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Algebraic Realization of Supersymmetric Quantum Mechanics for Cyclic Shape Invariant Potentials

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Abstract. We study in detail the spectrum of the bosonic oscillator Hamiltonian associated with the C_3-extended oscillator algebra $\mathcal{A}^{(3)}_{\alpha_0\alpha_1}$, where C_3 denotes a cyclic group of order three, and classify the various types of spectra in terms of the algebra parameters α_0, α_1. In such a classification, we identify those spectra having an infinite number of periodically spaced levels, similar to those of cyclic shape invariant potentials of period three. We prove that the hierarchy of supersymmetric Hamiltonians and supercharges, corresponding to the latter, can be realized in terms of some appropriately chosen $\mathcal{A}^{(3)}_{\alpha_0\alpha_1}$ algebras, and of Pauli spin matrices. Extension to period-λ spectra in terms of C_λ-extended oscillator algebras is outlined.

1 Introduction

When supplemented with the concept of shape invariance [1], supersymmetric quantum mechanics (SSQM) [2] has proved very useful for generating exactly solvable quantum mechanical models. Devising new approaches to construct shape invariant potentials is still under current investigation (for a recent review see Ref. [3]). A recent advance in this field has been the introduction of cyclic shape invariant potentials by Sukhatme *et al* [4], generalizing a previous work of Gangopadhyaya and Sukhatme [5].

In addition, SSQM has established a nice symmetry between bosons and fermions [2].

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Such a symmetry has been extended to some exotic statistics. Replacing fermions by parafermions [6], pseudofermions [7], or orthofermions [8], for instance, has led to parasupersymmetric (PSSQM) [9, 10], pseudosupersymmetric [7], or orthosupersymmetric [11] quantum mechanics, respectively.

The development of quantum groups and quantum algebras [12] during the last decade has proved very useful in connection with such problems. In particular, various deformations and extensions of the oscillator algebra have found a lot of applications to quantum mechanics, in general, and to SSQM and some of its generalizations, in particular.

Deformations of the oscillator algebra arose from successive generalizations of the Arik-Coon [13], and Biedenharn-Macfarlane [14] q-oscillators. Various attempts have been made to introduce some order in the various deformations by defining 'generalized deformed oscillator algebras' (GDOAs). Among them, one may quote the treatments due to Jannussis *et al* [15], Daskaloyannis [16], Irac-Astaud and Rideau [17], McDermott and Solomon [18], Meljanac *et al* [19], Katriel and Quesne [20], Quesne and Vansteenkiste [21, 22]. In the remainder of the present paper, we shall refer to GDOAs as defined in Ref. [21]. GDOAs have found some interesting applications to the algebraic treatment of some one-dimensional exactly solvable potentials [23, 24] or two-dimensional superintegrable systems [25], as well as to the description of systems with non-standard statistics [19, 26, 27, 28].

G-extended oscillator (or alternatively Heisenberg[3]) algebras, where G is some finite group, essentially appeared in connection with n-particle integrable models. It was shown that they provide an algebraic formulation [29, 30, 31] of the Calogero model [32] or some generalizations thereof [33]. In the former case, G is the symmetric group S_n [30]. For two particles, the abelian group S_2 can be realized in terms of Klein operator $K = (-1)^N$, where N denotes the number operator. The S_2-extended oscillator algebra is then known as the Calogero-Vasiliev [29], or modified [31] oscillator algebra.

The usefulness of GDOAs in connection with SSQM was pointed out by Bonatsos and Daskaloyannis [34]. Then Brzeziński *et al* [31], and Plyushchay [35] in more detail (see also Ref. [36]), showed that the Calogero-Vasiliev algebra provides a minimal bosonization of SSQM in terms of boson-like particles, instead of a combination of bosons and fermions, as is the case in the standard approach [2].

In a recent work [37], we introduced a new type of G-extended oscillator algebras $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$, where G is a cyclic group of order λ, $C_\lambda = \{I, T, T^2, \ldots, T^{\lambda-1}\}$, and α_0, α_1, $\ldots$, $\alpha_{\lambda-2}$ denote $\lambda - 1$ independent real parameters. Since C_λ is an Abelian group, its elements can be realized in terms of N only, so that $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ becomes a GDOA. The cyclic group C_2 being isomorphic to S_2, the C_2-extended oscillator algebra $\mathcal{A}^{(2)}_{\alpha_0}$ is equivalent to Calogero-Vasiliev algebra. Hence, new features only appear for $\lambda \geq 3$.

To each $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ algebra, one can associate a bosonic oscillator Hamiltonian H_0. That

[3]In both the oscillator and Heisenberg algebras, the creation and annihilation operators $a^\dagger$, a are considered as generators, but in the former the number operator appears as an additional independent generator, whereas in the latter it is defined in terms of $a^\dagger$, a as $N \equiv a^\dagger a$.

corresponding to $\mathcal{A}^{(2)}_{\alpha_0}$ is just the two-particle Calogero Hamiltonian, which has a very simple spectrum, coinciding with that of a shifted harmonic oscillator. For higher λ values, the situation is entirely different as, according to the parameter values, the spectrum may be nondegenerate, or may exhibit some $(\nu+1)$-fold degeneracies, where ν may take any value in the set $\{1, 2, \ldots, \lambda-1\}$, with in each case various possibilities for the level ordering.

In [37], we extended Plyushchay's work by showing that the C_3-extended oscillator algebra $\mathcal{A}^{(3)}_{\alpha_0\alpha_1}$ provides a minimal bosonization of Rubakov-Spiridonov PSSQM of order $p = 2$ [9]. More generally, it can be proved [38] that $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ leads to the same result for Rubakov-Spiridonov PSSQM of order $p = \lambda - 1$.

Here, we will address the problem of SSQM for cyclic shape invariant potentials of period λ. We will prove that the corresponding hierarchy of supersymmetric Hamiltonians and supercharges, which repeats after a cycle of λ iterations can be realized in terms of some appropriate $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ algebras, and of Pauli spin matrices. Although the detailed derivation will be carried out for the simplest nontrivial case corresponding to $\lambda = 3$, it will become clear that the arguments are still valid for arbitrary $\lambda \geq 3$.

To deal with this problem, after reviewing the definitions of the C_3-extended oscillator algebra, and of the corresponding oscillator Hamiltonian in section 2, we will study in detail the H_0 spectrum associated with $\mathcal{A}^{(3)}_{\alpha_0\alpha_1}$, and derive the complete classification of the different types of spectra in terms of the algebra parameters α_0, α_1, in section 3. In section 4 , we will then identify those spectra having an infinite number of periodically spaced levels, and show that for some of them one can obtain the searched for algebraic realization of SSQM. Section 5 contains some concluding remarks about the extension to period-λ spectra.

2 C_3-Extended Oscillator Algebra and Hamiltonian

Let us consider the bosonic oscillator Hamiltonian, defined (in units wherein $\hbar\omega = 1$) by [37]

$$H_0 \equiv \tfrac{1}{2}\left\{a, a^\dagger\right\}, \tag{2.1}$$

where the creation and annihilation operators $a^\dagger$, a satisfy the generalized relations

$$\begin{aligned} \left[N, a^\dagger\right] &= a^\dagger, \qquad [N, T] = 0, \qquad T^3 = I, \\ \left[a, a^\dagger\right] &= I + \kappa_1 T + \kappa_2 T^2, \qquad a^\dagger T = e^{-2\pi i/3}\, T a^\dagger, \end{aligned} \tag{2.2}$$

together with their Hermitian conjugates. Here, $N = N^\dagger$ is the number operator, $T = \left(T^\dagger\right)^{-1}$ is the (unitary) generator of a cyclic group $C_3 = \{I, T, T^2\}$, and κ_1, κ_2 are two complex constants, restricted by the condition $\kappa_2 = \kappa_1^*$ (deriving from the relation $T^2 = T^\dagger$).

In the present paper, we shall be concerned with a realization of T as a function of N, given by

$$T = e^{2\pi i N/3}, \tag{2.3}$$

in which case there only remain two nontrivial relations in equation (2.2), namely

$$\left[N, a^\dagger\right] = a^\dagger, \qquad \left[a, a^\dagger\right] = I + 2(\Re e\, \kappa_1) \cos \tfrac{2\pi}{3} N - 2(\Im m\, \kappa_1) \sin \tfrac{2\pi}{3} N. \tag{2.4}$$

According to [21], equation (2.4) defines a GDOA $\mathcal{A}(G(N))$, with

$$G(N) \equiv I + 2(\Re e\, \kappa_1) \cos \tfrac{2\pi}{3} N - 2(\Im m\, \kappa_1) \sin \tfrac{2\pi}{3} N. \tag{2.5}$$

Provided its parameters satisfy some conditions to be given below, the algebra possesses a bosonic Fock space $\mathcal{F} = \{\, |n\rangle \mid n = 0, 1, 2, \ldots \}$, spanned by the normalized eigenvectors of N,

$$N|n\rangle = n|n\rangle, \qquad \langle n|m\rangle = \delta_{n,m}, \tag{2.6}$$

which can be written as

$$|n\rangle = \mathcal{N}_n^{-1/2} \left(a^\dagger\right)^n |0\rangle, \qquad n = 0, 1, 2, \ldots, \tag{2.7}$$

where $\mathcal{N}_n$ is some normalization constant, and $|0\rangle$ is a vacuum state, i.e.,

$$a|0\rangle = 0. \tag{2.8}$$

From equation (2.4), it is clear that the operators $a^\dagger$, a act differently in the three subspaces $\mathcal{F}_\mu$, $\mu = 0, 1, 2$, of $\mathcal{F}$, defined by $\mathcal{F}_\mu \equiv \{\, |3k + \mu\rangle \mid k = 0, 1, 2, \ldots \}$, and such that $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2$. Actually, these three subspaces are the carrier spaces of the three inequivalent irreducible (one-dimensional) matrix representations of C_3, defined by $\Gamma^\mu(T) = \exp(2\pi i \mu/3)$, $\mu = 0, 1, 2$ [39]. The projection operators P_μ on the $\mathcal{F}_\mu$ subspaces are given by $P_\mu = \frac{1}{3} \sum_{\nu=0}^{2} \exp(-2\pi i \mu\nu/3)\, T^\nu$, or

$$\begin{aligned} P_0 &= \tfrac{1}{3}\left(I + 2\cos \tfrac{2\pi}{3} N\right), \qquad P_1 = \tfrac{1}{3}\left(I - \cos \tfrac{2\pi}{3} N + \sqrt{3} \sin \tfrac{2\pi}{3} N\right), \\ P_2 &= \tfrac{1}{3}\left(I - \cos \tfrac{2\pi}{3} N - \sqrt{3} \sin \tfrac{2\pi}{3} N\right). \end{aligned} \tag{2.9}$$

As it can easily be checked on equation (2.9), the P_μ's satisfy the relations

$$P_\mu P_\nu = \delta_{\mu,\nu} P_\mu, \qquad \sum_{\mu=0}^{2} P_\mu = I, \tag{2.10}$$

as it should be.

In terms of such operators, equation (2.4) can be rewritten as

$$\left[N, a^\dagger\right] = a^\dagger, \qquad \left[a, a^\dagger\right] = I + \alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2, \tag{2.11}$$

where α_μ, $\mu = 0, 1, 2$, are three real parameters, connected with κ_1 and $\kappa_2 = \kappa_1^*$ by the relations $\alpha_\mu = \sum_{\nu=1}^{2} \exp(2\pi i \mu\nu/3)\, \kappa_\nu$, or

$$\alpha_0 = 2\,\Re e\, \kappa_1, \quad \alpha_1 = -\,\Re e\, \kappa_1 - \sqrt{3}\, \Im m\, \kappa_1, \quad \alpha_2 = -\alpha_0 - \alpha_1 = -\,\Re e\, \kappa_1 + \sqrt{3}\, \Im m\, \kappa_1. \tag{2.12}$$

Hence, we may also express $G(N)$ as

$$G(N) = I + \alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2, \qquad \text{with } \alpha_0 + \alpha_1 + \alpha_2 = 0, \tag{2.13}$$

and denote the algebra $\mathcal{A}(G(N))$ by $\mathcal{A}^{(3)}_{\alpha_0\alpha_1}$, where the two independent real parameters α_0, α_1 are specified. In the remainder of this paper, we will assume $\alpha_\mu \equiv \alpha_{\mu \bmod 3}$, and $P_\mu \equiv P_{\mu \bmod 3}$ for arbitrary integer μ values.

For any GDOA, one may define a so-called structure function $F(N)$, which is the solution of the difference equation $F(N+1) - F(N) = G(N)$, such that $F(0) = 0$ [16, 20, 21, 22, 34]. In the present case, we get

$$F(N) = N + \beta_1 P_1 + \beta_2 P_2, \qquad \text{where } \beta_1 \equiv \alpha_0,\ \beta_2 \equiv \alpha_0 + \alpha_1. \tag{2.14}$$

In the bosonic Fock space $\mathcal{F}$, $F(N)$ satisfies the relations

$$a^\dagger a = F(N), \qquad a a^\dagger = F(N+1), \tag{2.15}$$

and the normalization coefficient $\mathcal{N}_n$ in equation (2.7) is given by $\mathcal{N}_n = \prod_{i=1}^n F(i)$, or

$$\begin{aligned}
\mathcal{N}_{3k} &= 3^{3k}\left[\Gamma(\overline{\beta}_1)\Gamma(\overline{\beta}_2)\right]^{-1}\Gamma(k+1)\Gamma(k+\overline{\beta}_1)\Gamma(k+\overline{\beta}_2), \\
\mathcal{N}_{3k+1} &= 3^{3k+1}\left[\Gamma(\overline{\beta}_1)\Gamma(\overline{\beta}_2)\right]^{-1}\Gamma(k+1)\Gamma(k+1+\overline{\beta}_1)\Gamma(k+\overline{\beta}_2), \\
\mathcal{N}_{3k+2} &= 3^{3k+2}\left[\Gamma(\overline{\beta}_1)\Gamma(\overline{\beta}_2)\right]^{-1}\Gamma(k+1)\Gamma(k+1+\overline{\beta}_1)\Gamma(k+1+\overline{\beta}_2),
\end{aligned} \tag{2.16}$$

in terms of gamma functions, and of $\overline{\beta}_1 \equiv (\beta_1+1)/3$, $\overline{\beta}_2 \equiv (\beta_2+2)/3$. The creation and annihilation operators act upon $|n\rangle$ as

$$a^\dagger|n\rangle = \sqrt{F(n+1)}\,|n+1\rangle, \qquad a|n\rangle = \sqrt{F(n)}\,|n-1\rangle. \tag{2.17}$$

Hence, from equation (2.14), it is obvious that $\mathcal{F}$ exists if and only if $F(1) > 0$ and $F(2) > 0$, or, in other words, the algebra parameters are restricted to those values for which

$$\alpha_0 > -1, \qquad \alpha_1 > -2 - \alpha_0. \tag{2.18}$$

We shall henceforth assume that these conditions are fulfilled. Note that $\alpha_0 = \alpha_1 = 0$ corresponds to the standard harmonic oscillator.

It is now straightforward to determine the action of the bosonic oscillator Hamiltonian H_0, defined in equation (2.1), in the bosonic Fock space $\mathcal{F}$. For such a purpose, it is useful to rewrite H_0 in the equivalent forms

$$H_0 = a^\dagger a + \tfrac{1}{2}\left(I + \alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2\right) = N + \tfrac{1}{2}I + \gamma_0 P_0 + \gamma_1 P_1 + \gamma_2 P_2 \tag{2.19}$$

by using equations (2.11), (2.14), and (2.15). In equation (2.19), the parameters γ_μ, $\mu = 0$, 1, 2, are defined by

$$\gamma_0 \equiv \tfrac{1}{2}\alpha_0, \qquad \gamma_1 \equiv \tfrac{1}{2}(2\alpha_0 + \alpha_1), \qquad \gamma_2 \equiv \tfrac{1}{2}(\alpha_0 + \alpha_1), \tag{2.20}$$

and satisfy the relation $\gamma_0 - \gamma_1 + \gamma_2 = 0$. The number operator eigenvectors $|n\rangle = |3k + \mu\rangle$ are also eigenvectors of H_0, corresponding to the eigenvalues

$$E_{3k+\mu} = 3k + \mu + \gamma_\mu + \tfrac{1}{2}, \qquad k = 0, 1, 2, \ldots, \qquad \mu = 0, 1, 2. \tag{2.21}$$

In each $\mathcal{F}_\mu$ subspace of $\mathcal{F}$, the spectrum of H_0 is therefore harmonic, but the three infinite sets of equally spaced energy levels, corresponding to $\mu = 0$, 1, 2, respectively, may be shifted with respect to each other by some amounts depending upon the algebra parameters α_0, α_1 through their linear combinations γ_0, γ_1, γ_2, defined in equation (2.20). We may therefore obtain nondegenerate spectra, as well as spectra with some double or triple degeneracies. In the next section, we will study such spectra in detail.

3 Classification of C_3-Extended Oscillator Hamiltonian Spectra

To obtain the various types of H_0 spectra, we shall proceed in two steps. We shall first determine the possible orderings of the H_0 ground states in $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$, corresponding to the eigenvalues E_0, E_1, and E_2, respectively. This will give rise to three general and two intermediate classes of spectra. Then, for each of these five possibilities, we shall successively study the relative order of the excited states in $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$ in the nondegenerate, doubly- and triply-degenerate cases.

Considering first E_0, E_1, and E_2, we obtain from equations (2.20) and (2.21)

$$E_1 - E_0 = \tfrac{1}{2}(\alpha_0 + \alpha_1 + 2), \qquad E_2 - E_1 = \tfrac{1}{2}(2 - \alpha_0), \qquad E_2 - E_0 = \tfrac{1}{2}(\alpha_1 + 4). \tag{3.1}$$

Since the parameter values are restricted by equation (2.18), it is obvious that the ground states in $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$ may either be nondegenerate, or exhibit a double degeneracy. In the former case, they may be ordered in three different ways, which we will refer to as (I), (II), and (III), respectively, as listed hereafter

$$\begin{array}{lll} \text{(I)} & E_0 < E_1 < E_2 & \text{if } -1 < \alpha_0 < 2 \text{ and } -2 - \alpha_0 < \alpha_1, \\ \text{(II)} & E_0 < E_2 < E_1 & \text{if } 2 < \alpha_0 \text{ and } -4 < \alpha_1, \\ \text{(III)} & E_2 < E_0 < E_1 & \text{if } 2 < \alpha_0 \text{ and } -2 - \alpha_0 < \alpha_1 < -4. \end{array} \tag{3.2}$$

In the latter case, their ordering is intermediate between classes (I) and (II), or (II) and (III), and are given by

$$\begin{array}{lll} \text{(I-II)} & E_0 < E_1 = E_2 & \text{if } \alpha_0 = 2 \text{ and } -4 < \alpha_1, \\ \text{(II-III)} & E_0 = E_2 < E_1 & \text{if } 2 < \alpha_0 \text{ and } \alpha_1 = -4, \end{array} \tag{3.3}$$

respectively.

Let us now consider the excited states in $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$, and distinguish between nondegenerate, doubly- and triply-degenerate spectra.

3.1 Nondegenerate spectra

For nondegenerate spectra, we have only to consider the three general classes (I), (II), and (III).

Starting with class (I), we note that since $E_3 - E_2 = (2 - \alpha_1)/2$, and $E_3 - E_1 = (4 - \alpha_0 - \alpha_1)/2$, we have three different possibilities for the ordering of E_3 with respect to E_1, and E_2:

$$\begin{aligned} &E_0 < E_1 < E_2 < E_3 && \text{if } -1 < \alpha_0 < 2 \text{ and } -2 - \alpha_0 < \alpha_1 < 2, \\ &E_0 < E_1 < E_3 < E_2 && \text{if } -1 < \alpha_0 < 2 \text{ and } 2 < \alpha_1 < 4 - \alpha_0, \\ &E_0 < E_3 < E_1 < E_2 && \text{if } -1 < \alpha_0 < 2 \text{ and } 4 - \alpha_0 < \alpha_1. \end{aligned} \tag{3.4}$$

Furthermore, since $E_4 - E_2 = (\alpha_0 + 4)/2$ is positive over the whole parameter range, in the first two cases the remainder of the spectrum is entirely determined, so that we obtain $E_0 < E_1 < E_2 < E_3 < E_4 < E_5 < E_6 < \cdots$, and $E_0 < E_1 < E_3 < E_2 < E_4 < E_6 < E_5 < \cdots$, respectively.

In the third case, we have to study the ordering of E_6 with respect to E_1, and E_2. As $E_6 - E_2 = (8 - \alpha_1)/2$, and $E_6 - E_1 = (10 - \alpha_0 - \alpha_1)/2$, there again appear three different possibilities:

$$\begin{aligned} &E_0 < E_3 < E_1 < E_2 < E_6 && \text{if } -1 < \alpha_0 < 2 \text{ and } 4 - \alpha_0 < \alpha_1 < 8, \\ &E_0 < E_3 < E_1 < E_6 < E_2 && \text{if } -1 < \alpha_0 < 2 \text{ and } 8 < \alpha_1 < 10 - \alpha_0, \\ &E_0 < E_3 < E_6 < E_1 < E_2 && \text{if } -1 < \alpha_0 < 2 \text{ and } 10 - \alpha_0 < \alpha_1, \end{aligned} \tag{3.5}$$

where for the first two the remainder of the spectrum is entirely determined.

By recursively carrying on such a classification, we get two nondegenerate spectra subclasses (I.1) and (I.2), themselves labelled by some index n running over 1, 2, 3, ...:

$$\begin{aligned} (\mathrm{I}.1.n) \quad & E_0 < E_3 < \cdots < E_{3n-3} < E_1 < E_2 < E_{3n} < E_4 < E_5 < \cdots \\ & \text{if } -1 < \alpha_0 < 2 \text{ and } 6n - \alpha_0 - 8 < \alpha_1 < 6n - 4, \\ (\mathrm{I}.2.n) \quad & E_0 < E_3 < \cdots < E_{3n-3} < E_1 < E_{3n} < E_2 < E_4 < E_{3n+3} \\ & < E_5 < \cdots \\ & \text{if } -1 < \alpha_0 < 2 \text{ and } 6n - 4 < \alpha_1 < 6n - \alpha_0 - 2. \end{aligned} \tag{3.6}$$

The parameter values in equation (3.6) can simply be obtained by combining those defining class (I) in equation (3.2) with the conditions $E_{3n-3} - E_1 = (6n - \alpha_0 - \alpha_1 - 8)/2 < 0$ for both subclasses, and either $E_{3n} - E_2 = (6n - \alpha_1 - 4)/2 > 0$ for the first one, or $E_{3n} - E_1 = (6n - \alpha_0 - \alpha_1 - 2)/2 > 0$ and $E_{3n} - E_2 = (6n - \alpha_1 - 4)/2 < 0$ for the second one.

It is worth noting that the parameter values corresponding to type (I.1.n) and (I.2.n) spectra cover all class (I) parameter range, but for $-1 < \alpha_0 < 2$, $\alpha_1 = 6n - 4$ or $\alpha_1 = 6n - \alpha_0 - 2$, where $n = 1, 2, 3, \ldots$.

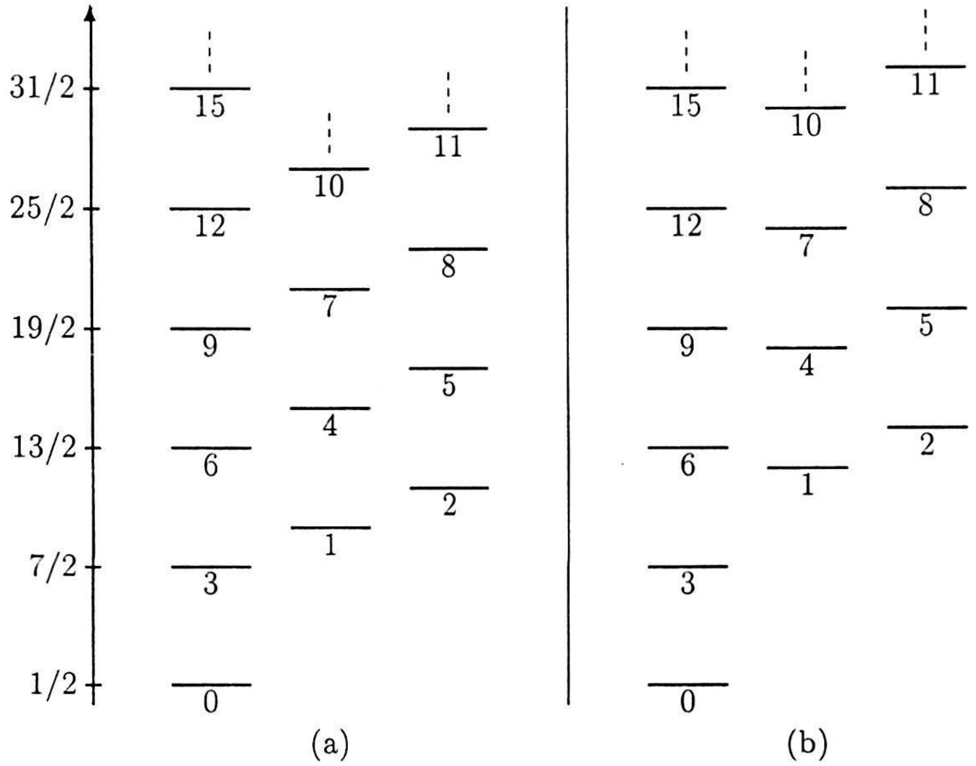

Figure 1: Examples of nondegenerate H_0 spectra belonging to class (I): (a) type (I.1.2) spectrum with $\alpha_0 = 0$, $\alpha_1 = 6$; (b) type (I.2.2) spectrum with $\alpha_0 = 0$, $\alpha_1 = 9$.

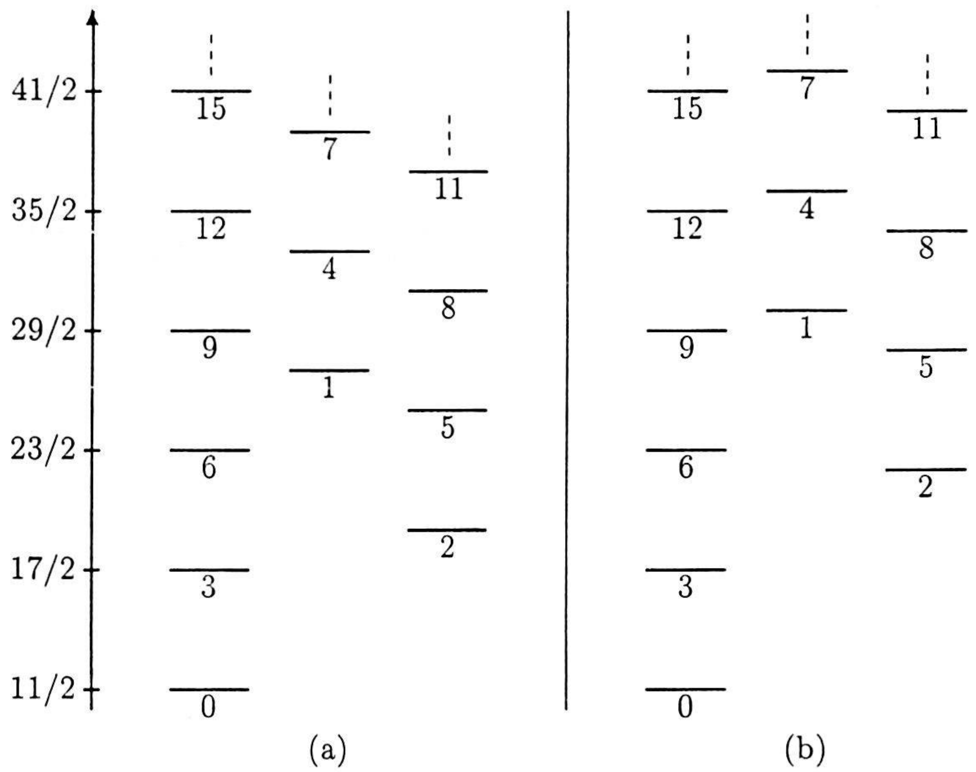

Figure 2: Examples of nondegenerate H_0 spectra belonging to class (II): (a) type (II.1.2.2) spectrum with $\alpha_0 = 10$, $\alpha_1 = 4$; (b) type (II.2.2.2) spectrum with $\alpha_0 = 10$, $\alpha_1 = 7$.

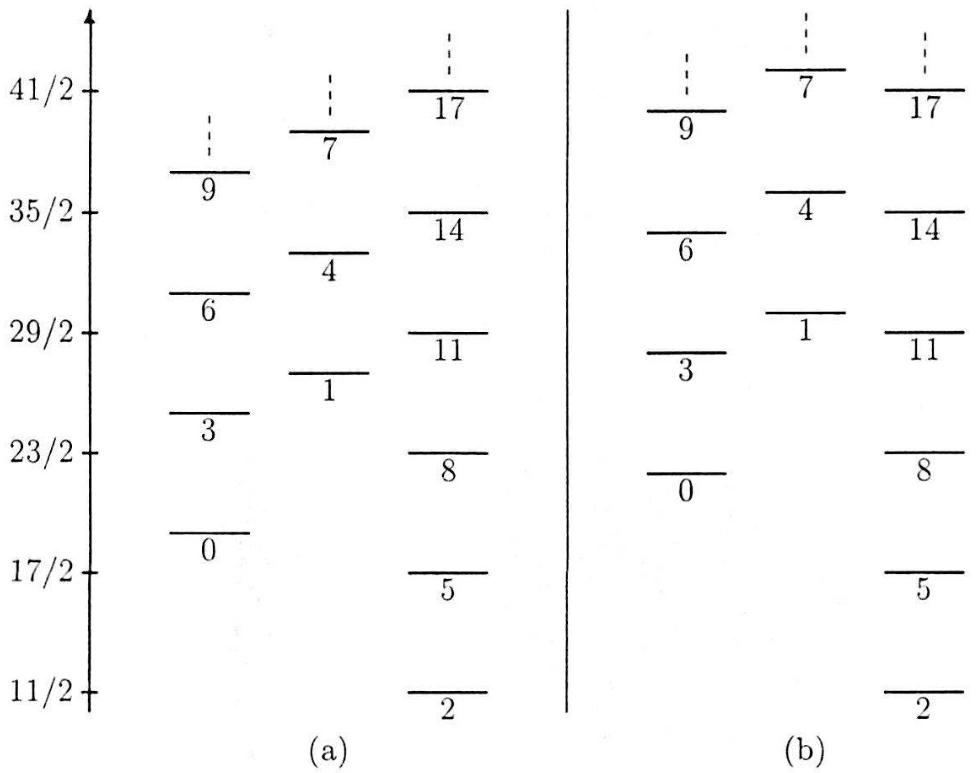

Figure 3: Examples of nondegenerate H_0 spectra belonging to class (III): (a) type (III.1.2.2) spectrum with $\alpha_0 = 18$, $\alpha_1 = -12$; (b) type (III.2.2.2) spectrum with $\alpha_0 = 21$, $\alpha_1 = -15$.

A similar procedure can be used for classes (II) and (III). Both of them separate into two subclasses (II.1), (II.2), and (III.1), (III.2), but the latter are now labelled by two integer indices m, $n = 1, 2, 3, \ldots$, instead of only one as for class (I). They are given by

$$\begin{aligned}
&(\text{II.1}.m.n) \quad E_0 < E_3 < \cdots < E_{3n-3} < E_2 < E_{3n} < E_5 < \cdots < E_{3m+3n-6} \\
&\qquad < E_{3m-1} < E_1 < E_{3m+3n-3} < E_{3m+2} < E_4 < \cdots \\
&\qquad \text{if } 6m - 4 < \alpha_0 < 6m + 2 \text{ and } 6n - 10 < \alpha_1 < 6m + 6n \\
&\qquad - \alpha_0 - 8, \\
&(\text{II.2}.m.n) \quad E_0 < E_3 < \cdots < E_{3n-3} < E_2 < E_{3n} < E_5 < E_{3n+3} < \cdots \\
&\qquad < E_{3m-1} < E_{3m+3n-3} < E_1 < E_{3m+2} < E_{3m+3n} < E_4 < \cdots \\
&\qquad \text{if } 6m - 4 < \alpha_0 < 6m + 2 \text{ and } 6m + 6n - \alpha_0 - 8 < \alpha_1 \\
&\qquad < 6n - 4,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&(\text{III.1}.m.n) \quad E_2 < E_5 < \cdots < E_{3n-1} < E_0 < E_{3n+2} < E_3 < \cdots < E_{3m+3n-4} \\
&\qquad < E_{3m-3} < E_1 < E_{3m+3n-1} < E_{3m} < E_4 < \cdots \\
&\qquad \text{if } 6m + 6n - 10 < \alpha_0 < 6m + 6n - 4 \text{ and } 6m - \alpha_0 - 8 < \alpha_1 \\
&\qquad < 2 - 6n, \\
&(\text{III.2}.m.n) \quad E_2 < E_5 < \cdots < E_{3n-1} < E_0 < E_{3n+2} < E_3 < E_{3n+5} < \cdots \\
&\qquad < E_{3m-3} < E_{3m+3n-1} < E_1 < E_{3m} < E_{3m+3n+2} < E_4 < \cdots
\end{aligned} \tag{3.8}$$

$$\text{if } 6m + 6n - 4 < \alpha_0 < 6m + 6n + 2 \text{ and } -4 - 6n < \alpha_1 < 6m - \alpha_0 - 2,$$

respectively.

The parameter values given in equations (3.7), and (3.8) can be checked in the same way as those in equation (3.6). Furthermore, those corresponding to type (II.1.m.n) and (II.2.m.n) spectra cover all class (II) parameter range, but for $6m - 4 < \alpha_0 < 6m + 2$, $\alpha_1 = 6n - 4$ or $\alpha_1 = 6m + 6n - \alpha_0 - 8$, where $m, n = 1, 2, 3, \ldots$, and $\alpha_1 > -4$, $\alpha_0 = 6m + 2$, where $m = 1, 2, 3, \ldots$. A similar remark applies to type (III.1.m.n) and (III.2.m.n) spectra, and class (III) parameter range, the exceptions being now $6m + 6n - 4 < \alpha_0 < 6m + 6n + 2$, $\alpha_1 = -4 - 6n$ or $\alpha_1 = 6m - \alpha_0 - 2$, where $m, n = 1, 2, 3, \ldots$, and $-2 - \alpha_0 < \alpha_1 < -4$, $\alpha_0 = 6n + 2$, where $n = 1, 2, 3, \ldots$.

Some examples of class (I), (II), and (III) nondegenerate spectra are displayed on figures 1, 2, and 3, respectively. One should remark that only type (I.1.1) spectra, for which $-1 < \alpha_0 < 2$ and $-2 - \alpha_0 < \alpha_1 < 2$, have the same level order as the standard harmonic oscillator, the spectrum of the latter being retrieved in the special case where $\alpha_0 = \alpha_1 = 0$.

3.2 Doubly-degenerate spectra

Doubly-degenerate spectra may appear as limiting cases of the nondegenerate ones of subsection 3.1, whenever two contiguous energies become equal, or they may directly result from the two intermediate classes, defined in equation (3.3). They belong to three different types, labelled by a, b, c, and corresponding to $\mathcal{F}_0$–$\mathcal{F}_1$, $\mathcal{F}_0$–$\mathcal{F}_2$, and $\mathcal{F}_1$–$\mathcal{F}_2$ degeneracies, respectively.

For class (I), for instance, we can obtain type a spectra by considering the limit $E_1 = E_{3n}$ in subclass (I.2.n), defined in equation (3.6), thereby getting the condition $\alpha_1 = 6n - \alpha_0 - 2$. The remaining two possibilities, namely $E_{3n-3} = E_1$ in subclass (I.1.n) or (I.2.n) for $n = 2$, $3, \ldots$, can be excluded because the former leads to the same types of spectra and parameter values as those already found, while the latter would imply the α_1 value $6n - \alpha_0 - 8$, lying outside the interval $(6n - 4, 6n - \alpha_0 - 2)$. Similarly, type b spectra can be obtained by considering the limit $E_2 = E_{3n}$ in subclass (I.1.n) or (I.2.n), thus giving the condition $\alpha_1 = 6n - 4$. On the contrary, type c spectra cannot be derived as limiting cases of class (I) spectra, as $E_1 < E_2$ by definition of the class, and $E_2 < E_4$ over the whole parameter range.

We conclude that class (I) doubly-degenerate spectra are given by

$$\begin{aligned}
&(\mathrm{I}.n.\mathrm{a}) \quad E_0 < E_3 < \cdots < E_{3n-3} < E_{3n} = E_1 < E_2 < E_{3n+3} = E_4 \\
&\qquad < E_5 < \cdots \\
&\qquad \text{if } -1 < \alpha_0 < 2 \text{ and } \alpha_1 = 6n - \alpha_0 - 2, \\
&(\mathrm{I}.n.\mathrm{b}) \quad E_0 < E_3 < \cdots < E_{3n-3} < E_1 < E_{3n} = E_2 < E_4 \\
&\qquad < E_{3n+3} = E_5 < \cdots \\
&\qquad \text{if } -1 < \alpha_0 < 2 \text{ and } \alpha_1 = 6n - 4,
\end{aligned} \tag{3.9}$$

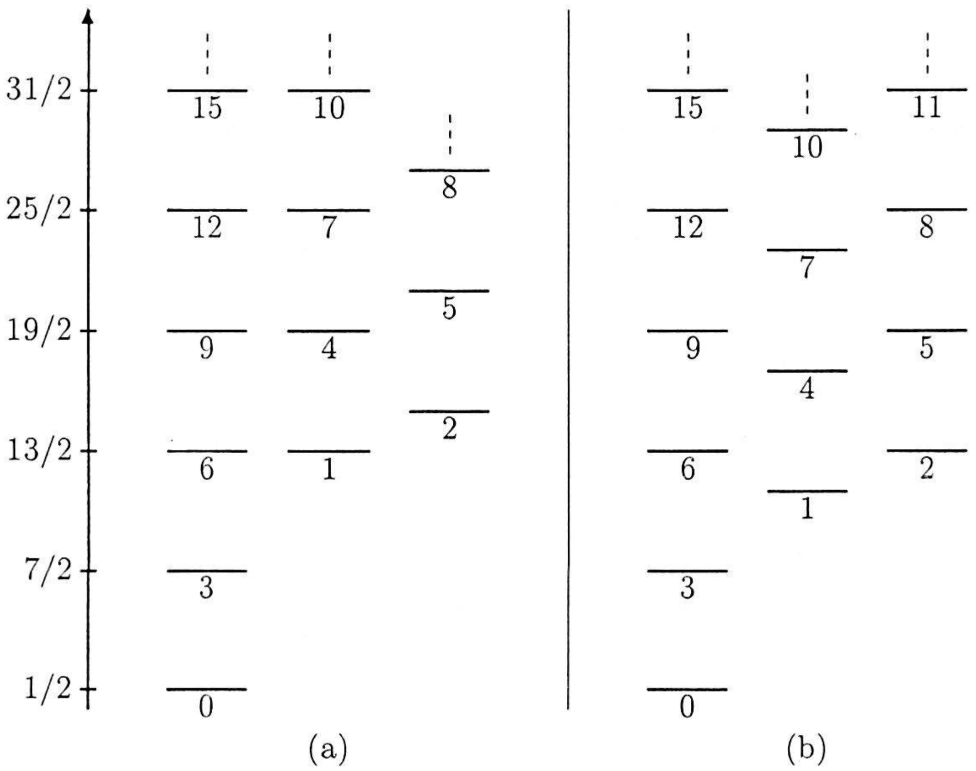

Figure 4: Examples of doubly-degenerate H_0 spectra belonging to class (I): (a) type (I.2.a) spectrum with $\alpha_0 = 0$, $\alpha_1 = 10$; (b) type (I.2.b) spectrum with $\alpha_0 = 0$, $\alpha_1 = 8$.

where n runs over 1, 2, 3, ... Together with type (I.1.n) and (I.2.n) nondegenerate spectra, they clearly exhaust all class (I) spectra.

By proceeding in the same way, the doubly-degenerate spectra, arising as limiting cases of class (III) nondegenerate ones, can be shown to separate into the following types:

$$\begin{array}{ll}
(\mathrm{III}.m.n.\mathrm{a}) & E_2 < E_5 < \cdots < E_{3n-1} < E_0 < E_{3n+2} < E_3 < E_{3n+5} < \cdots \\
& < E_{3m-3} < E_{3m+3n-1} < E_{3m} = E_1 < E_{3m+3n+2} \\
& < E_{3m+3} = E_4 < \cdots \\
& \text{if } 6m + 6n - 4 < \alpha_0 < 6m + 6n + 2 \text{ and } \alpha_1 = 6m - \alpha_0 - 2, \\
(\mathrm{III}.m.n.\mathrm{b}) & E_2 < E_5 < \cdots < E_{3n-1} < E_{3n+2} = E_0 < E_{3n+5} = E_3 < \cdots \\
& < E_{3m+3n-1} = E_{3m-3} < E_1 < E_{3m+3n+2} = E_{3m} < E_4 < \cdots \\
& \text{if } 6m + 6n - 4 < \alpha_0 < 6m + 6n + 2 \text{ and } \alpha_1 = -4 - 6n, \\
(\mathrm{III}.m.n.\mathrm{c}) & E_2 < E_5 < \cdots < E_{3n-1} < E_0 < E_{3n+2} < E_3 < \cdots \\
& < E_{3m+3n-4} < E_{3m-3} < E_{3m+3n-1} = E_1 < E_{3m} \\
& < E_{3m+3n+2} = E_4 < \cdots \\
& \text{if } \alpha_0 = 6m + 6n - 4 \text{ and } -4 - 6n < \alpha_1 < 2 - 6n,
\end{array} \tag{3.10}$$

where m, n run over 1, 2, 3, ... Together with type (III.1.$m.n$) and (III.2.$m.n$) nondegenerate spectra, they cover all class (III) parameter range, but for the discrete values $\alpha_0 = 6m + 6n + 2$, $\alpha_1 = -4 - 6n$, where m, $n = 1, 2, 3, \ldots$.

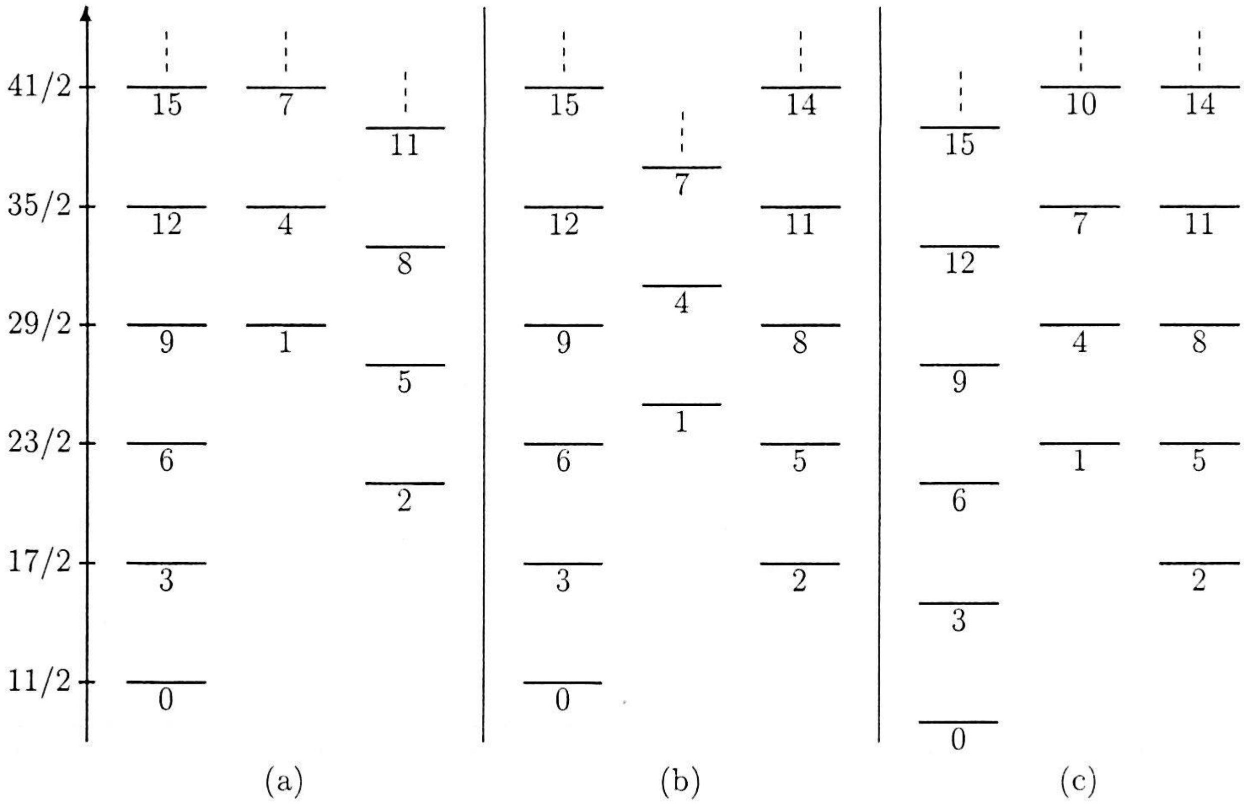

Figure 5: Examples of doubly-degenerate H_0 spectra belonging to class (II): (a) type (II.2.2.a) spectrum with $\alpha_0 = 10$, $\alpha_1 = 6$; (b) type (II.2.2.b) spectrum with $\alpha_0 = 10$, $\alpha_1 = 2$; (c) type (II.2.2.c) spectrum with $\alpha_0 = 8$, $\alpha_1 = 4$.

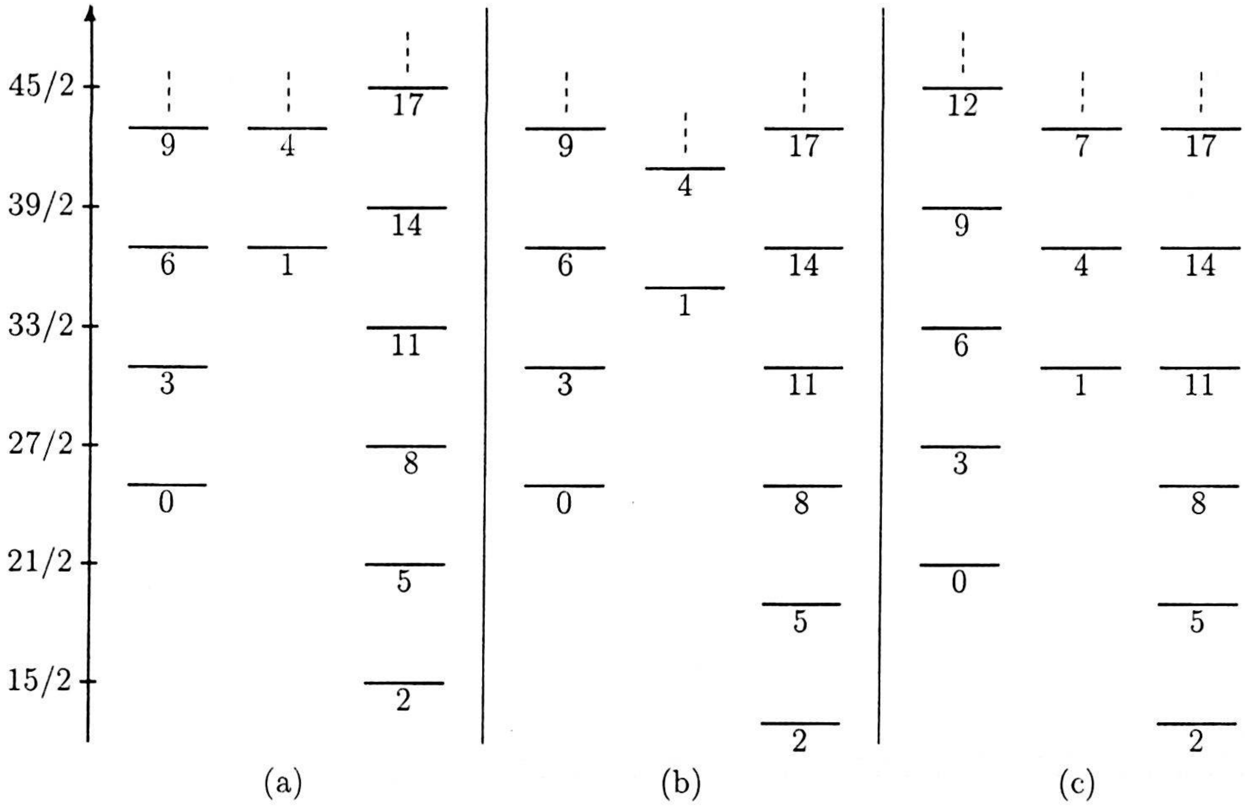

Figure 6: Examples of doubly-degenerate H_0 spectra belonging to class (III): (a) type (III.2.2.a) spectrum with $\alpha_0 = 24$, $\alpha_1 = -14$; (b) type (III.2.2.b) spectrum with $\alpha_0 = 24$, $\alpha_1 = -16$; (c) type (III.2.2.c) spectrum with $\alpha_0 = 20$, $\alpha_1 = -12$.

The doubly-degenerate spectra arising as limiting cases of class (II) nondegenerate ones can be grouped with those appearing in the intermediate classes (I-II), and (II-III) to provide the following types:

$$\begin{aligned}
&(\mathrm{II}.m.n.\mathrm{a}) && E_0 < E_3 < \cdots < E_{3n-3} < E_2 < E_{3n} < E_5 < \cdots < E_{3m+3n-6} \\
& && < E_{3m-1} < E_{3m+3n-3} = E_1 < E_{3m+2} < E_{3m+3n} = E_4 < \cdots \\
& && \text{if } 6m-4 < \alpha_0 < 6m+2 \text{ and } \alpha_1 = 6m+6n-\alpha_0-8, \\
&(\mathrm{II}.m.n.\mathrm{b}) && E_0 < E_3 < \cdots < E_{3n-3} = E_2 < E_{3n} = E_5 < \cdots \\
& && < E_{3m+3n-6} = E_{3m-1} < E_1 < E_{3m+3n-3} = E_{3m+2} < E_4 < \cdots \\
& && \text{if } 6m-4 < \alpha_0 < 6m+2 \text{ and } \alpha_1 = 6n-10, \\
&(\mathrm{II}.m.n.\mathrm{c}) && E_0 < E_3 < \cdots < E_{3n-3} < E_2 < E_{3n} < E_5 < E_{3n+3} < \cdots \\
& && < E_{3m-4} < E_{3m+3n-6} < E_{3m-1} = E_1 < E_{3m+3n-3} \\
& && < E_{3m+2} = E_4 < \cdots \\
& && \text{if } \alpha_0 = 6m-4 \text{ and } 6n-10 < \alpha_1 < 6n-4,
\end{aligned} \tag{3.11}$$

where m, n run over 1, 2, 3, ... Here we note that type (II.m.n.a), (II.m.n.b) (with $n \geq 2$), and (II.m.n.c) (with $m \geq 2$) spectra come from class (II), and together with type (II.1.m.n) and (II.2.m.n) nondegenerate spectra cover all class (II) parameter range, but for the discrete values $\alpha_0 = 6m+2$, $\alpha_1 = 6n-4$, where m, n = 1, 2, 3, ... On the contrary, type (II.1.n.c) [resp. (II.m.1.b)] spectra result from the intermediate class (I-II) [resp. (II-III)], and cover all the corresponding parameter range, but for the discrete values $\alpha_0 = 2$, $\alpha_1 = 6n-4$, where n = 1, 2, 3, ... [resp. $\alpha_1 = -4$, $\alpha_0 = 6m+2$, where m = 1, 2, 3, ...].

Some examples of doubly-degenerate spectra are displayed on figures 4, 5, and 6. One should note that the lowest doubly-degenerate state is the kth one, where $k = n+1$, $n+2$, $2m+n$, n, $2m+n-1$, $2m+n+1$, $n+1$, or $2m+n$ for type (I.n.a), (I.n.b), (II.m.n.a), (II.m.n.b), (II.m.n.c), (III.m.n.a), (III.m.n.b), or (III.m.n.c), respectively, and that above such a doubly-degenerate state, there always remain some nondegenerate ones. For type (II.m.1.b) spectra, and only for them, the ground state is doubly degenerate.

3.3 Triply-degenerate spectra

The allowed parameter values not encountered in subsections 3.1, 3.2 correspond to triply-degenerate spectra. The latter may be separated into the following three types:

$$\begin{aligned}
&(\mathrm{I}.n.\mathrm{abc}) && E_0 < E_3 < \cdots < E_{3n-3} < E_{3n} = E_1 = E_2 \\
& && < E_{3n+3} = E_4 = E_5 < \cdots \\
& && \text{if } \alpha_0 = 2 \text{ and } \alpha_1 = 6n-4, \\
&(\mathrm{II}.m.n.\mathrm{abc}) && E_0 < E_3 < \cdots < E_{3n-6} < E_{3n-3} = E_2 < E_{3n} = E_5 < \cdots \\
& && < E_{3m+3n-6} = E_{3m-1} < E_{3m+3n-3} = E_{3m+2} = E_1 \\
& && < E_{3m+3n} = E_{3m+5} = E_4 < \cdots \\
& && \text{if } \alpha_0 = 6m+2 \text{ and } \alpha_1 = 6n-10,
\end{aligned} \tag{3.12}$$

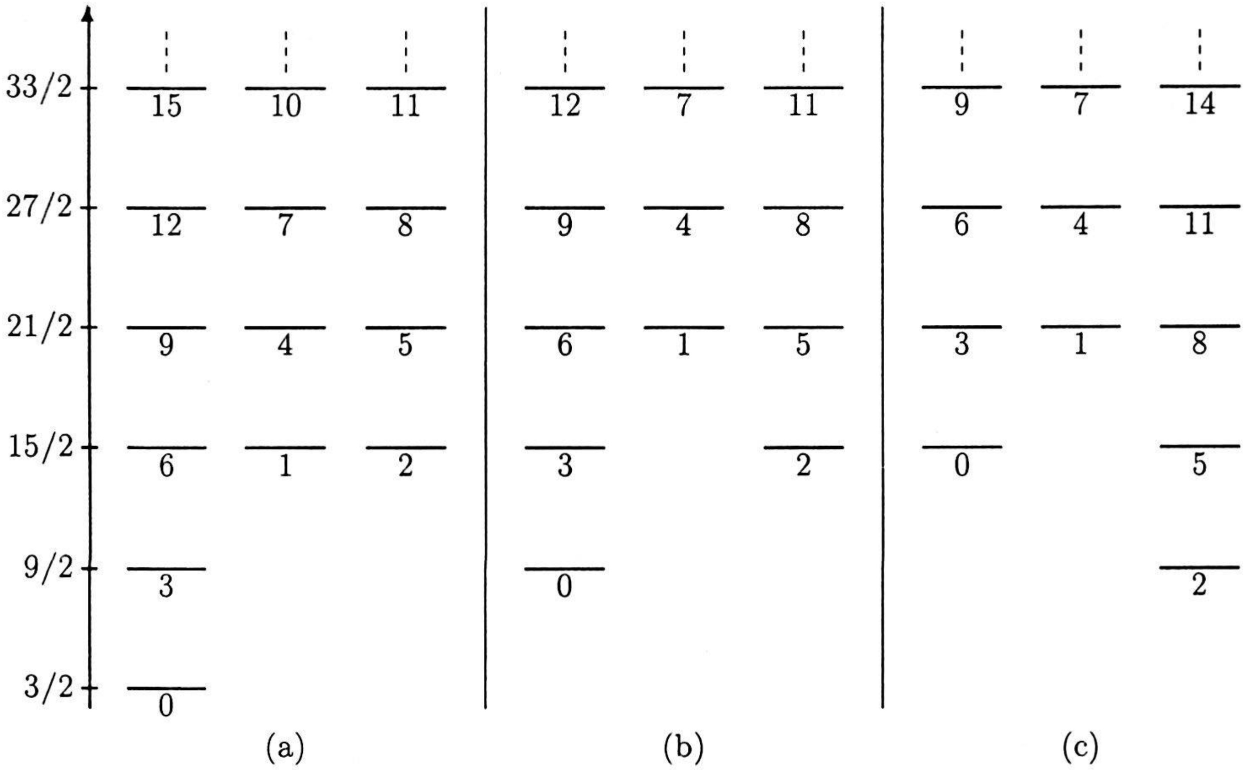

Figure 7: Examples of triply-degenerate H_0 spectra: (a) type (I.2.abc) spectrum with $\alpha_0 = 2$, $\alpha_1 = 8$; (b) type (II.1.2.abc) spectrum with $\alpha_0 = 8$, $\alpha_1 = 2$; (c) type (III.1.1.abc) spectrum with $\alpha_0 = 14$, $\alpha_1 = -10$.

$$\begin{aligned}
\text{(III.}m.n\text{.abc)}\quad & E_2 < E_5 < \cdots < E_{3n-1} < E_{3n+2} = E_0 < E_{3n+5} = E_3 < \cdots \\
& < E_{3m+3n-1} = E_{3m-3} < E_{3m+3n+2} = E_{3m} = E_1 \\
& < E_{3m+3n+5} = E_{3m+3} = E_4 < \cdots \\
& \text{if } \alpha_0 = 6m + 6n + 2 \text{ and } \alpha_1 = -4 - 6n,
\end{aligned}$$

where m, n run over 1, 2, 3, ... The first type comes from the intermediate class (I-II), the second one from class (II) or from the intermediate class (II-III), according to whether $n \geq 2$ or $n = 1$, while the third one results from class (III).

Some examples of triply-degenerate spectra are displayed on figure 7. Below the infinite set of triply-degenerate states, there appear n nondegenerate states in type (I.n.abc) spectra, while in the case of type (II.m.n.abc) [resp. (III.m.n.abc)] spectra, there are $n-1$ [resp. n] nondegenerate states, followed by m [resp. m] doubly-degenerate ones. For type (II.m.1.abc) spectra, and only for them, the ground state is doubly degenerate. No spectrum with a triply-degenerate ground state is obtained.

4 Period-Three Spectra and Supersymmetric Quantum Mechanics

From equations (3.6), (3.7), and (3.8), it results that type (I.1.1), (II.1.1.1), and (III.1.1.1) spectra, characterized by

$$\begin{array}{ll} (\mathrm{I}.1.1) & E_0 < E_1 < E_2 < E_3 < E_4 < E_5 < \cdots \\ & \text{if } -1 < \alpha_0 < 2 \text{ and } -2-\alpha_0 < \alpha_1 < 2, \\ (\mathrm{II}.1.1.1) & E_0 < E_2 < E_1 < E_3 < E_5 < E_4 < \cdots \\ & \text{if } 2 < \alpha_0 < 8 \text{ and } -4 < \alpha_1 < 4-\alpha_0, \\ (\mathrm{III}.1.1.1) & E_2 < E_0 < E_1 < E_5 < E_3 < E_4 < \cdots \\ & \text{if } 2 < \alpha_0 < 8 \text{ and } -2-\alpha_0 < \alpha_1 < -4, \end{array} \tag{4.1}$$

respectively, have an infinite number of periodically spaced levels. More precisely, the level spacings are given by ω_0, ω_1, ω_2, ω_0, ω_1, ω_2, ..., where ω_μ, $\mu = 0$, 1, 2, can be expressed in terms of the algebra parameters α_0, α_1, as

$$\begin{array}{llll} (\mathrm{I}.1.1) & \omega_0 = \frac{1}{2}(\alpha_0+\alpha_1+2), & \omega_1 = \frac{1}{2}(2-\alpha_0), & \omega_2 = \frac{1}{2}(2-\alpha_1), \\ (\mathrm{II}.1.1.1) & \omega_0 = \frac{1}{2}(\alpha_1+4), & \omega_1 = \frac{1}{2}(\alpha_0-2), & \omega_2 = \frac{1}{2}(4-\alpha_0-\alpha_1), \\ (\mathrm{III}.1.1.1) & \omega_0 = \frac{1}{2}(-\alpha_1-4), & \omega_1 = \frac{1}{2}(\alpha_0+\alpha_1+2), & \omega_2 = \frac{1}{2}(8-\alpha_0), \end{array} \tag{4.2}$$

respectively. In all three cases, the normalization of H_0 is such that $\Omega_3 \equiv \omega_0 + \omega_1 + \omega_2 = 3$.

Spectra of a similar type were recently encountered by Sukhatme *et al* [4] in the context of SSQM with cyclic shape invariant potentials of period three. In such a case, one may construct a hierarchy of supersymmetric Hamiltonians, and corresponding supercharges in terms of superpotentials that repeat after a cycle of three iterations. In terms of the operators

$$A_\mu = \frac{d}{dx} + W(x, b_\mu), \qquad A_\mu^\dagger = -\frac{d}{dx} + W(x, b_\mu), \qquad \mu = 0, 1, 2, \ldots, \tag{4.3}$$

where b_μ denotes a set of parameters such that $b_{\mu+3} = b_\mu$, and the superpotentials $W(x, b_\mu)$ satisfy the shape invariance conditions

$$W^2(x, b_\mu) + W'(x, b_\mu) = W^2(x, b_{\mu+1}) - W'(x, b_{\mu+1}) + \omega_\mu, \qquad \mu = 0, 1, 2, \tag{4.4}$$

the supersymmetric Hamiltonians $\mathcal{H}_\mu$, and supercharge operators $Q_\mu^\dagger$, Q_μ are defined by

$$\mathcal{H}_\mu = \begin{pmatrix} \mathcal{H}^{(\mu)} - \mathcal{E}_0^{(\mu)} I & 0 \\ 0 & \mathcal{H}^{(\mu+1)} - \mathcal{E}_0^{(\mu)} I \end{pmatrix}, \quad Q_\mu^\dagger = \begin{pmatrix} 0 & A_\mu^\dagger \\ 0 & 0 \end{pmatrix}, \quad Q_\mu = \begin{pmatrix} 0 & 0 \\ A_\mu & 0 \end{pmatrix}, \tag{4.5}$$

where

$$\begin{aligned} \mathcal{H}^{(0)} &= A_0^\dagger A_0, \\ \mathcal{H}^{(\mu)} &= A_{\mu-1} A_{\mu-1}^\dagger + \mathcal{E}_0^{(\mu-1)} I = A_\mu^\dagger A_\mu + \mathcal{E}_0^{(\mu)} I, \qquad \mu = 1, 2, \ldots, \end{aligned} \tag{4.6}$$

and $\mathcal{E}_0^{(\mu)}$ denotes the ground state energy of $\mathcal{H}^{(\mu)}$ (with $\mathcal{E}_0^{(0)} = 0$).

Since $A_3^\dagger = A_0^\dagger$, $A_3 = A_0$, $\mathcal{H}^{(3)} = \mathcal{H}^{(0)} + \mathcal{E}_0^{(3)} I$, one finds

$$\mathcal{H}_{\mu+3} = \mathcal{H}_\mu, \qquad Q_{\mu+3}^\dagger = Q_\mu^\dagger, \qquad Q_{\mu+3} = Q_\mu. \tag{4.7}$$

Hence, there are only three sets of independent operators $\{\mathcal{H}_\mu, Q_\mu^\dagger, Q_\mu\}$, corresponding to $\mu = 0$, 1, 2. Each one of them fulfils the defining relations of the sqm(2) superalgebra

$$\left(Q_\mu^\dagger\right)^2 = Q_\mu^2 = 0, \qquad \left[\mathcal{H}_\mu, Q_\mu^\dagger\right] = [\mathcal{H}_\mu, Q_\mu] = 0, \qquad \left\{Q_\mu, Q_\mu^\dagger\right\} = \mathcal{H}_\mu. \tag{4.8}$$

The eigenvalues $\mathcal{E}_n^{(\mu)}$, $n = 0, 1, 2, \ldots$, of $\mathcal{H}^{(\mu)}$, $\mu = 0, 1, 2$, satisfy the relations

$$\begin{aligned} \mathcal{E}_1^{(0)} &= \mathcal{E}_0^{(1)} = \omega_0, \\ \mathcal{E}_2^{(0)} &= \mathcal{E}_1^{(1)} = \mathcal{E}_0^{(2)} = \omega_0 + \omega_1, \\ \mathcal{E}_{3k+\nu}^{(0)} &= \mathcal{E}_{3k+\nu-1}^{(1)} = \mathcal{E}_{3k+\nu-2}^{(2)} = \mathcal{E}_{3(k-1)+\nu}^{(3)} = k\Omega_3 + \sum_{\rho=0}^{\nu-1} \omega_\rho, \end{aligned} \tag{4.9}$$

where $k = 1, 2, \ldots$, $\nu = 0, 1, 2$, and $\sum_{\rho=0}^{-1} \equiv 0$.

We shall now proceed to show that one may realize the operators defined in equations (4.5), (4.6), and satisfying equations (4.7), (4.8), in terms of creation and annihilation operators $a_\mu^\dagger$, a_μ, $\mu = 0, 1, 2$, belonging to C_3-extended oscillator algebras $\mathcal{A}^{(3)}_{\alpha_0^{(\mu)}\alpha_1^{(\mu)}}$, $\mu = 0$, 1, 2, whose parameters $\alpha_0^{(\mu)}$, $\alpha_1^{(\mu)}$ take some appropriate values corresponding to type (I.1.1) spectra. We shall actually prove that one may assume

$$A_\mu^\dagger = a_\mu^\dagger, \qquad A_\mu = a_\mu, \qquad \mu = 0, 1, 2. \tag{4.10}$$

For such a purpose, let us start with some algebra $\mathcal{A}^{(3)}_{\alpha_0\alpha_1}$, and from its generators let us construct the operators

$$\mathcal{H}^{(\mu)} = F(N + \mu) = N + \mu I + \alpha_0 P_{1-\mu} - \alpha_2 P_{2-\mu}, \tag{4.11}$$

where in the last step we used equations (2.9), and (2.14). It is straightforward to see that the eigenvalues $\mathcal{E}_n^{(\mu)}$ of $\mathcal{H}^{(\mu)}$ satisfy equation (4.9) with $\omega_\mu = 1 + \alpha_\mu$, $\mu = 0, 1, 2$, and $\Omega_3 = 3$. For this result to be meaningful, the conditions $\omega_\mu > 0$, $\mu = 0, 1, 2$, have to be fulfilled. The latter imply the following restrictions on α_0, α_1,

$$-1 < \alpha_0 < 2, \qquad -1 < \alpha_1 < 1 - \alpha_0. \tag{4.12}$$

The parameter values satisfying equation (4.12) form a subset of the set of allowed parameter values for type (I.1.1) spectra, as defined in equation (4.1).

From equation (2.15), it results that $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$, defined in equation (4.11), can be rewritten as $\mathcal{H}^{(0)} = a^\dagger a$ and $\mathcal{H}^{(1)} = aa^\dagger$, respectively. Comparing with equation (4.6), we conclude that equation (4.10) is valid for $\mu = 0$, provided we define $a_0^\dagger \equiv a^\dagger$, $a_0 \equiv a$, so that the corresponding algebra parameters are $\alpha_0^{(0)} = \alpha_0$, $\alpha_1^{(0)} = \alpha_1$.

Let us now define $a_1^\dagger$, a_1, and $a_2^\dagger$, a_2 in such a way that equation (4.10) is also valid for $\mu = 1$, and $\mu = 2$. From equations (4.6) and (4.11), we obtain

$$\begin{aligned} \mathcal{H}^{(1)} &= a_1^\dagger a_1 + (1+\alpha_0)I = N + I + \alpha_0 P_0 - \alpha_2 P_1, \\ \mathcal{H}^{(2)} &= a_1 a_1^\dagger + (1+\alpha_0)I = N + 2I - \alpha_2 P_0 + \alpha_0 P_2, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \mathcal{H}^{(2)} &= a_2^\dagger a_2 + (2+\alpha_0+\alpha_1)I = N + 2I - \alpha_2 P_0 + \alpha_0 P_2, \\ \mathcal{H}^{(3)} &= a_2 a_2^\dagger + (2+\alpha_0+\alpha_1)I = N + 3I + \alpha_0 P_1 - \alpha_2 P_2, \end{aligned} \tag{4.14}$$

from which we derive

$$\left[a_1, a_1^\dagger\right] = I + \alpha_1 P_0 + \alpha_2 P_1 + \alpha_0 P_2, \qquad \left[a_2, a_2^\dagger\right] = I + \alpha_2 P_0 + \alpha_0 P_1 + \alpha_1 P_2. \tag{4.15}$$

Finally, from equation (4.11), it results that $\mathcal{H}^{(3)} = \mathcal{H}^{(0)} + 3I$, so that $A_3^\dagger = A_0^\dagger = a_0^\dagger$, $A_3 = A_0 = a_0$, as it shoud be.

We conclude that the choice made in equations (4.10), (4.11), and (4.12) provides an algebraic realization of SSQM for any cyclic shape invariant potential of period three.[4] The matrix elements of the supersymmetric Hamiltonians and supercharges $\mathcal{H}_\mu$, $Q_\mu^\dagger$, Q_μ, $\mu = 0$, 1, 2, are expressed in terms of boson-like operators $a_\mu^\dagger$, a_μ, $\mu = 0, 1, 2$, belonging to C_3-extended oscillator algebras $\mathcal{A}^{(3)}_{\alpha_0\alpha_1}$, $\mathcal{A}^{(3)}_{\alpha_1,-\alpha_0-\alpha_1}$, $\mathcal{A}^{(3)}_{-\alpha_0-\alpha_1,\alpha_0}$, respectively, where α_0, α_1 are related to the level spacings through the relations $\omega_0 = 1+\alpha_0$, $\omega_1 = 1+\alpha_1$, $\omega_2 = 1-\alpha_0-\alpha_1$, and restricted to those values satisfying equation (4.12). The commutators of such operators $a_\mu^\dagger$, a_μ are given by

$$\left[a_\mu, a_\mu^\dagger\right] = I + \alpha_0^{(\mu)} P_0 + \alpha_1^{(\mu)} P_1 + \alpha_2^{(\mu)} P_2, \tag{4.16}$$

where the parameters $\alpha_\nu^{(\mu)} \equiv \alpha_{\nu+\mu}$, $\nu = 0, 1, 2$, fulfil relations similar to equation (4.12), i.e.,

$$-1 < \alpha_0^{(\mu)} < 2, \qquad -1 < \alpha_1^{(\mu)} < 1 - \alpha_0^{(\mu)}. \tag{4.17}$$

For different μ values, the sets $\{\alpha_0^{(\mu)}, \alpha_1^{(\mu)}, \alpha_2^{(\mu)}\}$ only differ from one another by a cyclic permutation.

As a final point, we would like to stress that the Hamiltonians $\mathcal{H}^{(\mu)}$, given in equation (4.11), differ from the corresponding bosonic oscillator Hamiltonians $H_0^{(\mu)} \equiv \frac{1}{2}\left\{a_\mu, a_\mu^\dagger\right\}$ through a linear combination of projection operators P_ν,

$$\mathcal{H}^{(\mu)} = H_0^{(\mu)} - \tfrac{1}{2}\sum_\nu \left(1 + \alpha_\nu^{(\mu)}\right) P_\nu + \mathcal{E}_0^{(\mu)} I, \tag{4.18}$$

[4]It is obvious that by an appropriate change of energy scale, one can get any Ω_3 value instead of $\Omega_3 = 3$, as considered here.

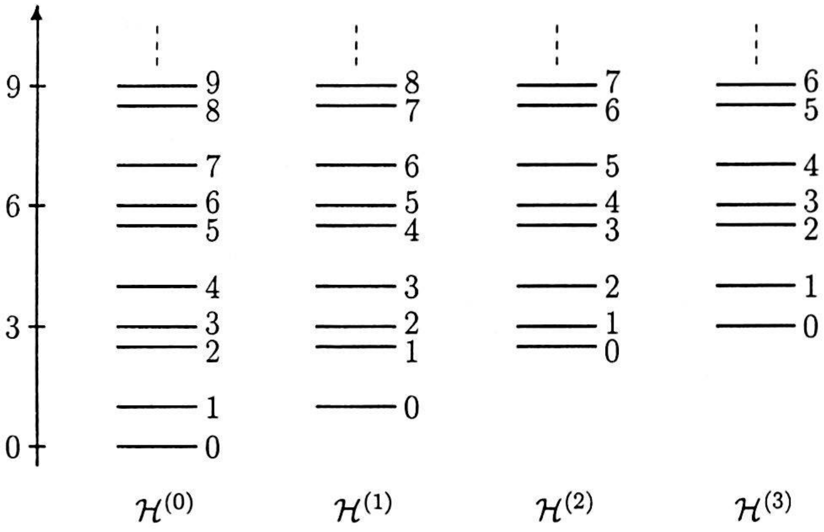

Figure 8: Spectra of the Hamiltonians $\mathcal{H}^{(\mu)}$, $\mu = 0,\ 1,\ 2,\ 3$, defined in equation (4.11), for $\alpha_0 = 0$, $\alpha_1 = \frac{1}{2}$.

or

$$\begin{aligned}
\mathcal{H}^{(0)} &= H_0^{(0)} - \tfrac{1}{2}\sum_\nu (1+\alpha_\nu)P_\nu, \\
\mathcal{H}^{(1)} &= H_0^{(1)} + \tfrac{1}{2}\sum_\nu (1+2\alpha_0-\alpha_{\nu+1})P_\nu \\
&= H_0^{(0)} + \tfrac{1}{2}\sum_\nu (1+\alpha_\nu)P_\nu, \\
\mathcal{H}^{(2)} &= H_0^{(2)} + \tfrac{1}{2}\sum_\nu (3-2\alpha_2-\alpha_{\nu+2})P_\nu \\
&= H_0^{(0)} + \tfrac{1}{2}\sum_\nu (3+\alpha_{\nu+1}-\alpha_{\nu+2})P_\nu.
\end{aligned} \tag{4.19}$$

This explains why the $\mathcal{H}^{(\mu)}$ and $H_0^{(\mu)}$ spectra, corresponding to parameter values satisfying equation (4.17), consist of periodically spaced levels characterized by different ω_ν values, although in both cases the level order is similar, and actually coincides with that of the standard harmonic oscillator.

On figure 8 are displayed the spectra of $\mathcal{H}^{(\mu)}$, $\mu = 0,\ 1,\ 2,\ 3$, for $\alpha_0 = 0$, and $\alpha_1 = \frac{1}{2}$. The corresponding values of ω_ν are $\omega_0 = 1$, $\omega_1 = \frac{3}{2}$, $\omega_2 = \frac{1}{2}$, and the associated C_3-extended oscillator algebras are $\mathcal{A}^{(3)}_{0,1/2}$, $\mathcal{A}^{(3)}_{1/2,-1/2}$, $\mathcal{A}^{(3)}_{-1/2,0}$, respectively.

5 Concluding Remarks

In the present paper, we considered a bosonic oscillator Hamiltonian H_0, associated with the C_3-extended oscillator algebra $\mathcal{A}^{(3)}_{\alpha_0\alpha_1}$ introduced in [37], and we studied its spectrum in terms of the algebra parameters α_0, α_1. We showed that such a spectrum has a very rich structure, contrary to what happens for the two-particle Calogero Hamiltonian, connected with the C_2 (or S_2)-extended oscillator algebra $\mathcal{A}^{(2)}_{\alpha_0}$ (also referred to as the Calogero-Vasiliev algebra). In particular, we obtained both nondegenerate spectra, with or without the same level order as the standard harmonic oscillator, and spectra exhibiting some double and/or triple degeneracies.

More importantly, we pointed out that some of the nondegenerate spectra, namely those of type (I.1.1), (II.1.1.1), and (III.1.1.1), have an infinite number of periodically spaced levels, as the spectra arising in SSQM when considering cyclic shape invariant potentials of period three [4]. We finally obtained a matrix realization of the supersymmetric Hamiltonians and supercharges associated with the latter in terms of creation and annihilation operators $a^\dagger_\mu$, a_μ, $\mu = 0, 1, 2$, belonging to C_3-extended oscillator algebras, whose parameters are obtained by cyclic permutations from a starting set $\{\alpha_0, \alpha_1, \alpha_2\}$, for which $-1 < \alpha_0 < 2$, $-1 < \alpha_1 < 1 - \alpha_0$, and $\alpha_2 = -\alpha_0 - \alpha_1$.

It is obvious that the results derived in the present paper can be extended to bosonic oscillator Hamiltonians H_0 associated with C_λ-extended oscillator algebras $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$, corresponding to λ values different from three. Although the complete classification of their possible types of spectra in terms of the algebra parameters α_0, α_1, …, $\alpha_{\lambda-2}$, becomes rather complicated for $\lambda > 3$, generalizing the results for spectra with periodically spaced levels is straightforward. In particular, it can easily be shown that the hierarchy of supersymmetric Hamiltonians and supercharges $\{\, \mathcal{H}_\mu, Q^\dagger_\mu, Q_\mu \mid \mu = 0, 1, \ldots, \lambda - 1 \,\}$ of [4], corresponding to cyclic shape invariant potentials of period $\lambda \geq 2$, can be built from creation and annihilation operators $a^\dagger_\mu$, a_μ, $\mu = 0, 1, \ldots, \lambda - 1$, belonging to C_λ-extended oscillator algebras, whose parameters are obtained by cyclic permutations from a starting set $\{\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-1}\}$, for which $-1 < \alpha_0 < \lambda - 1$, $-1 < \alpha_\mu < \lambda - \mu - 1 - \sum_{\nu=0}^{\mu-1} \alpha_\nu$ if $\mu = 1, 2, \ldots, \lambda - 2$, and $\alpha_{\lambda-1} = -\sum_{\nu=0}^{\lambda-2} \alpha_\nu$.

A very interesting open question is the possibility of realizing C_λ-extended oscillator algebras in terms of differential operators. Since one-dimensional Hamiltonians are known to have no degeneracies in their bound state spectrum, the existence of degeneracies in the H_0 spectrum for some parameter values shows that such a realization should at least involve two variables.

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