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Quantum Algebras, Observables, and Random Variables

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Abstract. In an earlier paper, the author introduced the concept of *statistical* and *deterministic predictability* for an investigation of the quantum physical measuring process and defined *quantum algebras* (*Q-algebras*) as the appropriate algebraic structure for this investigation. In this paper, Q-algebras are studied in detail, and the concept of *observables* is presented.

Various topologies, an order relation and the weak completion are introduced for Q-algebras and used to elaborate on their relations with C*-algebras, W*-algebras and monotone sequentially complete C*-algebras. The system of (orthogonal) projections in the weak sequential completion of a Q-algebra forms a σ-complete orthomodular lattice, and the probabilities resulting from the statistical predictability are σ-additive on this lattice. Observables with values in an abstract measurable space can then be defined and, using the spectral theorem, bounded real-valued observables can be identified with self-adjoint elements of the weak sequential completion of the Q-algebra.

The observables, considered here, include both the self-adjoint operators of quantum physics as well as the random variables of (Kolmogorovian) mathematical probability theory. This seems to render possible a universal axiomatic theory comprising Kolmogorovian probability theory and a quantum model although there remains a major difference between certain quantum probabilities and Kolmogorovian probability.

1 Introduction

Q-algebras have been introduced in [1]; they seem to be the appropriate mathematical structure to study the quantum physical measurement process. A Q-algebra $\mathcal{A}$ is defined as a complex algebra with unit element $\mathbb{I}$ and an involution * fulfilling the following two conditions:

(i) $X \in \mathcal{A}, X^*X=0 \Rightarrow X=0$
(ii) For all $X \in \mathcal{A}$ there exists an *atom* E and a $Y \in \mathcal{A}$ with $XYE \neq 0$.

A self-adjoint element $E \in \mathcal{A}$ with $E^2=E$ is called an (orthogonal) *projection* (the system of projections is denoted by $\mathcal{E}$), and an *atom* is a projection with $E\mathcal{A}E=\mathbb{C}\,E$. A partial order relation is introduced on $\mathcal{E}$ by the definition: $E \le F :\Leftrightarrow EF=E$ $(E,F \in \mathcal{E})$. The system $\mathcal{E}$ is interpreted as system of logical propositions with the relation $\le$ representing the "*logical implication*". The "*negation*" is represented by the operation $E \to E' := \mathbb{I} - E$.

An element $X \in \mathcal{A}$ is said to be *statistically predictable* under a projection E if $EXE=\lambda E$ for some $\lambda \in \mathbb{C}$; λ is called the *expectation value* of X under E and denoted by $\mathbb{E}(X|E)$. If E is an atom, each X is statistically predictable under E. For a projection F, $\mathbb{E}(F|E)$ is a real number from the unit interval [0,1] which is interpreted as a probability and is denoted by $\mathbb{P}(F|E)$.

Furthermore,

$$\mathbb{E}(X^*X|E) \ge 0 \text{ and } \mathbb{E}(X^*|E) = \overline{\mathbb{E}(X|E)} \text{ for all } X \in \mathcal{A},$$

$\mathbb{E}(X|E)$ is a real number if $X=X^*$, and $\mathbb{E}(XY|E) = \mathbb{E}(X|E)\,\mathbb{E}(Y|E)$ if E commutes with X or Y $(X, Y \in \mathcal{A})$.

If $E, Y \in \mathcal{A}$, E an atom with $YE \ne 0$, then $EY^*YE \ne 0$, $\mathbb{E}(Y^*Y|E) \ne 0$, $EY^* \ne 0$, $YEY^* \ne 0$ and

$$F := \frac{1}{\mathbb{E}(Y^*Y|E)} YEY^* \ne 0 \qquad (*)$$

is an atom with

$$\mathbb{E}(X|F) = \frac{\mathbb{E}(Y^*XY|E)}{\mathbb{E}(Y^*Y|E)} \qquad (*)$$

for all $X \in \mathcal{A}$, since $F=F^*$,

$$F^2 = \frac{1}{\mathbb{E}(Y^*Y|E)^2} Y(EY^*YE)Y^* = \frac{1}{\mathbb{E}(Y^*Y|E)} YEY^* = F \text{ and}$$

$$FXF = \frac{1}{\mathbb{E}(Y^*Y|E)^2} Y(EY^*XYE)Y^* = \frac{\mathbb{E}(Y^*XY|E)}{\mathbb{E}(Y^*Y|E)^2} YEY^* = \frac{\mathbb{E}(Y^*XY|E)}{\mathbb{E}(Y^*Y|E)} F.$$

The above two equations marked with (*) will repeatedly be applied in this paper. For each $0 \ne X \in \mathcal{A}$, there are $Y, E \in \mathcal{A}$, E an atom, with $XYE \ne 0$ (condition (ii) above). Then

$$YE \ne 0, EY^*YE \ne 0, EY^* \ne 0, YEY^* \ne 0,\ EY^*X^*XYE \ne 0,\ \mathbb{E}(Y^*X^*XY|E) \ne 0$$

$$\Rightarrow YEY^*X^*XYEY^* = \mathbb{E}(Y^*X^*XY|E)YEY^* \ne 0 \Rightarrow XYEY^* \ne 0$$

$$\Rightarrow XF \ne 0 \text{ with the atom } F = \frac{1}{\mathbb{E}(Y^*Y|E)} YEY^*.$$

Condition (ii) in the definition of a Q-algebra is therefore equivalent to and can be replaced by the following condition:

$$0 \neq X \in \mathcal{A} \Rightarrow \text{There exists in atom } E \text{ in } \mathcal{A} \text{ with } XE \neq 0.$$

Furthermore, for each $0 \neq E \in \mathcal{E}$, there is an atom F with $F \leq E$ (i.e. $\mathcal{E}$ is *atomic*[1]), since first there is an atom D with $ED \neq 0$ and then

$$F := \frac{1}{\mathbb{E}(E|D)} EDE$$

is an atom with $EF=F$.

This abstract algebraic formalism represents a kind of non-Boolean logic. It has been used in [1] to study quantum measurement and quantum phenomena like indeterminism and interference.

In the present paper, the mathematical structure of Q-algebras is analyzed in detail. First, three different topologies as well as a partial order relation $\leq$ (extending the above-mentioned order relation on the system of projections) are introduced. Then, the completion of a Q-algebra in the weakest one among the three topologies is studied and used to investigate the relations to C*-algebras, W*-algebras and monotone sequentially complete C*-algebras. A major result is that the system of orthogonal projections in a proper sequential completion forms a σ-complete lattice. This σ-complete lattice will finally be the framework to study observables, and it will be seen that "bounded real-valued" observables can be identified with self-adjoint elements in the Q-algebra or in its weak completion.

The mathematical modeling of propositions and observables used here includes both the model of quantum physics (orthogonal projections and operators on a Hilbert space) as well as the Kolmogorov model of mathematical probability theory (σ-algebras and random variables), although the probabilities considered here differ from those studied in mathematical probability theory.

2 Topologies and an order relation for Q-algebras

The following definition and the lemma provide the mathematical tools for the consideration of topological structures and an order structure on a Q-algebra.

Definition 1: *Let $\mathcal{A}$ be a Q-algebra with $X, Y \in \mathcal{A}$.*

(i) $|X|_E := |\mathbb{E}(X|E)|$ *for an atom E.*

(ii) $\|X\|_E := \sqrt{\mathbb{E}(X^* X|E)}$ *for an atom E.*

(iii) $\|X\| := \sup_{\text{all atoms } E} \|X\|_E$.

(iv) *X is called bounded if* $\|X\| < \infty$.

(v) $\mathcal{A}_b$ *denotes the set of bounded elements in* $\mathcal{A}$.

(vi) *X is called positive* ($X \geq 0$) *if* $\mathbb{E}(X|E) \geq 0$ *for all atoms E.*

(vii) $X \leq Y :\Leftrightarrow Y - X \geq 0$.

[1] In [1], it was still conjectured that the system of projections in a Q-algebra need not be atomic.

Lemma 1: *Let $\mathcal{A}$ be a Q-algebra and $X, Y \in \mathcal{A}$.*

(i) *Let E be an atom. Then* $|XY|_E \leq \frac{1}{4}\left(\lambda_1 |X|_{F_1} + \lambda_2 |X|_{F_2} + \lambda_3 |X|_{F_3} + \lambda_4 |X|_{F_4}\right)$ *with*

$$F_1 = \frac{1}{\lambda_1}(\mathbb{I} + Y)E(\mathbb{I} + Y)^*,\ \lambda_1 = \mathbb{E}\left((\mathbb{I} + Y)^*(\mathbb{I} + Y)|E\right),$$

$$F_2 = \frac{1}{\lambda_2}(\mathbb{I} - Y)E(\mathbb{I} - Y)^*,\ \lambda_2 = \mathbb{E}\left((\mathbb{I} - Y)^*(\mathbb{I} - Y)|E\right),$$

$$F_3 = \frac{1}{\lambda_3}(\mathbb{I} - iY)E(\mathbb{I} - iY)^*,\ \lambda_3 = \mathbb{E}\left((\mathbb{I} - iY)^*(\mathbb{I} - iY)|E\right), \text{ and}$$

$$F_4 = \frac{1}{\lambda_4}(\mathbb{I} + iY)E(\mathbb{I} + iY)^*,\ \lambda_4 = \mathbb{E}\left((\mathbb{I} + iY)^*(\mathbb{I} + iY)|E\right).$$

(ii) $\mathbb{E}(X|E)=0$ *(i.e.* $|X|_E=0$*) for all atoms E* $\Rightarrow X=0$.

(iii) $\left|\mathbb{E}(X^*Y|E)\right| \leq \mathbb{E}(X^*X|E)^{\frac{1}{2}}\ \mathbb{E}(Y^*Y|E)^{\frac{1}{2}}$ *for all atoms E.*

(iv) $|X|_E \leq \|X\|_E \leq \|X\|$ *for all atoms E.* $\|X\|_E = 0$ *for all atoms E* $\Rightarrow X=0$. $\|X\| = 0 \Rightarrow X=0$.

(v) $\|XY\|_E = \|X\|_F \|Y\|_E$ with $F = \frac{1}{\mathbb{E}(Y^*Y|E)} YEY^*$ for $YE \neq 0$ and $\|YX\|_E \leq \|Y\|\ \|X\|_E$.

(vi) $\|XY\| \leq \|X\|\ \|Y\|$, $\|X^*\| = \|X\|$ and $\|X^*X\| = \|X\|^2$.

Proof: (i) With the above equations (*), (i) immediately follows from the identity

$$4\,XY = (\mathbb{I} + Y)^*X(\mathbb{I} + Y) - (\mathbb{I} - Y)^*X(\mathbb{I} - Y) + i\,(\mathbb{I} - iY)^*X(\mathbb{I} - iY) - i\,(\mathbb{I} + iY)^*X(\mathbb{I} + iY).$$

(ii) Using (i) with $Y=X^*$, we get from $0=\mathbb{E}(X|E)=|X|_E$ for all atoms E:

$$0 = |XX^*|_E = \mathbb{E}(XX^*|E) \quad \text{for all atoms E}$$

$$\Rightarrow 0 = EXX^*E = (X^*E)^* X^*E \quad \text{for all atoms E}$$

$$\Rightarrow 0 = X^*E \quad \text{for all atoms E}$$

$$\Rightarrow X^* = 0 \Rightarrow X = 0.$$

(iii) $\rho(X,Y) := \mathbb{E}(X^*Y|E)$ for $X, Y \in \mathcal{A}$ defines a non-negative hermitean form on $\mathcal{A}$, and the Cauchy-Schwarz inequality yields (iii).

(iv) The first inequality follows from (iii) with $Y=\mathbb{I}$ and the second inequality is obvious. Then apply (ii).

(v) Using the above equations (*) we get:

$$\|XY\|_E = \mathbb{E}(Y^*X^*XY|E)^{1/2} = \mathbb{E}(Y^*Y|E)^{1/2}\ \mathbb{E}(X^*X|F)^{1/2} = \|Y\|_E \|X\|_F$$

and then

$$\|YX\|_E = \|X\|_E \|Y\|_D \leq \|X\|_E \|Y\| \text{ with } D = XEX^* / \mathbb{E}(X^*X|E) \text{ for } XE \neq 0.$$

With $XE=0$, both sides equal 0.

(vi) $\|XY\| \leq \|X\|\ \|Y\|$ immediately follows from (v). Furthermore, with (iv):

$$\|X^*\|^2 = \sup_{\text{all atoms E}} \mathbb{E}(XX^*|E) = \sup_{\text{all atoms E}} |XX^*|_E \leq \|XX^*\| \leq \|X\|\ \|X^*\| \text{ for all } X \in \mathcal{A}$$

$$\Rightarrow \|X^*\| \le \|X\| \text{ for all } X \in \mathcal{A} \Rightarrow \|X\| = \|(X^*)^*\| \le \|X^*\| \text{ for all } X \in \mathcal{A}$$

$$\Rightarrow \|X^*\| = \|X\| \text{ for all } X \in \mathcal{A}$$

and

$$\|X^* X\| \le \|X^*\|\, \|X\| = \|X\|^2 = \sup\{\, \mathbb{E}(X^* X|E) \mid E \in \mathcal{A}, E \text{ atom}\}$$

$$= \sup\{|X^* X|_E \mid E \in \mathcal{A}, E \text{ atom}\} \le \|X^* X\| \text{ for all } X \in \mathcal{A},$$

which concludes the proof of Lemma 1. □

From Lemma 1 (ii) we immediately get for $X \in \mathcal{A}$:

$$X = X^* \Leftrightarrow \mathbb{E}(X|E) \in \mathbb{R} \text{ for all atoms } E$$

and

$$X \ge 0 \Rightarrow X = X^*.$$

The family of semi-norms $|\ |_E$, E an atom, defines a locally convex topology τ_w (*weak topology*) on $\mathcal{A}$. The involution * and, with $Y \in \mathcal{A}$ fixed, the mappings $\mathcal{A} \ni X \to XY \in \mathcal{A}$ and $\mathcal{A} \ni X \to YX \in \mathcal{A}$ are continuous in this topology ($|X|_E = |X^*|_E$, Lemma 1 (i), and $YX = (X^*Y^*)^*$).

Another locally convex topology τ_s on $\mathcal{A}$ is defined by the semi-norms $\|\ \|_E$, E an atom. With $Y \in \mathcal{A}$ fixed, the mapping $\mathcal{A} \ni X \to XY \in \mathcal{A}$ is τ_s-continuous, and the mapping $\mathcal{A} \ni X \to YX \in \mathcal{A}$ is τ_s-continuous only if Y is bounded (Lemma 1 (v)). In general, the involution is not τ_s-continuous. τ_s is stronger than τ_w.

$\|\ \|$ is a norm on the linear space of bounded elements $\mathcal{A}_b$. From Lemma 1 (vi) we now obtain that $\mathcal{A}_b$ is a normed involutive algebra fulfilling the C*-condition[2],[3]. On $\mathcal{A}$, $\|\ \|$ does not provide a norm since ∞ is a possible value, but nevertheless, $\|\ \|$ defines a topology τ_n on $\mathcal{A}$. This topology is stronger than τ_w and τ_s. The restriction of τ_n on $\mathcal{A}_b$ is the norm topology.

For a projection F in $\mathcal{A}$, we have $0 \le \mathbb{E}(F|E) \le 1$ and hence $\|F\| \le 1$. Lemma 1 (vi) implies that either $\|F\| = 1$ or $F=0$. Therefore, the set $\mathcal{A}_b$ of bounded elements in $\mathcal{A}$ includes all projections from $\mathcal{A}$ and is itself a Q-algebra.

Definition 1 (vi) defines a partial order relation on $\mathcal{A}$ as well as on $\mathcal{A}_b$. Lemma 1 (ii) yields the antisymmetry of the relation. We shall now prove that this order relation is an extension of the order relation on the system $\mathcal{E}$ of (orthogonal) projections in $\mathcal{A}$, mentioned in the introduction. For $E,F \in \mathcal{E}$ we have:

$$EF = E \Rightarrow F - E = (F - E)^2 \Rightarrow \mathbb{E}(F - E|D) \ge 0 \text{ for all atoms } D \in \mathcal{A},$$

and vice versa (again using the equations (*)),

$$\mathbb{E}(F - E|D) \geq 0 \text{ for all atoms } D \in \mathcal{A}$$

$$\Rightarrow \mathbb{E}(EFE - E|D) = \mathbb{E}(E(F-E)E|D) = \mathbb{E}(E|D)\,\mathbb{E}\left(F - E\left|\frac{1}{\mathbb{E}(E|D)}EDE\right.\right) \geq 0$$

for all atoms D

$\Rightarrow$ *EFE-E* is positive.

However, $E\text{-}EFE=E(\mathbb{I}-F)E=((\mathbb{I}-F)E)^*((\mathbb{I}-F)E)$ is positive as well, and therefore $EFE=E$, which implies $EF=E$ (see [1]).

Example 1: Let $\mathcal{A}_1$ be the Q-algebra consisting of all complex-valued measurable functions on a set Ω with a σ-algebra ϑ such that the singletons $\{\omega\}$ belong to ϑ for each $\omega\in\Omega$. In this case, the norm defined above is given by

$$\|X\| = \sup_{\omega\in\Omega}|X(\omega)| \quad \text{for } X\in\mathcal{A}_1.$$

The weak topology (τ_w) and the τ_s-topology coincide (as they always do if the Q-algebra is commutative), and convergence in one of these two topologies coincides with point-wise convergence. Furthermore, $X\geq 0$ if and only if $X(\omega)\geq 0$ for all $\omega\in\Omega$ ($X\in\mathcal{A}_1$).

Example 2: Let $\mathcal{A}_2$ be the Q-algebra consisting of all bounded linear operators on a Hilbert space. Then the norm defined above coincides with the usual operator norm, the weak topology coincides with the weak operator topology and the τ_s-topology coincides with the strong operator topology. Furthermore, $X\geq 0$ if and only if $\langle\xi|X\xi\rangle\geq 0$ for all Hilbert space vectors ξ ($X\in\mathcal{A}_2$).

Example 3: Let $\mathcal{H}_0$ be a pre-Hilbert-space and let $\mathcal{A}_3$ be the system of those linear operators X from $\mathcal{H}_0$ to $\mathcal{H}_0$ for which there is a linear operator X^* from $\mathcal{H}_0$ to $\mathcal{H}_0$ with

$$\langle\eta|X\xi\rangle = \langle X^*\eta|\xi\rangle \text{ for all } \eta,\xi\in\mathcal{H}_0.$$

$\mathcal{A}_3$ is a Q-algebra. The norm defined above coincides with the usual operator norm, but $\mathcal{A}_3$ can contain unbounded operators. If $\mathcal{H}_0$ is complete, $\mathcal{A}_3$ contains bounded operators only and coincides with $\mathcal{A}_2$.

The last theorem in [1] gave a representation Π of any Q-algebra $\mathcal{A}$ as linear operators on a pre-Hilbert-space $\mathcal{H}_0$. The following lemma provides further topological properties of this representation.

Lemma 2: *Let X_α be a net (generalized sequence) in $\mathcal{A}$ and $X\in\mathcal{A}$.*

(i) $\|\Pi(X)\| = \|X\|$.

(ii) $X_\alpha \xrightarrow[\tau_s]{} X \Leftrightarrow \|\Pi(X_\alpha)\xi - \Pi(X)\xi\| \longrightarrow 0$ for all $\xi\in\mathcal{H}_o$.

(iii) $X_\alpha \xrightarrow[\tau_w\text{ (weak)}]{} X \Leftrightarrow \langle\xi|\Pi(X_\alpha)\xi\rangle \longrightarrow \langle\xi|\Pi(X)\xi\rangle$ for all $\xi\in\mathcal{H}_o$.

(iv) $X\geq 0 \Leftrightarrow \langle\xi|\Pi(X)\xi\rangle\geq 0$ for all $\xi\in\mathcal{H}_0$.

Proof: For each atom E there is a vector $\xi\in\mathcal{H}_0$ with $\|\xi\|=1$ and $\mathbb{E}(X|E)=\langle\xi|\Pi(X)\xi\rangle$ for all $X\in\mathcal{A}$. Hence $\|\Pi(X)\|\geq\|X\|$ and the implications "$\Leftarrow$" of (ii), (iii), and (iv).
On the other hand, for each vector $\xi\in\mathcal{H}_0$, there are $n\in\mathbb{N}, n$ atoms $E_1,\dots,E_n$ and $Y_1,\dots,Y_n\in\mathcal{A}$ with $Y_kE_k\neq 0$ and

$$\langle\xi|\Pi(X)\xi\rangle=\sum_{k=1}^{n}\mathbb{E}\left(Y_k^*XY_k|E_k\right)=\sum_{k=1}^{n}\mathbb{E}\left(Y_k^*Y_k|E_k\right)\mathbb{E}\left(X\left|\frac{1}{\mathbb{E}\left(Y_k^*Y_k|E_k\right)}Y_kE_kY_k^*\right.\right)$$

for all $X\in\mathcal{A}$ (again with (*)). $\|\xi\|^2=\sum_{k=1}^{n}\mathbb{E}\left(Y_k^*Y_k|E\right)$. From this, we get $\|\Pi(X)\|\leq\|X\|$ and the implications "$\Rightarrow$" of (ii), (iii), and (iv). □

3 The weak completion of a Q-algebra

If the Q-algebra $\mathcal{A}$ is τ_n-complete, $\mathcal{A}_b$ is norm complete and thus becomes a C*-algebra[2],[3]. It is well-known that each C*-algebra has a representation as bounded operators on a Hilbert space (Gelfand Naimark theorem), which again provides us with a representation of $\mathcal{A}_b$ and particularly of $\mathcal{E}$ on a Hilbert space while Π is a pre-Hilbert space representation of $\mathcal{A}$ including the elements that are not bounded (and the construction[1] of Π does not require C*-algebra theory).

If a Q-algebra $\mathcal{A}$ has a finite dimension, it is automatically complete and each element is bounded. Thus it is a finite-dimensional C*-algebra, and therefore it is the finite direct sum of matrix algebras[2]:

$$\mathcal{A}=\bigoplus_{k=1}^{m}M_{n_k}\text{ , where } M_{n_k} \text{ is the algebra of complex } n_k\times n_k-\text{matrices.}$$

Now, we consider the weak (τ_w) completion $\overline{\mathcal{A}}$ of a Q-algebra $\mathcal{A}$. $\overline{\mathcal{A}}$ is a linear space comprising $\mathcal{A}$, but the product XY is not defined in $\overline{\mathcal{A}}$. XY is defined only if at least one of the two elements X,Y lies in $\mathcal{A}$. For $0\neq X\in\overline{\mathcal{A}}$ there is an atom E in $\mathcal{A}$ with $XE\neq 0$ since $0\neq X\in\overline{\mathcal{A}}$ implies $0\neq|X|_E=\mathbb{E}(X|E)$ for some atom E in $\mathcal{A}$, thus $EXE\neq 0$ and $XE\neq 0$.

If $\mathcal{A}$ is commutative, $|XY|_E=|X|_E|Y|_E$ and the product XY can be defined for all $X,Y\in\overline{\mathcal{A}}$. Then $\overline{\mathcal{A}}$ is a Q-algebra.

Lemma 3: *Let $\mathcal{A}$ be a Q-algebra.*
(i) *Each monotone increasing sequence $0\leq X_n\leq X_{n+1}\leq Y$ ($X_n\in\overline{\mathcal{A}}$) with an upper bound Y in $\overline{\mathcal{A}}$ weakly converges against its lowest upper bound $\sup X_n\in\overline{\mathcal{A}}$ and $\mathbb{E}(\sup X_n|E)=\sup\mathbb{E}(X_n|E)$ for all atoms $E\in\mathcal{A}$.*
(ii) *Each monotone increasing net $0\leq X_\alpha\leq Y$ ($X_\alpha\in\overline{\mathcal{A}}$) with an upper bound Y in $\overline{\mathcal{A}}$ weakly converges against its lowest upper bound $\sup X_\alpha\in\overline{\mathcal{A}}$ and $\mathbb{E}(\sup X_\alpha|E)=\sup\mathbb{E}(X_\alpha|E)$ for all atoms $E\in\mathcal{A}$.*

Proof: (i) $0 \le X_n \le X_{n+1} \le Y \Rightarrow 0 \le \mathbb{E}(X_n|E) \le \mathbb{E}(X_{n+1}|E) \le \mathbb{E}(Y|E) \Rightarrow \mathbb{E}(X_n|E)$ converges in $\mathbb{R}$. Since

$$\left|X_n - X_m\right|_E = \left|\mathbb{E}(X_n|E) - \mathbb{E}(X_m|E)\right|$$

for all atoms E in $\mathcal{A}$, X_n is a Cauchy sequence in the weak topology and thus weakly converges against a $Z \in \overline{\mathcal{A}}$. From

$$\mathbb{E}(Z|E) = \lim_{n\to\infty} E(X_n|E) \quad \text{for all atoms } E \in \mathcal{A}$$

we get:

$$X_m \le Z \le Y \text{ for all } m \in \mathbb{N}.$$

Since Y is an arbitrary upper bound for the sequence X_n, this means that Z is the lowest upper bound sup X_n.
(ii) Replace the sequence by a net (generalized sequence) in the proof of (i). □

Definition 2A: *Let $\mathcal{A}$ be a Q-algebra.*
(i) $\mathcal{A}_{\Sigma^*} := \left\{ X \in \overline{\mathcal{A}} \;\middle|\; X \text{ is the weak limit of a norm-bounded sequence } Y_n \text{ in } \mathcal{A}_b \right\}$.
(ii) $\mathcal{A}_{W^*} := \left\{ X \in \overline{\mathcal{A}} \;\middle|\; X \text{ is the weak limit of a norm-bounded net } Y_\alpha \text{ in } \mathcal{A}_b \right\}$.

We again consider the representation Π on the pre-Hilbert space $\mathcal{H}_0$. Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$. $\mathcal{H}$ is a Hilbert space. Since the bounded linear operators on $\mathcal{H}_0$ can uniquely be extended to bounded linear operators on $\mathcal{H}$ and since, for each norm-bounded net of bounded linear operators T_α,

$$\langle \xi | T_\alpha \xi \rangle \text{ is a Cauchy net for all } \xi \in \mathcal{H}_o \Leftrightarrow \langle \xi | T_\alpha \xi \rangle \text{ is a Cauchy net for all } \xi \in \mathcal{H},$$

we get from Lemma 2 that $\mathcal{A}_{W^*}$ is isomorphic to the weak closure (weak operator topology) of $\Pi(\mathcal{A}_b)$ in the space $\mathcal{B}(\mathcal{H})$ consisting of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ (the space $\mathcal{B}(\mathcal{H})$ is complete in the weak operator topology) and that $\mathcal{A}_{\Sigma^*}$ is isomorphic to the weak sequential closure of $\Pi(\mathcal{A}_b)$ in $\mathcal{B}(\mathcal{H})$[2]. Thus we get the following theorem.

Theorem 1: *Let $\mathcal{A}$ be any Q-algebra. Then $\mathcal{A}_{W^*}$ is a W*-algebra and $\mathcal{A}_{\Sigma^*}$ is a Baire-*-algebra with $\mathcal{A}_b \subseteq \mathcal{A}_{\Sigma^*} \subseteq \mathcal{A}_{W^*}$. Furthermore, $\mathcal{A}_{W^*}$ and $\mathcal{A}_{\Sigma^*}$ are Q-algebras.*

A C*-algebra is called a W*-algebra if each monotone increasing bounded net has a lowest upper bound and if a separating family of normal functionals exists (or, which is an equivalent definiton, if it is the dual space of a Banach space). C*-algebras and W*-algebras have extensively been studied since John von Neumann's first publications on this subject in the 1930ies; [2] and [3] are only two among a huge variety of books available now on the subject.

A C*-algebra is called a Baire-*-algebra if each monotone increasing bounded sequence of self-adjoint elements has a lowest upper bound (i.e. the C*-algebra is monotone sequentially

[2] Operator algebras that are sequentially closed in the weak operator topology are studied under the name Σ*-algebras in [4].

complete) and if a separating family of σ-normal functionals exists. Baire-*-algebras are very interesting from a measure theoretic viewpoint and have been studied in [5-11].

$\mathcal{A}_{\Sigma^*}$ is a Baire-*-algebra, but there may be a smaller Baire-*-algebra containing $\mathcal{A}_b$. Therefore, we add to Definiton 2A:

Definition 2B: *Let $\mathcal{A}$ be a Q-algebra. A subalgebra $\mathcal{B}$ of $\mathcal{A}_{W^*}$ is called monotone sequentially closed if $\lim X_n \in \mathcal{B}$ for each bounded, monotone sequence of self-adjoint elements in $\mathcal{B}$ weakly converging against $\lim X_n$ in $\mathcal{A}_{W^*}$. The smallest monotone sequentially closed subalgebra of $\mathcal{A}_{W^*}$ containing $\mathcal{A}_b$ is denoted by $\mathcal{A}_{Baire^*}$.*

$\mathcal{A}_{\text{Baire}^*}$ is a Baire-*-algebra (in fact, the smallest one containing the bounded elements of $\mathcal{A}$), and we have:

$$\mathcal{A}_b \subseteq \mathcal{A}_{Baire^*} \subseteq \mathcal{A}_{\Sigma^*} \subseteq \mathcal{A}_{W^*} \subseteq \overline{\mathcal{A}}.$$

An important feature of W*-algebras and Baire-*-algebras is that the spectral decomposition theorem can be applied (which is not possible in C*-algebras). Furthermore, the system of projections in a W*-algebra forms a complete lattice, and the system of projections in a Baire-*-algebra forms a σ-complete lattice.

This provides two important properties of the system of projections $\mathcal{E}_{\text{Baire}^*}$ in $\mathcal{A}_{\text{Baire}^*}$. First, $\mathcal{E}_{\text{Baire}^*}$ is a lattice; i.e. $E \wedge F$ and $E \vee F$ exist in $\mathcal{E}_{\text{Baire}^*}$ for all $E,F \in \mathcal{E}_{\text{Baire}^*} \supseteq \mathcal{E}$. In $\mathcal{E}$, $E \wedge F$ and $E \vee F$ exist only for commuting pairs E,F. Second, $\mathcal{E}_{\text{Baire}^*}$ is σ-complete; i.e.

$$\bigwedge_{n=1}^{\infty} E_n \quad \text{and} \quad \bigvee_{n=1}^{\infty} E_n$$

exist in $\mathcal{E}_{\text{Baire}^*}$ for each sequence E_n in $\mathcal{E}_{\text{Baire}^*}$. The σ-completeness is important for the introduction of observables in the next paragraph. Furthermore, it follows from Lemma 3 that the probabilities $\mathbb{P}(F|E)$ are σ-additive in F on $\mathcal{E}_{\text{Baire}^*}$.

If $\mathcal{A}$ is weakly complete, $\mathcal{A}_b = \mathcal{A}_{\text{Baire}^*} = \mathcal{A}_{\Sigma^*} = \mathcal{A}_{W^*}$, and $\mathcal{E}$ is a complete lattice. If $\mathcal{A}$ is sequentially τ_w-complete, $\mathcal{A}_b = \mathcal{A}_{\text{Baire}^*} = \mathcal{A}_{\Sigma^*}$, and $\mathcal{E}$ is a σ-complete lattice.

If the Q-algebra $\mathcal{A}$ is commutative and sequentially complete in the weak topology, $\mathcal{E}$ is an atomic σ-complete Boolean lattice and can therefore be represented as a σ-algebra (consisting of subsets of the set of atoms). The σ-algebras play a fundamental role in modern mathematical probability theory based on Kolmogorov's axioms.

Let us have a look again at the examples considered in the last paragraph. The Q-algebra $\mathcal{A}_1$ of measurable functions is sequentially complete, but in general not complete in the weak topology. The Q-algebra of all bounded measurable functions is not sequentially complete in the weak topology if the σ-algebra contains an infinite number of atoms. The Q-algebra $\mathcal{A}_2$ of all bounded linear operators on a Hilbert space is always complete in the weak topology. In general, $\mathcal{A}_3$ is not weakly complete, and its weak completion is an abstract space containing elements that cannot be represented as operators[3].

[3] The weak completion of $\mathcal{A}_3$ can be represented as the linear space consisting of the *sesqui-linear forms* on $\mathcal{H}_0$. A sesqui-linear form is a mapping ρ: $\mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{C}$ such that $\rho(\eta,\xi)$ is linear in the second and anti-linear in the first component.

For each norm-bounded sequence $0 \leq X_n \leq X_{n+1}$ in a weakly complete Q-algebra $\mathcal{A}$ we have

$$\Pi\left(\sup_{n=1}^{\infty} X_n\right) = \sup_{n=1}^{\infty} \Pi\left(X_n\right),$$

from which we can now conclude that a σ-complete orthomodular lattice has a Hilbert space representation that is compliant not only with finite, but also with countably infinite operations ∧ and ∨ if and only if has a σ-additive embedding in a weakly complete Q-algebra. A complete orthomodular lattice has a Hilbert space representation that is compliant with arbitrary infinite operations ∧ and ∨ if and only if it has a completely additive embedding in a weakly complete Q-algebra.

4 Observables

A way how an observable can formally be defined when a system of logical propositions is given only is shown in [12]: In the case of a "real-valued" observable, it should be possible to allocate to each interval I a logical proposition E_I or moreover to each Borel set B a logical proposition E_B. The mapping $B \rightarrow E_B$ should be compatible with the logical operations. Observables with values in any abstract set Ω with a σ-algebra ϑ over Ω (the pair Ω,ϑ is a measurable space) can be considered as well.

Definition 3: *Let $\mathcal{E}$ be an orthomodular σ-complete lattice (e.g. the system of projections in a Baire-*-algebra) and let Ω,ϑ be a measurable space. An observable on $\mathcal{E}$ with values in Ω,ϑ is a mapping $X{:}\vartheta \rightarrow \mathcal{E}$ with the following properties:*
(i) $X(\Omega) = 1$
(ii) $X(B^c) = X(B)'$ *for all* $B \in \vartheta$[4]
(iii) $X\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigvee_{n=1}^{\infty} X\left(B_n\right)$ *for each sequence of sets* B_n *in* ϑ.

Further properties that follow from this definition are: $X(\varnothing) = 0$, $X(A) \leq X(B)$ for $A \subseteq B$ ($A, B \in \vartheta$) and, for each sequence of sets B_n in ϑ:

$$X\left(\bigcap_{n=1}^{\infty} B_n\right) = \bigwedge_{n=1}^{\infty} X\left(B_n\right).$$

Furthermore, $X(\vartheta)$ forms a σ-complete Boolean sublattice of $\mathcal{E}$.

Note that the observables considered here include the random variables of mathematical probability theory[13]. A random variable is a measurable function f and defines an observable X_f via $X_f(B) := f^{-1}(B)$. A closer look at mathematical probability theory shows that the mapping $B \rightarrow f^{-1}(B)$ between two σ-algebras is much more important for the theory than the mapping $\omega \rightarrow f(\omega)$ between two point sets itself.

[4] $B^c = \Omega - B$ is the set theoretic complement, and the operation $E \rightarrow E'$ is the orthocomplementation in $\mathcal{E}$. If $\mathcal{E}$ is the system of projections in a Baire-*-algebra, then $E' = \mathbb{I} - E$.

Vice versa, for each observable X on a σ-algebra Σ over a set Δ with values in $\mathbb{R}^n$ (with the Borel σ-algebra), a measurable $\mathbb{R}^n$-valued function f on Δ can be found such that $X=X_f$: For $\tau\in\Delta$ let δ_τ denote the Dirac measure on Δ concentrated in the point τ. Then $\delta_\tau\circ X$ is a $\{0,1\}$-valued measure on $\mathbb{R}^n$ and thus itself a Dirac measure in some point $s\in\mathbb{R}^n$. With f(τ):=s we have: $f^{-1}(B)=\{\tau\in\Delta \mid \delta_\tau(X(B))=1\}=X(B)$. This also implies that f is measurable.

Definition 4: *Let X be an observable (under the assumptions of Definition 3).*

(i) *Let* Δ,Σ *be another measurable space and let* $f:\Omega\to\Delta$ *be a measurable mapping. Then* $\Delta\ni D\to X(f^{-1}(D))$ *defines an observable on* $\mathcal{E}$ *with values in* Δ. *This observable is denoted by* $f(X)$.

(ii) *An element* $\omega\in\Omega$ *is called eigen-value of X if* $X(B)\neq 0$ for all $B\in\vartheta$ *with* $\omega\in B$. *If* $\{\omega\}\in\vartheta$, *this is equivalent to:* $X(\{\omega\})\neq 0$.

(iii) *Let* Ω *be a topological space and let* ϑ *be the Borel σ-algebra over* Ω *(i.e. the σ-algebra generated by the topology on* Ω*). An element* $\omega\in\Omega$ *is called spectral-value of X if* $X(B)\neq 0$ *for all open sets* $B\subseteq\Omega$ *with* $\omega\in B$. *The set of all spectral-values is the spectrum of X denoted by* $spc(X)$.

(iv) *Let* Ω *be the real continuum. X is called bounded if* $spc(X)$ *is bounded.*

Under the assumptions of (i) with Ω,Δ being topological spaces, ϑ,Σ the Borel σ-algebras and f a continuous mapping, we have: $spc(f(X))=f(spc(X))$.

We will now study observables X on $\mathcal{E}_{Baire^*}$ which is the system of projections in the Baire-*-algebra $\mathcal{A}_{Baire^*}$ generated by a Q-algebra $\mathcal{A}$. In this case, $X(A)$ and $X(B)$ commute for all $A,B\in\vartheta$ since

$$A=(A\cap B)\cup(A\cap B^c)$$
$$\Rightarrow X(A)=X((A\cap B)\cup(A\cap B^c))=(X(A)\wedge X(B))\vee(X(A)\wedge X(B)')$$
$$\Rightarrow X(A)X(B)=X(B)X(A) \quad \text{(see [1])}.$$

Definition 5: *Let X be an observable on the system* $\mathcal{E}_{Baire^*}$ *of projections in the Baire-*-algebra* $\mathcal{A}_{Baire^*}$ *generated by a Q-algebra* $\mathcal{A}$. *If* $X(B)$ *is statistically predictable under* $E\in\mathcal{E}_{Baire^*}$ *for some* $B\in\vartheta$, *we define:*

$$\mathbb{P}^{X|E}(B):=\mathbb{P}(X(B)|E).$$

If $X(B)$ *is statistically predictable under* $E\in\mathcal{E}_{Baire^*}$ *for all* $B\in\vartheta$ *(e.g. when E is an atom), we thus get a σ-additive probability measure* $\mathbb{P}^{X|E}$ *on* ϑ *which is called the distribution of X under E.*

There is a simple relation between these distributions and the eigen-values of an observable. Assuming that the singleton $\{\omega\}$ belongs to the σ-algebra ϑ, ω is an eigen-value of X if and only if there is an $E\neq 0$ in $\mathcal{E}$ with

$$\mathbb{P}^{X|E}=\delta_\omega \quad \text{(Dirac measure concentrated in } \omega\text{)}.$$

If ω is an eigen-value, select $E:=X(\{\omega\})\neq 0$. $X(\{\omega\})X(B)X(\{\omega\})=X(\{\omega\}\cap B)$ which equals $X(\{\omega\})$ if $\omega\in B$ and which equals 0 if $\omega\notin B$, i.e. $\mathbb{P}^{X|E}(B)$ equals 1 if $\omega\in B$ and 0 if $\omega\notin B$. Vice versa, if

$\mathbb{P}^{X|E}$ is the Dirac measure in ω for $0 \neq E \in \mathcal{E}$, we have $\mathbb{P}^{X|E}(B)=1$, i.e. $EX(B)E=E$, for all B with $\omega \in B$, therefore $X(B) \neq 0$ for all B with $\omega \in B$, i.e. ω is an eigen-value.

The following two theorems show the relation between the distributions of an observable and its spectrum.

Theorem 2: *Let X be an observable on the system $\mathcal{E}_{Baire^*}$ of projections in the Baire-*-algebra generated by a Q-algebra $\mathcal{A}$ with values in the metric space Ω. Then are equivalent:*
(i) $\omega \in spc(X)$.
(ii) *There is a sequence of atoms E_n in $\mathcal{E}_{Baire^*}$ with*

$$\lim_{n\to\infty} \mathbb{P}^{X|E_n}(B) = \begin{cases} 1 & \textit{if } \omega \textit{ is an inner point of } B \in \vartheta \\ 0 & \textit{if } \omega \textit{ is an inner point of } B^c \in \vartheta \end{cases}$$

Proof: (i)⇒(ii) Let d denote the metric on Ω. For $U_n := \{\omega' \in \Omega | d(\omega',\omega) < 1/n\}$, we have $X(U_n) \neq 0$, and hence there is an atom E_n in $\mathcal{E}_{Baire^*}$ with $E_n \leq X(U_n)$. Then for all n with $U_n \subseteq B$:

$$E_n X(B) E_n = E_n X(U_n) X(B) E_n = E_n X(U_n \cap B) E_n = E_n X(U_n) E_n = E_n \Rightarrow \mathbb{P}^{X|E_n}(B) = 1.$$

(ii)⇒(i) Let U be an open set with $\omega \in U$. Then, there is an atom E_n with $\mathbb{P}^{X|E_n}(U) > 0$.

$$\Rightarrow E_n X(U) E_n \neq 0 \Rightarrow X(U) \neq 0. \; \square$$

Theorem 3: *Let X be a real-valued observable on the system $\mathcal{E}_{Baire^*}$ of projections in the Baire-*-algebra generated by a Q-algebra $\mathcal{A}$. Then are equivalent:*
(i) $\lambda \in spc(X)$.
(ii) *There is a sequence of atoms E_n in $\mathcal{E}_{Baire^*}$ such that the expectation values of the probability distributions $\mathbb{P}^{X|E_n}$ converge against λ and their variances converge against 0.*

Proof: (i)⇒(ii) Let E_n be an atom with $E_n \leq X(U_n)$ and $U_n := \{s | |s-\lambda| < 1/n\}$. Then:

$$\mathbb{P}^{X|E_n}(U_n) = 1 \Rightarrow \int_{IR} (s-\lambda)^2 d\mathbb{P}^{X|E_n}(s) = \int_{U_n} (s-\lambda)^2 d\mathbb{P}^{X|E_n}(s) \leq \frac{1}{n^2}.$$

This implies that the variance of the probability distribution $\mathbb{P}^{X|E_n}$ is smaller than $1/n^2$ and that the distance between λ and the expectation value of $\mathbb{P}^{X|E_n}$ is smaller than $1/n$.
(ii)⇒(i) If U is an open set with $\lambda \in U$, there is $\varepsilon > 0$ with $\{s: |s-\lambda| < \varepsilon\} \subseteq U$. Then choose n such that we have for the variance $Var(\mathbb{P}^{X|E_n})$ and for the expectation value $Exp(\mathbb{P}^{X|E_n})$:

$$Var(\mathbb{P}^{X|E_n}) < \varepsilon^2/4 \text{ and } |Exp(\mathbb{P}^{X|E_n}) - \lambda| < \varepsilon/2.$$

Then $U \supseteq \{s: |s-\lambda| < \varepsilon\} \supseteq \{s: |s - Exp(\mathbb{P}^{X|E_n})| < \varepsilon/2\}$ and, from the Tschebyscheff inequality:

$$\mathbb{P}^{X|E_n}(U) \geq \mathbb{P}^{X|E_n}\left(\left\{s: \left|s - Exp\left(\mathbb{P}^{X|E_n}\right)\right| < \varepsilon/2\right\}\right)$$

$$= 1 - \mathbb{P}^{X|E_n}\left(\left\{s: \left|s - Exp\left(\mathbb{P}^{X|E_n}\right)\right| \geq \varepsilon/2\right\}\right) \geq 1 - \frac{Var\left(\mathbb{P}^{X|E_n}\right)}{\varepsilon^2/4} > 0$$

$$\Rightarrow E_n X(U) E_n \neq 0 \Rightarrow X(U) \neq 0\,. \square$$

If X is an observable with values in the topological space Ω and $\lambda \in \Omega \setminus spc(X)$, there is an open set V containing λ with $X(V)=0$. Therefore

$$\mathbb{P}^{X|E}(V)=0$$

for all $E\in\mathcal{E}$; i.e. a measurement of the observable X will never provide the value λ. This and the last two theorems justify the interpretation of the spectrum as the set of possible outcomes of a measurement.

Observables on the system $\mathcal{E}_{Baire^*}$ of projections in the Baire-*-algebra $\mathcal{A}_{Baire^*}$ generated by a Q-algebra $\mathcal{A}$ are spectral measures in the Baire-*-algebra $\mathcal{A}_{Baire^*}$, and the spectral theorem provides a one-to-one correspondence between the bounded real-valued (complex-valued) observables and the self-adjoint (normal[5]) elements of $\mathcal{A}_{Baire^*}$ which we will identify with each other from now on. In addition to the interpretation of the projections in $\mathcal{A}$ as logical propositions considered in [1], this now gives us also an interpretation of the self-adjoint bounded elements in $\mathcal{A}$ as real-valued observables. Furthermore, we have found a way to compute the probability distribution of a self-adjoint bounded X in $\mathcal{A}$ in addition to the expectation value and variance under $E\in\mathcal{E}$, and we have the identity:

$$\mathbb{E}(X|E) = \int \lambda \; d\,\mathbb{P}^{X|E}.$$

For an observable X with values in Ω,ϑ and a real-valued measurable function f on Ω, $f(X)$ is a real-valued observable with:

$$\mathbb{E}(f(X)|E) = \int \lambda \; d\,\mathbb{P}^{f(X)|E} = \int f(\lambda) \; d\,\mathbb{P}^{X|E}.$$

So far, we have two ways of constructing X^n for a self-adjoint X in $\mathcal{A}_{Baire^*}$: multiplying X n times with itself, and $f(X)$ with the function $f(\lambda)=\lambda^n$ (Definition 4 (i)). It follows from spectral theory that both ways provide the same result.

Let's have a look again at our second example. Let $\mathcal{A}_2$ be the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. $\mathcal{A}_2$ is weakly complete, and $\mathcal{E}$ is the lattice of orthogonal projections on $\mathcal{H}$. Let X be an observable on $\mathcal{E}$ with values in a measurable space Ω,ϑ, and let

[5] X is called normal if it commutes with its adjoint: $XX^*=X^*X$.

$E \in \mathcal{E}$ be the orthogonal projection on a one-dimensional linear subspace of $\mathcal{H}$. Then we have for any normed vector $\xi \in E(\mathcal{H})$[1]:

$$I\!P^{X|E}(B) = \langle \xi | X(B)\xi \rangle \quad \forall B \in \vartheta .$$

If Ω is the real continuum, the mapping $B \rightarrow X(B)$ is the spectral measure of a self-adjoint closed linear operator $\widetilde{X}$ on $\mathcal{H}$ which is bounded if and only if the observable X is bounded (for the sake of clarity let's, for a moment, make a distinction between the operator and the observable). In both cases, we have for any normed vector $\xi \in E(\mathcal{H})$:

$$\int \lambda \; d\, I\!P^{X|E} = \int \lambda \; d\langle \xi | X(\bullet)\xi \rangle = \langle \xi | \widetilde{X}\xi \rangle .$$

In the unbounded case, we only have to assume that ξ belongs to the dense linear subspace of $\mathcal{H}$ where $\widetilde{X}$ is defined to ensure integrability.

With a commutative Q-algebra, the distributions considered above provide only the probabilities 1 and 0; values between 0 and 1 are not possible[1]. The distribution of an observable with values in $I\!R^n$ under an atom E then reduces to a measure concentrated in a single point (Dirac measure).

Now we turn to the question whether there the same relations between unbounded elements of a Q-algebra and unbounded observables as in the bounded case studied above. For this purpose, we need the *algebraic spectrum*.

For $X \in \mathcal{A}$, the *algebraic spectrum* of X is the complement of the set of all $\lambda \in \mathbb{C}$ such that $(X-\lambda\, \mathbb{I})^{-1}$ exists in $\mathcal{A}_{Baire^*}$. The algebraic spectrum is defined for any $X \in \mathcal{A}$, X need not be self-adjoint or normal. It follows from spectral theory that the algebraic spectrum of X coincides with spc(X) if X has a spectral decomposition.

An unbounded real-valued observable X need not necessarily provide an element in $\mathcal{A}$ or $\overline{\mathcal{A}}$. For instance, if $\mathcal{H}$ is a Hilbert space with infinite dimension and $\mathcal{A}$ is the algebra of all bounded linear operators on $\mathcal{H}$, unbounded real-valued observables exist, but do not belong to $\mathcal{A}=\overline{\mathcal{A}}$. However, if $\mathcal{A}$ is the algebra of all complex-valued measurable functions on a measurable space Ω,ϑ, all real-valued observables including the unbounded ones belong to self-adjoint elements of $\mathcal{A}$.

Vice versa, does an unbounded self-adjoint element X in a Q-algebra $\mathcal{A}$ represent an observable? i.e. does it have a spectral decomposition? The answer is yes if X fulfills an additional condition (which bounded self-adjoint elements do anyway): the algebraic spectrum of X must contain real numbers only. Like in the theory of unbounded symmetrical Hilbert space operators, this can be proved by using the Cayley transformation $C_X:=(X-i)(X+i)^{-1}$. C_X is unitary ($C_X^*=C_X^{-1}$), thus bounded and normal and has a spectral decomposition from which the spectral decomposition of X can be derived with the function $f(\lambda):=i(\lambda-1)(\lambda+1)^{-1}$.

Lemma 4: *If a Q-algebra $\mathcal{A}$ is commutative and sequentially complete in the weak topology, the algebraic spectrum of each $X \in \mathcal{A}$ with $X=X^*$ contains real numbers only.*

Proof: In the commutative case, we have for a polynomial mapping p:

$$|p(X)|_E = |\mathbb{E}(p(X)|E)| = |p(\mathbb{E}(X|E))| \text{ for each atom } E.$$

Now let p_n be a sequence of polynomial mappings with $\lim_{n\to\infty} p_n(s) = \frac{1}{1+s^2}$ $\forall s \in \mathbb{R}$, and $\mathcal{A} \ni X = X^*$. Since $\mathbb{E}(X|E)$ is real, $p_n(X)$ is a weak Cauchy sequence and thus weakly converges against some $Z \in \mathcal{A}$ with

$$Z(\mathbb{I} + X^2) = \lim_{weak} p_n(X)(\mathbb{I} + X^2) = \mathbb{I}.$$

$$\Rightarrow \quad (X-i)(X+i)Z = (\mathbb{I} + X^2)Z = \mathbb{I}$$

$$\Rightarrow \quad (X+i)Z = (X-i)^{-1}.$$

Furthermore:

$$1 = \mathbb{E}((X-i)(X-i)^{-1}|E) = \mathbb{E}((X-i)|E)\,\mathbb{E}((X-i)^{-1}|E) \text{ for all atoms } E$$

$$\Rightarrow \left|\mathbb{E}((X-i)^{-1}|E)\right| = \frac{1}{|\mathbb{E}(X|E) - i|} \le 1 \text{ for all atoms } E$$

$$\Rightarrow \left\|(X-i)^{-1}\right\| \le 1, \text{ since } \|Y\| = \sup_{E \text{ atom}} |\mathbb{E}(Y|E)| \text{ for } Y \in \mathcal{A} \text{ in the commutative case.}$$

For $\lambda=\alpha+i\beta$ with real α,β and $\beta\neq0$, we can apply the above to $(X-\alpha)/\beta$ to see that

$$X - \lambda = \beta\left(\frac{X-\alpha}{\beta} - i\right)$$

has a bounded inverse. Therefore, λ does not belong to the algebraic spectrum of X. □

Theorem 4: *Each commutative Q-algebra $\mathcal{A}$ that is sequentially complete in the weak topology is isomorphic to the algebra of measurable complex-valued functions on some measurable space Ω,ϑ with an atomic σ-algebra ϑ.*

Proof: Let Ω be the set of atoms of $\mathcal{A}$. For $X\in\mathcal{A}$ define a complex-valued function on Ω by

$$\hat{X}(E) := \mathbb{E}(X|E).$$

The mapping $X \to \hat{X}$ is linear, multiplicative, injective, and commutes with the involution[1]. Weak convergence in $\mathcal{A}$ corresponds with pointwise convergence of the functions on Ω. Each projection F in $\mathcal{A}$ defines a subset $B_F := \{E\in\Omega \mid E\le F\} \subseteq \Omega$ and the system of all B_F forms an atomic σ-algebra ϑ.
It follows from the spectral decomposition of a self-adjoint element X of $\mathcal{A}$ (which is possible due to Lemma 4) that X can be weakly approximated by a sequence Y_n of finite linear combinations of projections in $\mathcal{A}$. Obviously, the $\hat{Y}_n$ are measurable and so is their pointwise limit $\hat{X}$. Since each $X\in\mathcal{A}$ can be written as $X=Y+iZ$ with $Y,Z\in\mathcal{A}$ self-adjoint, $\hat{X}$ is measurable for all $X\in\mathcal{A}$.

For any complex-valued measurable function f on Ω there is a sequence of simple measurable functions g_n on Ω with $f(E)=lim\ g_n(E)$ for all $E\in\Omega$. A simple function is a finite linear combination of indicator functions of measurable sets from ϑ. Since the indicator function of B_F equals $\hat{F}$, there is a $Y_n\in\mathcal{A}$ for each g_n with $g_n = \hat{Y}_n$, and since the g_n converge, the Y_n form a weak Cauchy sequence. Let X be the limit. Then $\hat{X} = f$. □

5 An extension of the concept of statistical predictability

What is $\mathbb{E}(Y|X=x)$ or $\mathbb{P}(F|X=x)$ for observables X, Y and a projection F? If x is an eigen-value of X and if the singleton $\{x\}$ belongs to the σ-algebra ϑ, we have $X(\{x\})\neq0$ and we can define $\mathbb{E}(Y|X=x):=\mathbb{E}(Y|X(\{x\}))$ or $\mathbb{P}(F|X=x):=\mathbb{P}(F|X(\{x\})$ if Y or F are statistically predictable under the projection $X(\{x\})$. However, we would like to define $\mathbb{E}(Y|X=x)$ or $\mathbb{P}(F|X=x)$ for other x that are not eigen-values, too. As we will see now, this is possible for any x belonging to the spectrum of X although $X(\{x\})=0$.

A realistic physical measurement never provides a real number as result, because of the limited precision the result is always an interval. The interval can be made narrower by improving precesion, and only by doing this infinitely many times, a real number can be achieved as measurement result. This infinite process, of course, is not possible in practice, but represents a theoretical and idealistic measurement process. Without assuming such a theoretical possibility, there would be no need to explain $\mathbb{E}(Y|X=x)$ or $\mathbb{P}(F|X=x)$.

Let $\mathcal{A}$ be a Q-algebra, and let X be an observable on $\mathcal{E}_{Baire^*}$ (the system of projections in $\mathcal{A}_{Baire^*}$) with values in a topological space Ω. Furthermore, assume $\omega\in spc(X)$ and let U be an open set in Ω with $\omega\in U$. Then, $X(U)\neq0$, and we can define:

$$\mathcal{D}_U := \left\{ Y\in\mathcal{A} \middle| \ Y \text{ is statistically predictable under } X(U) \right\} \quad \text{and}$$

$$\mathcal{D}_{X=\omega} := \bigcup_{U \text{ open with } \omega\in U} \mathcal{D}_U .$$

For open sets U, V with $U\subseteq V$ we have $X(U)\leq X(V)$ and therefore $\mathcal{D}_V\subseteq\mathcal{D}_U$ and $\mathbb{E}(Y|X(U)) = \mathbb{E}(Y|X(V))$ for all $Y\in\mathcal{D}_V$ (see [1]). For open sets U, V with $\omega\in U$, $\omega\in V$ and $Y\in\mathcal{D}_V\cap\mathcal{D}_U$ we thus have:

$$\mathbb{E}(Y|X(U)) = \mathbb{E}(Y|X(U\cap V)) = \mathbb{E}(Y|X(V)).$$

For $Y\in\mathcal{D}_{X=\omega}$ we can therefore define: $\mathbb{E}(Y|X=\omega) := \mathbb{E}(Y|X(U))$, where U is any open set with $\omega\in U$ and $Y\in\mathcal{D}_U$. $\mathcal{D}_U$ and $\mathcal{D}_{X=\omega}$ are linear subspaces of $\mathcal{A}$ (but not subalgebras), and the mapping $Y\rightarrow\mathbb{E}(Y|X=\omega)$ is linear on $\mathcal{D}_{X=\omega}$. Since

$$\begin{aligned} |\mathbb{E}(Y|X=\omega)| &= |\mathbb{E}(Y|X(U))| = \|\mathbb{E}(Y|X(U))X(U)\| = \|X(U)YX(U)\| \\ &\leq \|X(U)\|\,\|Y\|\,\|X(U)\| = \|Y\|, \end{aligned}$$

the mapping $Y \to \mathbb{E}(Y|X=\omega)$ can uniquely be extended to the τ_n-completion $\overline{\mathcal{D}}_{X=\omega}$ of $\mathcal{D}_{X=\omega}$. Then $\overline{\mathcal{D}}_{X=\omega} \subseteq \overline{\mathcal{A}}$. This extension is again denoted by $\mathbb{E}(Y|X=\omega)$.

Definition 6: (i) *The elements* $Y \in \overline{\mathcal{D}}_{X=\omega}$ *are called statistically predictable under* $X=\omega$ *with the expectation value* $\mathbb{E}(Y|X=\omega)$. *For* $E \in \mathcal{E}_{Baire^*} \cap \overline{\mathcal{D}}_{X=\omega}$, *this expectation value is interpreted as probability and denoted by* $\mathbb{P}(E|X=\omega)$.

(ii) *If Y is an observable with values in a measurable space* Δ,Σ *and* $B \in \Sigma$ *such that* $Y(B)$ *is statistically predictable under* $X=\omega$, *we define*:

$$\mathbb{P}^{Y|X=\omega}(B) := \mathbb{P}(Y(B)|X=\omega).$$

Generally, this probability is defined only for a subset of Σ, and in those cases when it is defined for all $B \in \Sigma$, it is not necessarily σ-additive, it may be only finitely additive.

Example: We assume that $Y=f(X)$ with a measurable mapping f between Ω and another topological space Δ and that f is continuous in ω. Then, for a Borel subset D in Δ:

$$\mathbb{P}^{Y|X=\omega}(D) := \begin{cases} 1 & \text{if } f(\omega) \text{ is an inner point of } D \\ 0 & \text{if } f(\omega) \text{ is an inner point of } D^c \end{cases}$$

Proof: Let $f(\omega)$ be an inner point in D, i.e. there is an open set $V \subseteq \Delta$ with $f(\omega) \in V \subseteq D$, and let U be an open set in Ω such that $\omega \in U$ and $f(U) \subseteq V$. Then $Y(D)=X(f^{-1}(D))$ and $X(U)$ commute and $X(U)Y(D)X(U)=X(f^{-1}(D)) \wedge X(U)=X(f^{-1}(D) \cap U)=X(U)$, i.e. $\mathbb{P}(Y(D)|X(U))=1$. If $f(\omega)$ is an inner point in D^c, we get in the same way: $\mathbb{P}(Y(D^c)|X(U))=1$, i.e. $\mathbb{P}(Y(D)|X(U))=0$. □

In general, the probability in the example above cannot be extended to those Borel subsets D in Δ where $f(\omega)$ is not an inner point of D or D^c.

6 Conclusions

The concept of statistical predictablity introduced in [1], on the one hand, can be derived from an analysis of quantum measurement and, on the other hand, can be taken as a starting point from which the quantum theoretic mathematical formalism can be developed. The minimum mathematical structure necessary to consider statistical predictablity is the Q-algebra. Each Q-algebra has a representation as linear (not necessarily bounded) operators on a pre-Hilbert-space.

An axiomatic approach to quantum theory would thus include the definition of the following three items only:

- statistical predictability[1] (the formal definition and its interpretation as probability),
- Q-algebras[1], and
- observables (§4).

The definition of observables requires a σ-complete lattice, and this is the reason why we had to consider the Baire-*-algebra generated by a Q-algebra in §3.

This axiomatic approach to quantum theory may not be more intelligible but it presumes, at least mathematically, less structure than the traditional Hilbert space approach, and the interpretation of λ with $EFE=\lambda E$ (which implies $0\le\lambda\le 1$) as a probability does not appear to be less intelligible than the postulate that

$$\frac{|\langle\xi|\eta\rangle|^2}{\|\xi\|\,\|\eta\|}$$

(with Hilbert space vectors η,ξ and a complex number $<\eta|\xi>$) represents a probability.

With this approach, the mathematical modeling of propositions and observables includes both the model of quantum physics (orthogonal projections and operators on a Hilbert space) as well as the Kolmogorov model of mathematical probability theory (σ-algebras and measurable functions; here the propositions and observables are called *events* and *random variables*[13]). The algebra of measurable functions is a Baire-*-algebra, but not a W*-algebra. From this point of view, Baire-*-algebras seem to be a more appropriate structure for quantum theory than W*-algebras the use of which, however, is more common among mathematically oriented physicists.

A major difference between this approach and other approaches to quantum theory is the concept of statistical predictability. It is this concept where the probabilities typical for quantum theory arise from. These probabilities differ from those studied in mathematical probability theory, what has extensively been discussed in [1]. So far, the commonalities with mathematical probability theory include propositions and observables, but not the probability.

Nevertheless, the introduction of states on the Q-algebra $\mathcal{A}$ (σ-additive positive linear functionals φ on $\mathcal{A}_{Baire^*}$ with $\varphi(\mathbb{I})=1$) would provide a second type of probability corresponding to the one considered in mathematical probability theory. This would result in Kolmogorov's approach to probability theory if $\mathcal{A}$ is assumed to be commutative, and in a non-Boolean extension of probability theory if $\mathcal{A}$ is not commutative. Statistical quantum mechanics (quantum thermodynamics) deals with both types of probabilities and needs the non-Boolean extension of probability theory.

For a projection E we can define $\varphi_E(X) := \mathbb{E}(X|E)$ for those X that are statistically predictable under E. φ_E is defined on all $\mathcal{A}_{Baire^*}$ and thus becomes a state if and only if E is an atom. The φ_E (E an atom) provide a special type of states, but there are many other states not arising from atoms.

A real-valued observable X can have the property: $X(N)=0$ for all Lebesgue-negligible Borel sets N. Then, each distribution of X is Lebesgue-continuous. This is impossible with a real-valued random variable defined on a measurable space. Thus, the assumption of dominated classes of distributions that is very often needed in mathematical statistics becomes much more natural in the more general framework of observables.

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