ and the variable conjugate to $\bar{c}$ is $(\partial_+ c)$. In constructing the quantum Hamiltonian density $\mathcal{H}_{BRST}$ from the Lagrangian density in the usual way one has to keep in mind that the former has to be Hermitian. Accordingly, we have [14,15,9,7,17] :

$$\begin{aligned} \mathcal{H}_{BRST} &:= \Pi^+(\partial_+ A^-) + \Pi^-(\partial_+ A^+) + \Pi(\partial_+ \phi) + \Pi_\theta(\partial_+ \theta) \\ &+ \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) + \Pi_c(\partial_+ c) + (\partial_+ \bar{c})\Pi_{\bar{c}} - \mathcal{L}_{BRST} \\ &= \frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_- A^-) + e^2(A^-)^2 - 2e^2 A^+ A^- - 2e^2 A^-(\partial_- \theta) \\ &- \frac{1}{4e^2}[\Pi_\theta - (1 - 2e + 2e^2)(\partial_- \theta) + \partial_- \phi - 2e(e-1)A^+ \\ &+ 2e^2 A^-]^2 - \frac{1}{2}(\Pi^+)^2 - \Pi^+(\partial_- \theta + A^+ - \phi) + \Pi_c \Pi_{\bar{c}} + \bar{c}c \; . \end{aligned} \tag{3.44}$$

We can check the consistency of (3.43) with (3.44) by looking at Hamilton's equations for the fermionic variables i.e. (cf. Ref. [14,15,9,7,17])

$$\partial_+ c = \frac{\vec{\partial}}{\partial \Pi_c} \mathcal{H}_{BRST} \; ; \partial_+ \dot{\bar{c}} = \mathcal{H}_{BRST} \frac{\partial^{\leftarrow}}{\partial \Pi_{\bar{c}}} \tag{3.45}$$

thus we see that

$$\partial_+ c = \frac{\vec{\partial}}{\partial \Pi_c} \mathcal{H}_{BRST} = \Pi_{\bar{c}} \; ; \partial_+ \bar{c} = \mathcal{H}_{BRST} \frac{\partial^{\leftarrow}}{\partial \Pi_{\bar{c}}} = \Pi_c \tag{3.46}$$

is in agreement with (3.43). For the operators c, $\bar{c}$, $\partial_+ c$ and $\partial_+ \bar{c}$, one needs to specify the anti-commutation relations of $\partial_+ c$ with $\bar{c}$ or of $\partial_+ \bar{c}$ with c, but not of c with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that [14,15,9,7,17] :

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0 \; ; \partial_+ \{\bar{c}, c\} = 0 \tag{3.47a}$$

$$\{\partial_+ \bar{c}, c\} = -\{\partial_+ c, \bar{c}\} \tag{3.47b}$$

where { , } means an anticommutator. We thus see that the anticommutators in (3.47b) are non-trivial and need to be fixed. In order to fix these, we demand that c satisfy the Heisenberg equation [14,15,9,7,17] :

$$[c, \mathcal{H}_{BRST}] = i\partial_+ c \tag{3.48}$$

and using the property $c^2 = \bar{c}^2 = 0$, one obtains

$$[c, \mathcal{H}_{BRST}] = \{\partial_+ \bar{c}, c\} \partial_+ c \tag{3.49}$$

Equations (3.47)-(3.49) then imply

$$\{\partial_+ \bar{c}, c\} = -\{\partial_+ c, \bar{c}\} = i \ . \tag{3.50}$$

Here the minus sign in the above equation is non-trivial and implies the existence of states with negative norm in the space of state vectors of the theory [14,15,9,7,17].

3D3. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformation (3.36). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators that satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy :

$$[\Pi, Q] = (-\partial_- c - \partial_- \partial_+ c) \ ; [\phi, Q] = \partial_+ c \tag{3.51a}$$

$$[A^+, Q] = \partial_- c \ ; [A^-, Q] = \partial_+ c \tag{3.51b}$$

$$[\theta, Q] = -c \ ; [\Pi_\theta, Q] = (1 - 2e)[\partial_- c + \partial_- \partial_+ c] \tag{3.51c}$$

$$\{\bar{c}, Q\} = \partial_- \phi + 2eA^+ - \Pi^+ - \Pi - (1 - 2e)(\partial_- \theta) \tag{3.51d}$$

$$[\Pi^-, Q] = 2e(c + \partial_+ c) \tag{3.51e}$$

$$\{\partial_+ \bar{c}, Q\} = (1 - 2e)\partial_- \theta - \partial_- \Pi^- - \Pi_\theta - \partial_- \phi - 2eA^+ \ . \tag{3.51f}$$

All other commutators and anti-commutators involving Q vanish. In view of (3.51), the BRST charge operator for the present theory can be written as [14,15,9,7,17] :

$$\begin{aligned} Q = \int dx^- [& ic\{\partial_- \Pi^- + \Pi_\theta - (1 - 2e)\partial_- \theta + \partial_- \phi + 2eA^+\} \\ & - i(\partial_+ c)\{\Pi^+ + \Pi - \partial_- \phi - 2eA^+ + (1 - 2e)\partial_- \theta\}] \end{aligned} \tag{3.52}$$

this equation implies that the set of states satisfying the condition

$$\Pi^+ |\psi\rangle = 0 \tag{3.53a}$$

$$[\Pi - \partial_{-}\phi - 2eA^{+} + (1-2e)\partial_{-}\theta]|\psi\rangle = 0 \tag{3.53b}$$

$$[\partial_{-}\Pi^{-} + \Pi_{\theta} - (1-2e)\partial_{-}\theta + \partial_{-}\phi + 2eA^{+})]|\psi\rangle = 0 \tag{3.53c}$$

belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belongs to the set of BRST-invariant states.

In order to understand the condition needed for recovering the physical states of the theory we write the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider Eq. (3.41) (namely, $-\partial_{+}\partial_{+}c = c$). The solution of this equation gives the Heisenberg operator $c(\tau)$ where $\tau(\equiv x^{+})$ is the light-cone time variable, (and correspondingly $\bar{c}(\tau)$) as [14,15,9,7,17] :

$$c(\tau) = e^{i\tau}B + e^{-i\tau}D \; ; \bar{c}(\tau) = e^{-i\tau}B^{\dagger} + e^{i\tau}D^{\dagger} \tag{3.54}$$

which at time $\tau = 0$ imply

$$c \equiv c(o) = B + D ; \bar{c} \equiv \bar{c}(o) = B^{\dagger} + D^{\dagger} \tag{3.55}$$

$$\partial_{+}c \equiv \partial_{+}c(o) = i(B - D) \; ; \partial_{+}\bar{c} \equiv \partial_{+}\bar{c}(o) = -i(B^{\dagger} - D^{\dagger}) \; . \tag{3.56}$$

By imposing the conditions [14,15,9,7,17] :

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\partial_{+}\bar{c}, \partial_{+}c\} = 0 \; ; \tag{3.57}$$

$$\{\partial_{+}\bar{c}, c\} = i = -\{\partial_{+}c, \bar{c}\} \tag{3.58}$$

one then obtains

$$B^2 + \{B, D\} + D^2 = B^{\dagger 2} + \{B^{\dagger}, D^{\dagger}\} + D^{\dagger 2} = 0 \tag{3.59a}$$

$$\{B, B^{\dagger}\} + \{D, D^{\dagger}\} + \{B, D^{\dagger}\} + \{B^{\dagger}, D\} = 0 \tag{3.59b}$$

$$\{B, B^{\dagger}\} + \{D, D^{\dagger}\} - \{B, D^{\dagger}\} - \{B^{\dagger}, D\} = 0 \tag{3.59c}$$

$$\{B, B^{\dagger}\} - \{D, D^{\dagger}\} - \{B, D^{\dagger}\} + \{D, B^{\dagger}\} = -1 \tag{3.59d}$$

$$\{B, B^{\dagger}\} - \{D, D^{\dagger}\} + \{B, D^{\dagger}\} - \{D, B^{\dagger}\} = -1 \tag{3.59e}$$

with the solution

$$B^2 = D^2 = B^{\dagger 2} = D^{\dagger 2} = 0 \tag{3.60a}$$

$$\{B, D\} = \{B^\dagger, D\} = \{B, D^\dagger\} = \{B^\dagger, D^\dagger\} = 0 \tag{3.60b}$$

$$\{B^\dagger, B\} = \frac{-1}{2} \; ; \{D^\dagger, D\} = +\frac{1}{2} \; . \tag{3.60c}$$

We now let $|0\rangle$ denote the fermionic vacuum for which

$$B|0\rangle = D|0\rangle = 0 \; . \tag{3.61}$$

Defining $|0\rangle$ to have norm one, (3.60c) implies

$$\langle 0|BB^\dagger|0\rangle = -1/2 \; ; \langle 0|DD^\dagger|0\rangle = +1/2 \tag{3.62}$$

so that

$$B^\dagger|0\rangle \neq 0 \; ; D^\dagger|0\rangle \neq 0 \; . \tag{3.63}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{BRST}$ is however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of fermionic annihilation and creation operators the quantum Hamiltonian density is

$$\begin{aligned}
\mathcal{H}_{BRST} = {} & \frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_- A^-) + e^2(A^-)^2 - 2e^2A^+A^- - 2e^2A^-(\partial_-\theta) \\
& - \frac{1}{4e^2}[\Pi_\theta - (1 - 2e + 2e^2)(\partial_-\theta) + \partial_-\phi - 2e(e-1)A^+ \\
& + 2e^2A^-]^2 - \frac{1}{2}(\Pi^+)^2 - \Pi^+(\partial_-\theta + A^+ - \phi) \\
& + 2(B^\dagger B + D^\dagger D)
\end{aligned} \tag{3.64}$$

and the BRST charge operator Q is

$$\begin{aligned}
Q = \int dx^- [& iB\{(\partial_-\Pi^- + \Pi_\theta - (1-2e)\partial_-\theta + \partial_-\phi + 2eA^+) \\
& - i(\Pi^+ + \Pi - \partial_-\phi - 2eA^+ + (1-2e)\partial_-\theta)\} \\
& + iD\{(\partial_-\Pi^- + \Pi_\theta - (1-2e)\partial_-\theta + \partial_-\phi + 2eA^+) \\
& + i(\Pi^+ + \Pi - \partial_-\phi - 2eA^+ + (1-2e)\partial_-\theta)\}] \; .
\end{aligned} \tag{3.65}$$

Now because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (3.53) holds, but also additional states for which

$$B|\psi\rangle = D|\psi\rangle = 0 \tag{3.66a}$$

$$\Pi^+|\psi\rangle \neq 0 \tag{3.66b}$$

$$[\Pi - \partial_-\phi - 2eA^+ + (1-2e)\partial_-\theta]|\psi\rangle \neq 0 \tag{3.66c}$$

$$[\partial_-\Pi^- + \Pi_\theta - (1-2e)\partial_-\theta + \partial_-\phi + 2eA^+]|\psi\rangle \neq 0 \; . \tag{3.66d}$$

The Hamiltonian is, however, also invariant under the anti-BRST transformations (in which the role of c and $-\bar{c}$ gets interchanged) given by [14,15,9,7,17] :

$$\bar{\bar{\delta}}\phi = \bar{\bar{\delta}}\theta = \bar{c} \; ; \bar{\bar{\delta}}A^+ = -\partial_-\bar{c} \; ; \bar{\bar{\delta}}A^- = -\partial_+\bar{c} \; ; \bar{\bar{\delta}}u = -\partial_+\partial_+\bar{c} \; ; \bar{\bar{\delta}}v = \partial_+\bar{c} \; , \tag{3.67a}$$

$$\bar{\bar{\delta}}\Pi = \bar{\bar{\delta}}\Pi_\theta \; ; \bar{\bar{\delta}}\Pi^+ = \bar{\bar{\delta}}\Pi^- = \bar{\bar{\delta}}\Pi_u = \bar{\bar{\delta}}\Pi_v = 0 \tag{3.67b}$$

$$\bar{\bar{\delta}}\bar{c} = 0 \; ; \bar{\bar{\delta}}c = -b \; , \bar{\bar{\delta}}b = 0 \tag{3.67c}$$

with generator or anti-BRST charge

$$\begin{aligned} \bar{Q} = \int dx^- [& -i\bar{c}\{\partial_-\Pi^- + \Pi_\theta - (1-2e)\partial_-\theta + \partial_-\phi + 2eA^+\} \\ & + i(\partial_+\bar{c})\{\Pi^+ + \Pi - \partial_-\phi - 2eA^+ + (1-2e)\partial_-\theta\}] \end{aligned} \tag{3.68}$$

$$\begin{aligned} = \int dx^- [& -iB^\dagger\{(\partial_-\Pi^- + \Pi_\theta - (1-2e)\partial_-\theta + \partial_-\phi + 2eA^+) \\ & + i(\Pi^+ + \Pi - \partial_-\phi - 2eA^+ + (1-2e)\partial_-\theta)\} \\ & - iD^\dagger\{(\partial_-\Pi^- + \Pi_\theta - (1-2e)\partial_-\theta + \partial_-\phi + 2eA^+) \\ & - i(\Pi^+ + \Pi - \partial_-\phi - 2eA^+ + (1-2e)\partial_-\theta)\}] \; . \end{aligned} \tag{3.69}$$

We also have

$$[Q, H_{BRST}] = [\bar{Q}, H_{BRST}] = 0 \tag{3.70a}$$

$$H_{BRST} = \int dx^- \mathcal{H}_{BRST} \tag{3.70b}$$

and we further impose the dual condition that both Q and $\bar{Q}$ annihilate physical states, implying that :

$$Q|\psi\rangle = 0 \tag{3.71a}$$

$$\bar{Q}|\psi\rangle = 0 \; . \tag{3.71b}$$

The states for which (3.53) hold, satisfy both the above conditions (3.71a) and (3.71b) and, in fact, are the only states satisfying both of these conditions since, although with (3.60)

$$2(B^{\dagger}B + D^{\dagger}D) = -2(BB^{\dagger} + DD^{\dagger}) \tag{3.72}$$

there are no states of this operator with $B^{\dagger}|0\rangle = 0$ and $D^{\dagger}|0\rangle = 0$ (cf. (3.63)), and hence no free eigenstates of the fermionic part of $\mathcal{H}_{BRST}$ which are annihilated by each of B, $B^{\dagger}$, D, $D^{\dagger}$. Thus the only states satisfying (3.71) are those satisfying the constraints (3.21) and (3.25). Further, the states for which (3.53) hold, satisfy both of the conditions (3.71a) and (3.71b) and in fact, are the only states satisfying both of these conditions (3.71a) and (3.71b), because in view of (3.57) and (3.58), one can not have simultaneously, c, $\partial_+ c$, and $\bar{c}$, $\partial_+ \bar{c}$, applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (3.71) are those that satisfy the constraints of the theory (3.21) and (3.25), and they belong to the set of BRST-invariant and anti-BRST-invariant states.

One can understand the above point in terms of fermionic annihilation and creation operators as follows. The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which (3.53) holds, but also additional states for which (3.66) holds. However, $\bar{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\bar{Q}$ contains only the states for which (3.53) holds, simply because $B^{\dagger}|\psi\rangle \neq 0$ and $D^{\dagger}|\psi\rangle \neq 0$. Thus, in this alternative way also we see that the states satisfying $Q|\psi\rangle = \bar{Q}|\psi\rangle = 0$ (i.e., satisfying (3.71)) are only those that satisfy the constraints of the theory (3.21) and (3.25) and also that these states belong to the set of BRST invariant and anti-BRST-invariant states.

Towards the end, we like to make an important observation that some interesting work on the same model has been done in Refs. [18,19]. The instant-form of the chiral Schwinger model discovered in [10] correspond to a regularization different from those involved in the class of models studied earlier [1-7]. The gauge field becomes massive once again, but the massless excitation that remains in this case appears to be chiral from the counting of degrees of freedom. In [18], it has been shown that the Pauli-Villars method can accomodate Lorentz noninvariant regularizations and thereby lead to the bosonized instant-form action of [10]. Gauge invariant reformulation of the instant-form model [10] has also been studied in [18]. In [19], the instant-form model [10] has been solved and its exact fermion propagator has been derived in a path integral approach. Further, the operator solutions of the instant-form theory in the bosonized and fermionic forms have also been obtained in [19]. For the details of this work we refer to the work of Refs [18,19].

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