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Splitting solitons on a torus

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Abstract

New CP^1-soliton behaviour on a flat torus is reported. Defined by the Weierstrass elliptic function and numerically evolved from rest, one has topological charge-two solitons that initially show either the expected two energy lumps or, notably, four. In the former case, each soliton splits up in two lumps which eventually get back together; they keep splitting up and reuniting as time progresses.

1 Introduction

The CP^1 model in (2+1) dimensions appears as a low dimensional analogue of non-abelian gauge field theories in four dimensional space-time. This analogy relies on common properties like conformal invariance, existence of topological solitons, hidden symmetry and asymptotic freedom. Amongst various applications, CP^1 models have been used in the study of the quantum Hall effect and high-T_c superconductivity. In differential geometry, the soliton-solutions of CP^1 models are known as harmonic maps, a rich industry of research on its own.

The classical (2+0)-dimensional CP^1 or non-linear $O(3)$ model on the extended plane $\Re_2 \cup \{\infty\} \approx S_2$, where the soliton solutions are harmonic maps $S_2 \mapsto S_2$, has been amply discussed in the literature [1, 2]. In (2+1) dimensions the model is not integrable, and the study of its dynamics is done with the aid of numerical simulations. Due to the conformal invariance of the theory on the plane, the $O(3)$ solitons are unstable in the sense that they change their size under any small perturbation, either explicit or introduced by the discretisation procedure. It can make the solitons shrink indefinitely and, when their width is comparable to the lattice spacing, the numerical code breaks down [3]. However, such instability can be cured by the addition of two extra terms to the lagragian [4]. The first one resembles the term introduced by Skyrme in his nuclear model in four dimensional space-time [5], and the second one is a potential term. The fields of the planar Skyrme model (skyrmions) produce stable lumps which repel each other when started off from rest [4, 6].

In a recent paper [7] we considered both the pure and modified CP^1 schemes imposing periodic boundary conditions, which amounts to defining the system on a torus T_2. The corresponding soliton configurations are harmonic maps $T_2 \mapsto S_2$. We found (using the Weierstrass' $\sigma(z)$ function to define the solitons) that in contradistinction with the familiar theory on S_2, the toroidal model: • has no *analytical* single-soliton solution [this is because elliptic functions, in terms of which the toroidal solitons must be expressed, are at least of the second order; or, in the language of differential geometry, because genus(torus)=1]; • needs only a Skyrme term to stabilise the solitons (thus the lagrangian retains its $O(3)$ invariance: on S_2, the latter is broken by the additional potential term); • does not require a damping set-up for the numerical simulation (a radiation-absorbing device is implemented for the model evolved on the compactified plane in order to prevent the reflection of kinetic waves from the boundaries); • has perfectly static skyrmions when their initial speed v_0 is zero. This holds for any value of the Skyrme term θ_1, although the lumps are unstable when $\theta_1 < 7 \times 10^{-5}$, approximately. (As already pointed out, on S_2 the skyrmions move away from each other when $v_0 = 0$.); • possesses no critical velocity below which the skyrmions scatter back-to-back in head-on collisions. They always scatter at right angles provided $v_0 \neq 0$. Also, on T_2 the skyrmions scatter any number of times (multi-scattering), as they keep encountering each other in the periodic grid.

In the present work we study periodic CP^1 configurations started off from rest and defined through the Weierstrass' elliptic function $\wp(z)$. Being of order two, it naturally defines a soliton in the topological charge two sector.

2 Periodic skyrmion model

Our model is given by the lagrangian density

$$\mathcal{L} = \frac{|\partial_t W|^2 - 2|\partial_z W|^2}{(1+|W|^2)^2} + 8\theta_1 \frac{|\partial_z W|^2}{(1+|W|^2)^4}(|\partial_t W|^2 - |\partial_z W|^2), \tag{1}$$

$z = x + iy \in T_2$, which is the pure CP^1 model plus an additional Skyrme, θ_1-term ($\theta_1 \in \Re^+$). The complex field W obeys the periodic boundary condition

$$W[z + (m + in)L] = W(z), \qquad \forall t, \tag{2}$$

where $m, n = 0, 1, 2, ...$ and L is the size of a square torus. The static solitons (skyrmions) are elliptic functions which may be written as

$$W = \lambda\, \wp(z - a) + b, \qquad \lambda, a, b \in \mathcal{Z}, \tag{3}$$

$\wp(z)$ being the elliptic function of Weierstrass. Within a fundamental cell of length L, $\wp$ possesses the expansion [8]

$$\wp(z) = z^{-2} + \xi_2 z^2 + \xi_3 z^4 + ... + \xi_j z^{2j-2} + ..., \qquad \xi_j \in \Re. \tag{4}$$

This function is of the second order, hence (3) represents solitons of topological index 2. (In general, the product of n $\wp$'s will give solitons of even topological number $2n$.). Note that (3) is an approximate solution of the model (1), except in the pure CP^1 limit (θ_1=0) where it exactly solves the corresponding static field equation. Therefore, we expect our solitons to evolve only for a non-zero Skyrme parameter.

In reference [7] we computed the periodic solitons through

$$W = \prod_{j=1}^{\kappa} \frac{\sigma(z - a_j)}{\sigma(z - b_j)}, \qquad \sum_{j=1}^{\kappa} a_j = \sum_{j=1}^{\kappa} b_j, \tag{5}$$

employing a subroutine that numerically calculates $\sigma(z)$ [the parameters entering equation (5) are defined independently of those entering equation (3)]. Now, via the formula below [8], in this paper we use the same subroutine to compute $\wp(z)$:

$$\wp(z) = -\frac{d^2}{dz^2} \ln[\sigma(z)], \tag{6}$$

where the Laurent expansion for σ reads

$$\sigma(z) = \sum_{j=0}^{\infty} c_j z^{4j+1}, \qquad c_j \in \Re. \tag{7}$$

3 Basic numerical procedure

We treat configurations of the form (3) as the initial conditions for our time evolution, studied numerically. Our simulations run in the ϕ-formulation of the model, whose field equation follows from the lagrangian density (1) with the help of the stereographic projection

$$W = \frac{\phi_1 + i\phi_2}{1 - \phi_3}, \tag{8}$$

where the real scalar field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ satisfies $\vec{\phi}.\vec{\phi} = 1$.

We compute the series (7) up to the fifth term, the coefficients c_j being in our case negligibly small for $j \geq 6$. We employ the fourth-order Runge-Kutta method and approximate the spatial derivatives by finite differences. The laplacian is evaluated using the standard nine-point formula and, to further check our results, a 13-point recipe is also utilised. Our results showed unsensitiveness to either method. The discrete model evolves on a 200×200 periodic lattice ($n_x = n_y = 200$) with spatial and time steps δx=δy=0.02 and δt=0.005, respectively. The size of our fundamental, toroidal network is $L = n_x \times \delta x = 4$.

Unavoidable round-off errors gradually shift the fields away from the constraint $\vec{\phi}.\vec{\phi} = 1$. So we rescale $\vec{\phi} \to \vec{\phi}/\sqrt{\vec{\phi}.\vec{\phi}}$ every few iterations. Each time, just before the rescaling operation, we evaluate the quantity $\mu \equiv \vec{\phi}.\vec{\phi} - 1$ at each lattice point. Treating the maximum of the absolute value of μ as a measure of the numerical errors, we find that $\max|\mu| \approx 10^{-8}$. This magnitude is useful as a guide to determine how reliable a given numerical result is. Usage of an unsound numerical procedure in the Runge-Kutta evolution shows itself as a rapid growth of $\max|\mu|$; this also occurs, for instance, in the $O(3)$ limit ($\theta_1 = 0$) when the unstable lumps of energy become infinitely spiky.

4 Splitting lumps

The static energy density associated with the field (3) reads

$$E = \epsilon(1 + 4\theta_1\epsilon), \quad \epsilon = 8\lambda^2 \frac{|\wp(z-a)||\wp^2(z-a) - \wp^2(L/2)|}{[1 + |\lambda\wp(z-a) + b|^2]^2}, \tag{9}$$

where the identity

$$[\frac{d\wp(z-a)}{dz}]^2 = 4\wp(z-a)[\wp(z-a)^2 - \wp^2(L/2)], \tag{10}$$

has been used [8]. The parameter λ is related to the size of the solitons, b determines their mutual separation and a merely shifts the solution on the torus. Without loss of generality, we may take the values of these parameters according to numerical convenience. Let us take λ=1 and a=(2.025, 2.05) throughout. Now, for b=0 we have $W = \wp(z-a)$, whose energy density gives two indistinguishable lumps on top of each other, as depicted in figure 1 (top-left). The value b=1 gives two lumps separated along the ordinates [figure 1 (top-right)], whereas b=-1 positions them along the abscissas. A pure imaginary b places our extended structures on a diagonal bisecting the toroidal grid. For b with non-zero real and imaginary parts the lumps are situated in an arbitrary diagonal of the cell. These set-ups are true regardless of θ_1, which we have put equal to 0.001.

Our numerical simulations show that the skyrmions are stable. Their stability is reflected on the left-hand side of the nether half of figure 1, which shows the evolution of the maximum value of the system's total energy density (E_{max}) for b=0,1. In the $O(3)$ limit the lumps are no longer stable, as can be appreciated from the bottom-right graph of figure 1. In this case, as expected, the solitons remain static with the passing of time.

But in the stable, Skyrme situation, the lumps evolve in novel fashion. Let us first consider the configuration when the extended entities are on top of each other at $t = 0$. As time elapses, the system splits in four equal lumps, each progressing towards its nearest lattice corner. There they meet and coalesce, for all corners are nothing but the same point. Then the solitons split up once more and the 'fractional-skyrmions' make their way back to the centre of the nett, in a cycle that repeats itself indefinitely. The foregoing event is illustrated in the superior half of figure 2, with the trajectory of the four energy peaks in the $x-y$ plane. The accompanying 3-D picture captures the moment when the skyrmion quartet, having concurred at the corners and split afresh, begin to motion towards the centre of the network. Worthy of remark is that the trajectory in question resembles the usual head-on collision course and subsequent 90° scattering of two solitons [9], in spite of ours being energy chunks with no initial velocity. Also, we have thoroughly verified that the topological number is 2 all along the numerical evolution, which suggests that each 'fractional soliton' carries a topological charge of 1/2. This behaviour contrasts with the one on S_2, where two lumps on top of each other were found to move away from one another, two evolving lumps in mutual repulsion [4].

A more involved trajectory occurs for two initially well-separated skyrmions. In the bottom-left plot of figure 2, the labels $a-g$ indicate the itinerary of one of the entities (call the corresponding symmetrical points $a' - g'$). At $t = 0$ a full lump is at point a but it soon halves under the numerical simulation. One of its fractional offspring moves following curve b, whereas its counterpart proceeds in the opposite sense. At $x = 0 = 4$ they get back together into one full structure, which runs vertically up before separating anew. One of these components cruises along c and, at site d, reunites back with its peer travelling from the left. Before dividing itself according to curve e, the skyrmion is seen to shift towards the centre, as one can tell from the small leg connecting curves d-e (the full lump started at a' undergoes a similar process). The bottom-right diagram of figure 2 exhibits the skyrmions heading centrewards from d and d'. Thence our system continues through $f - g - h$ and returns to its $t = 0$ coordinates. Observe that at g a half-soliton from lump a undistinguishably coalesces with a half-soliton from lump a', so actually we do not know which bit is ascending (descending) along h (h'). We terminated our simulations when a like cycle was about to commence, as evidenced by the small vertical lines emerging from a, a'. Our $\wp$-solitons undergo both repulsive and attractive forces.

Our research has been constraint to systems with zero inital speed. Important mathematical aspects of CP^1 solitons given by equation (3) [the $O(3)$ case only] have been recently analysed in [10] using the geodesic approximation. Consistently, the presence of four energy peaks rather than two is therein observed as well.

It is important to note that our results hold for any initial data with v_0=0. As remarked earlier, they do not depend qualitatively on the values of λ, a, b in equation (3). Nor do our results depend on the torus being square or non-square. For instance, the occurrence of four peaks in the charge-two topological sector (a notable outcome of the present work) is unaltered. In effect, the four peaks arise due to the symmetry of the energy density under $\wp \to -\wp$ [see equation (9) for $b = 0$] and the fact that $\wp$ is even. What square periodic boundary conditions (the so-called lemnistcatic case in pure mathematics) do is to greatly simplify the computations and lead to the relatively simple Laurent expansion (7).

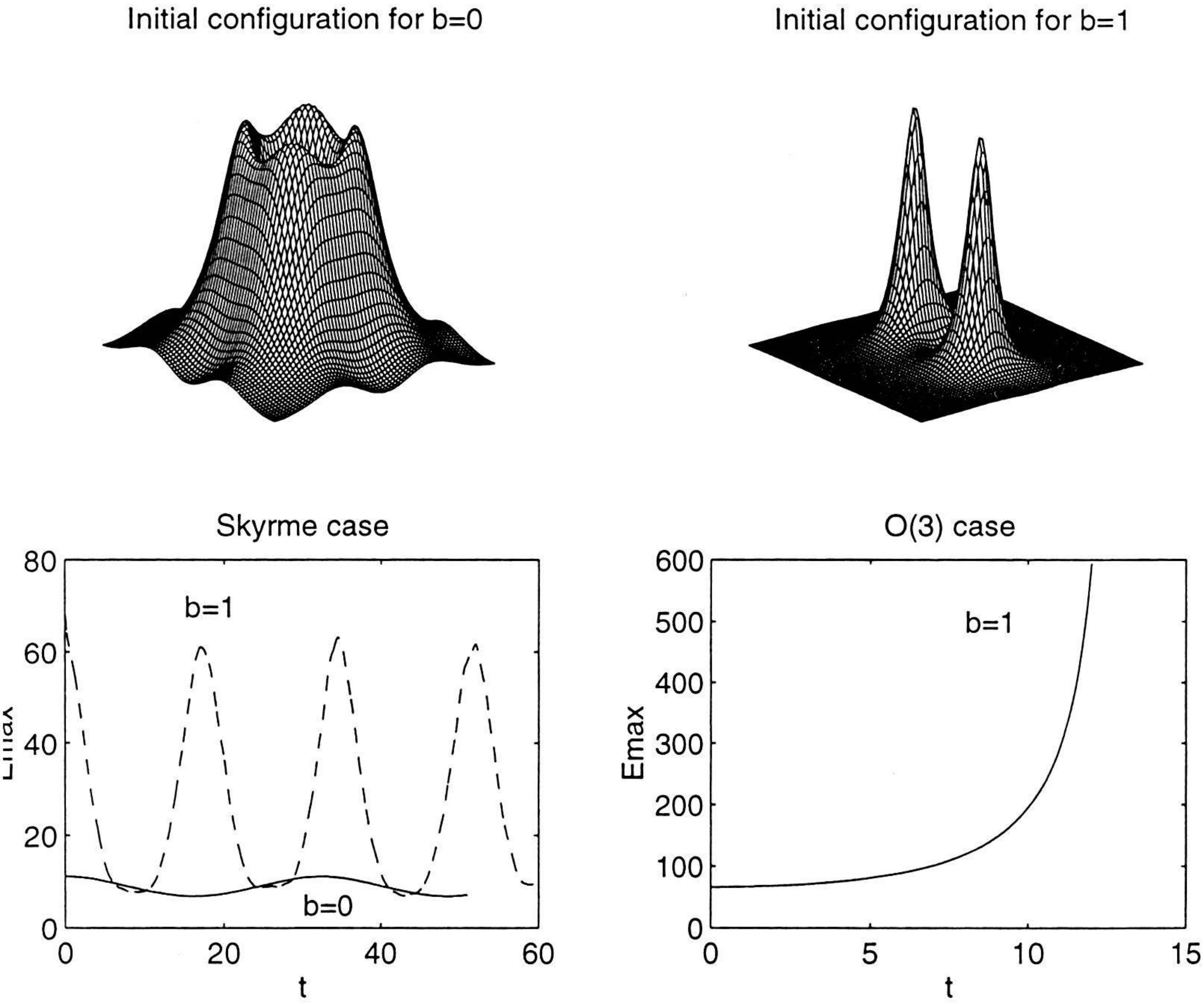

Figure 1: Energy density configurations at $t = 0$ and the evolution of their peaks. The bottom-right graph corresponds to $\theta_1=0$, when the lumps are unstable and shrink non-stoppingly.

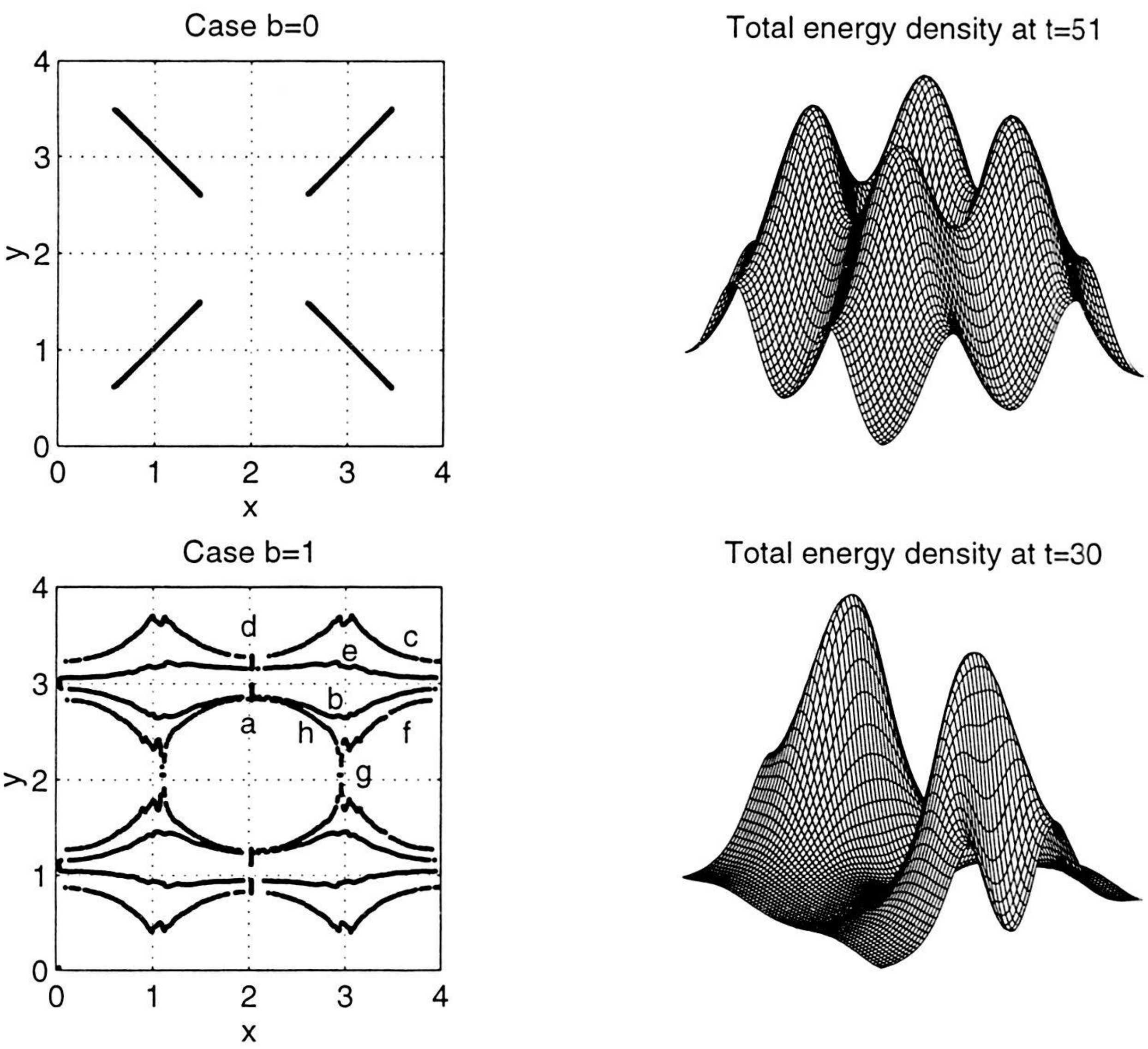

Figure 2: Above: Trajectory of the skyrmions initially on top of each other. They split up in four lumps heading to the corners where they coalesce and break-off again, moving back to the centre of the lattice, as in the illustration for t=51. Below: The initially separated skyrmions also divide each in two, but transit more complicated paths; the labels $a - h$ refer to one of the 'half-lumps'. The t=30 picture is shortly after the fractional progeny have reunited at d (and at its symmetrical point) and begun to travel centrewards.

5 Concluding remarks

The CP^1 model in (2+1) dimensions is variegated. More so its stable, skyrmionic version on T_2, which in this work has been shown to possess qualitatively different features as compared to the familiar model on the compactified plane.

A particularity of our periodic solitons is that they have no analytic representative of degree one, limitation dictated by their elliptic nature. In (2+0) dimensions, the CP^1 model on S_2 is known to have soliton solutions in all topological classes.

Another peculiarity of the toroidal model, one herein discovered, is that the properties of the skyrmions depend on the elliptic function used to define them. Thus, skyrmion fields expressed in terms of $\sigma(z)$, equation (5), evolve differently than those expressed through $\wp(z)$, equation (3). In the former case, energy chunks started off from rest stay still in their initial positions as time goes by. In the latter case, the system splits up in four lumps that stroll the network acted upon by repulsive-attractive forces. It is our concern to analyse the solitons on T_2 by employing alternative elliptic functions, and classify the different characteristics exhibited (let us remind that on S_2 no new soliton traits arise from casting two-soliton configurations in different ways, *e.g.*, $W = z^2, z^{-2}, \frac{(z-a)(z-b)}{(z-c)(z-d)}$). Comparison of the various elliptic solitons among themselves (and with the spherical lumps) should help us understand better the mechanisms underlying the CP^1 dynamics. This of course must include investigation on the appealing question of collisions. On T_2, the fields (5) have yielded -in the Skyrme case- the outcome of always scattering off at 90° when impinged with a non-zero initial speed [7]. On S_2, unsimilarly, the existence of a critical speed, below which the skyrmions scatter at 180° to the initial direction of motion, has long been a landmark of the planar model .

Finally, that the topological charge of our splitting system is very well conserved (=2) as time progresses, invites speculation on whether each 'fractional-soliton' carries a non-integer degree. Clearly, further research on this matter is required.

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