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Dual Killing-Yano symmetry and multipole moments in electromagnetism and mechanics of continua

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Abstract. In this work we introduce the Killing-Yano symmetry on the phase space and we investigate the symplectic structure on the space of Killing-Yano tensors. We perform the detailed analyze of the n -dimensional flat space and the Riemannian manifolds with constant scalar curvature. We investigate the form of some multipole tensors, which arise in the expansion of a system of charges and currents, in terms of second-order Killing-Yano tensors in the phase space of classical mechanics. We find some relations between these tensors and the generators of dynamical symmetries like the angular momentum, the mass-inertia tensor, the conformal operator and the momentum conjugate Runge-Lenz vector.

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1 Introduction

Killing tensors are indispensable tools in the quest for exact solutions in many branches of general relativity as well as classical mechanics. Killing tensors can also be important for solving the equations of motion in particular space-time [1]. Killing-Yano tensors were introduced in 1952 by Yano from a mathematical point of view on the configuration space [2]. When a manifold admits a Killing-Yano tensor we can construct a Killing tensor and a new constant of motion in the case of geodesic motions [3, 4]. It was a big success of Gibbons et al. to have been able to show that Killing-Yano tensors, which had long been known for relativistic systems as a rather mysterious structure, can be understood as an object generating a "non-generic symmetry", i.e. a supersymmetry appearing only in the specific space-time [5]. On the other hand Lax pairs tensors can be viewed as a generalization of Killing-Yano tensors [6]. The relation between Killing-Yano tensors and Nambu tensors was found in [7]. As it is well known in the expansion of charge and currents in electromagnetism, three families of multipole moments arise : the charge, the magnetic and the toroid moments [8]. Among the first members of these multipolar families, the time derivative of the charge dipole $\dot{\mathbf{d}}$, charge quadrupole \dot{Q}_{ij} and the magnetic dipole $\boldsymbol{\mu}$, correspond to infinitesimal translations, shears and rotations of the points of a continuous distribution of charged matter. For example the charge multipole moments, $Q_{i_1 i_2 \dots i_n}$, are related to the n -th order inertia moments of a continuous distribution of mass [9]. In view of the correspondence between the electric charge e , which is connected to gauge invariance and the gravitational mass m , which is related to the Poincaré invariance, we make the formal change of the current density \mathbf{j} by the momentum vector \mathbf{p} . In this way we obtain the following associations for these tensors

$$\dot{d}_i \longrightarrow p_i, \quad (1.1)$$

$$Q_{ij} \longrightarrow x_i x_j - \frac{1}{3} \mathbf{r}^2 \delta_{ij}, \quad (1.2)$$

$$\dot{Q}_{ij} \longrightarrow x_i p_j + x_j p_i - \frac{2}{3} (\mathbf{r} \cdot \mathbf{p}) \delta_{ij}, \quad (1.3)$$

$$\mu_i \longrightarrow L_i. \quad (1.4)$$

These tensors can be found as generators of many Lie dynamical symmetries like for example the three-dimensional rotation group $SO(3)$, which is generated by the three components of the angular momentum L_i , the group of the rigid rotator $Rot(3)$, generated by the mass quadrupole tensor Q_{ij} and L_i [10], or the linear motion group $SL(3)$ which in turn is generated by the shear tensor $S_{ij} \equiv \dot{Q}_{ij}$ and L_i [11]. It is then natural to seek for the symmetries and the geometrical features of the higher-rank tensors arising in the multipole expansion. An important point that one should mention is that the above tensors are written in the configuration space. Unfortunately the components of the higher-rank multipoles do not satisfy the closure relations for the Lie symmetry. For our purposes it will turn out to be useful to consider the same tensors in the momentum space too. It is more convenient to write these tensors in terms of purely geometric quantities in a form which will allow the generalization to higher dimensions. On the other hand the Killing-Yano symmetry was defined only on the configuration space.

For all these reasons the extension of the Killing-Yano symmetry on the phase space is very interesting to investigate.

The plan of the paper is as follows:

In Section 2 the extension of the Killing-Yano symmetry on the phase space is presented. In Section 3 multipole and dynamical symmetry tensors are investigated using dual Killing-Yano tensors. In Section 4 we present our concluding remarks.

2 Dual Killing-Yano symmetry

A Killing-Yano tensor [2] is an antisymmetric tensor which satisfies the equation

$$D_\lambda f_{\nu\mu} + D_\nu f_{\lambda\mu} = 0. \quad (2.1)$$

Here D denotes the covariant derivative.

For a given metric $g_{\mu\nu}$ instead of x_μ we consider the momentum p_μ . In this way we have obtained the metric $\tilde{g}_{\mu\nu}$ on the momentum space. Performing the operation of mapping x_μ to p_μ twice, leads back to the original metric $g_{\mu\nu}$. We call $\tilde{g}_{\mu\nu}(p)$ dual to $g_{\mu\nu}(x)$.

The existence of a Killing-Yano tensor on a given manifold is deeply related to the existence of a new supersymmetry in the case of geodesic motion on the spinning space [3]. We know that the action for the geodesic of spinning space has the form [3]

$$S = \int_a^b d\tau \left(\frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \frac{D\psi^\nu}{D\tau} \right). \quad (2.2)$$

The overdot denotes an ordinary proper-time derivative $d/d\tau$ whilst the covariant derivative of a Grassmann variable ψ^μ is defined by $D\psi^\mu/D\tau = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma_{\lambda\nu}^\mu \psi^\nu$. In general, the symmetries of a spinning-particle model can be divided into two classes. First, there are conserved quantities which exist in any theory and these are called *generic* constants of motion. It has been shown that for a spinning particle model defined by the action (2.2) there are four generic symmetries (for more details see [5]). By construction we have obtained four *generic* symmetries on the momentum space too. The second kind of conserved quantities, called *non-generic*, depend on the explicit form of the metric $g_{\mu\nu}(x)$. The existence of a Killing-Yano tensor $f_{\mu\nu}$ of the bosonic manifold is equivalent to the existence of a supersymmetry for the spinning particle with supercharge $Q_f = f_a^\mu \Pi_\mu \psi^a - \frac{1}{3} i H_{abc} \psi^a \psi^b \psi^c$ satisfies $\{Q, Q_f\} = 0$, where $H_{\mu\nu\lambda} = D_\lambda f_{\mu\nu}$, $\Pi_\mu = g_{\mu\nu} \dot{x}^\nu$ whereas the supercharge Q has the form $Q = \Pi_\mu \psi^\mu$ (see [5] for more details). Because the dual metric $\tilde{g}_{\mu\nu}$ admits a Killing-Yano tensor $\tilde{f}_{\mu\nu}$ the corresponding *non-generic* supersymmetries is defined by the supercharge $\tilde{Q}_f = f_a^\mu \tilde{\Pi}_\mu \psi^a - \frac{1}{3} i \tilde{H}_{abc} \psi^a \psi^b \psi^c$ satisfying $\{\tilde{Q}, \tilde{Q}_f\} = 0$. Here $\tilde{H}_{\mu\nu\lambda} = \tilde{D}_\lambda \tilde{f}_{\mu\nu}$, the canonical momentum is $\tilde{\Pi}_\mu = \tilde{g}_{\mu\nu} \dot{p}^\nu$ and the supercharge Q has the form $\tilde{Q} = \tilde{\Pi}_\mu \psi^\mu$. We mention that \tilde{D}_λ means the covariant derivative on the momentum space.

Using dual Killing-Yano symmetry we have obtained a pair of Killing-Yano tensors (f, \tilde{f}) defined on the phase space.

2.1 Examples

2.1.1. Flat space case

In the case of three-dimensional flat space these tensors have the following form :

$$f_{ij} = \varepsilon_{kij}x_k \quad \tilde{f}_{ij} = \varepsilon_{kij}p_k. \quad (2.3)$$

Eq.(2.3) can be reversed and thus one may express the position x and momentum p variables in terms of f and \tilde{f}

$$x_i = \frac{1}{2}\varepsilon_{ijk}f_{jk}, \quad p_i = \frac{1}{2}\varepsilon_{ijk}\tilde{f}_{jk}. \quad (2.4)$$

The Poisson bracket of f and \tilde{f} reads

$$\{f_{ij}, \tilde{f}_{kl}\} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \quad (2.5)$$

A scalar product can be defined for these Killing-Yano tensors. For example, the square of f can be written as follows

$$f^2 \equiv f \cdot f \equiv f_{ij}f_{ij}. \quad (2.6)$$

Equation (2.4) enables us to construct the phase-space in terms of the Nambu tensor ε_{ijk} and the Killing-Yano tensors f_{ij} and \tilde{f}_{ij} . When a manifold admits a Killing-Yano tensor f_{ij} then we can construct a Killing tensor $K_{ij} = x_i x_j - r^2 \delta_{ij}$. This Killing tensor corresponds to a constant of motion $K = K_{ij}p_i p_j$. The Nambu tensor ε_{ijk} is of rank three and it defines a Nambu mechanics with the constants of motion $H = p^2$ and K [12, 13]. These results can be generalized in the flat space of an arbitrary dimension [7]. In this case we have

$$x_i = \frac{1}{n!}\varepsilon_{i_1 \dots i_n} f_{i_1 \dots i_n}, \quad p_i = \frac{1}{n!}\varepsilon_{i_1 \dots i_n} \tilde{f}_{i_1 \dots i_n}. \quad (2.7)$$

Equation (2.7) enables us to construct the phase space in terms of Nambu tensor $\varepsilon_{i_1 \dots i_n}$ and Killing-Yano tensors $f_{i_1 \dots i_n}$ and $\tilde{f}_{i_1 \dots i_n}$.

2.1.2. Riemannian manifold with constant scalar curvature

It is well known that any n -dimensional Riemannian manifold with constant scalar curvature admits $\frac{n(n-1)}{2}$ Killing-Yano tensors of order two [14]. For example in the three dimensional case, the corresponding metric with constant curvature has the form

$$ds^2 = \left(1 + \frac{Kr^2}{4}\right)^{-2} \sum_{i=1}^3 (dq^i)^2. \quad (2.8)$$

where $r = \sqrt{\sum_{i=1}^3 (q^i)^2}$ and $q^i (i = 1, 2, 3)$ are the coordinates, whereas K is a real constant denoting curvature of the configuration space [15]. In this case we have three Killing-Yano

tensors $f_{\mu\nu}$ and three $\tilde{f}_{\mu\nu}$. When q^i are the spherical coordinates the expressions of the Killing-Yano tensors looks like

$$\begin{aligned} f_{12} &= \frac{r \sin \phi}{16(1 + \frac{Kr^2}{4})^2}, f_{13} = \frac{r \sin 2\theta \cos \phi}{32(1 + \frac{Kr^2}{4})^2}, f_{23} = \frac{r^2 \sin \theta^2 \cos \phi (Kr^2 - 4)}{(4 + Kr^2)(1 + \frac{Kr^2}{4})^2}, \\ \tilde{f}_{12} &= \frac{16p \sin \phi}{(1 + \frac{Kp^2}{4})^2}, \tilde{f}_{13} = \frac{p \sin 2\theta \cos \phi}{32(1 + \frac{Kp^2}{4})^2}, \tilde{f}_{23} = \frac{p^2 \sin \theta^2 \cos \phi (Kp^2 - 4)}{(1 + \frac{Kp^2}{4})^2(4 + Kp^2)}. \end{aligned} \quad (2.9)$$

We are able to express the components of Runge-Lenz vector and the energy level of the Kepler problem [15] in terms of purely geometric quantities $f_{\mu\nu}$ and $\tilde{f}_{\mu\nu}$.

2.2 Symplectic structure

In this subsection the symplectic structure to the space of Killing-Yano tensors is constructed. Let us consider for the beginning the n -dimensional flat space case. From (2.1) we found that the Killing-Yano tensors f and \tilde{f} are $f_{i_1 \dots i_{n-1}} = \epsilon_{i_n \dots i_1 \dots i_{n-1}} x^{i_n}$, $\tilde{f}_{i_1 \dots i_{n-1}} = \epsilon_{i_n \dots i_1 \dots i_{n-1}} p^{i_n}$. Because each of the antisymmetric tensors $f_{i_1 \dots i_{n-1}}$ and $\tilde{f}_{i_1 \dots i_{n-1}}$ has n independent components we can consider \mathbf{f} and $\tilde{\mathbf{f}}$ as a n -dimensional vectors. We can combine the vectors \mathbf{f} and $\tilde{\mathbf{f}}$ into a $2n$ dimensional vector $\mathbf{x} = (\mathbf{f}, \tilde{\mathbf{f}})$, interpret the quantities $(\frac{\partial H}{\partial \mathbf{f}_j}, \frac{\partial H}{\partial \tilde{\mathbf{f}}_k})$, as a

$2n$ -dimensional vector ∇H , and introduce a $2n \times 2n$ matrix $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, where I is the $n \times n$ identity matrix. With this notation Hamilton's equations can be unified in the form $\dot{\mathbf{x}} = J \cdot \nabla H(\mathbf{x})$.

Let us consider a even-dimensional Riemannian manifold having constant scalar curvature. It is well known that all symplectic structures have locally the same structure.

A precise formulation of this assertion is given by Darboux's theorem [16]. Due to this theorem, any statement of local nature which is invariant under symplectic transformation and has been proved for the standard phase space ($M = \mathbf{R}^{2n}, \omega = \sum_{k=1}^n dp_j \wedge dq^j$) can be extended to all symplectic manifolds. On the other hand from equation (2.1) we found that an antisymmetric covariant constant tensor $f_{\mu\nu}$ is a solution of Killing-Yano equations. If the corresponding form is non-degenerate then we have a symplectic structure on this manifold. By construction the dual manifold admits a symplectic structure. If a manifold admits a non-degenerate covariant constant Killing-Yano, then we can construct a symplectic structure to the space of Killing-Yano tensors. As an example we mention here the self-dual Taub-NUT metric [17]. In this case we have four Killing-Yano tensors. Three of these, denoted by f_i are special because they are covariant constant and non-degenerate. The dual manifold has three covariant constant non-degenerate Killing-Yano tensors \tilde{f}_i . In the two-form notation, using the spherical co-ordinates (r, θ, ϕ) and respectively (p, θ, ϕ) the explicit expressions are:

$$\begin{aligned} f_i &= 4m(d\psi + \cos \theta d\phi) dx_i - \epsilon_{ijk} (1 + \frac{2m}{r}) dx_j \wedge dx_k, \\ \tilde{f}_i &= 4m(d\psi + \cos \theta d\phi) dp_i - \epsilon_{ijk} (1 + \frac{2m}{p}) dp_j \wedge dp_k. \end{aligned} \quad (2.10)$$

3 Multipole and dynamical symmetry tensors

In this section the expressions of the multipole tensors in terms of purely geometric quantities (f, \tilde{f}) are presented. The first step consists in writing the square of the radius r^2 and of the impulse p^2 in terms of (f, \tilde{f}).

$$r^2 = \frac{1}{2}f^2 \quad p^2 = \frac{1}{2}\tilde{f}^2, \quad (3.1)$$

The magnetic dipole tensor is given by

$$\mu_i = L_i = \frac{1}{2}\varepsilon_{klm}f_{ki}\tilde{f}_{lm}, \quad (3.2)$$

and the dilatation has the form

$$D \equiv \mathbf{r} \cdot \mathbf{p} = \frac{1}{2}f_{ij}\tilde{f}_{ij}. \quad (3.3)$$

The quadrupole mass-inertia tensor reads

$$Q_{ij} = \frac{1}{4}(f_{im}f_{mj} - \frac{1}{3}\delta_{ij}f^2). \quad (3.4)$$

The toroid dipole tensor, a quantity related to the poloidal currents on a torus, can be written in the following manner [18]

$$T_i = \frac{1}{10}(x_i D - 2r^2 p_i) = \frac{1}{40}\varepsilon_{ijk}\{f_{jk}(f \cdot \tilde{f}) - 2\tilde{f}_{jk}f^2\}. \quad (3.5)$$

In the case of purely transversal velocity fields this expression gets a simplified form [19]:

$$T_i = \frac{1}{2}d_i D = \frac{1}{8}\varepsilon_{ijk}f_{jk}(f \cdot \tilde{f}). \quad (3.6)$$

Next we pass to other tensors, related to dynamical symmetries. Consider first the conformal operator

$$C_i = 2x_i D - r^2 p_i = \frac{1}{4}\varepsilon_{ijk}\{2f_{jk}(f \cdot \tilde{f}) - \tilde{f}_{jk}f^2\}. \quad (3.7)$$

Together with the angular momentum L_i , C_i is a generator of a symmetry group which obeys commutation relations isomorphic to those of $SO(4)$. This is a subgroup of the group $SO(4,2)$ [20, 21] (isomorphic to the conformal group in Minkowski space) which leaves invariant the free Maxwell's equations [22]. The other generators of this larger group are the impulse p_i and the dilatation D which were defined above.

Another interesting tensor is the following particular form of the Runge-Lenz vector [21] with components

$$A_i = \frac{1}{2}x_i p^2 - p_i D - \frac{1}{2}x_i. \quad (3.8)$$

This vector, together with the orbital angular momentum L_i , the dilatation D and other two vectors and two scalars generates the $SO(4,2)$ group which contains as a subgroup the

symmetry group of the Hydrogen atom, i.e. $SO(4)$. Thus, from the algebraic point of view the properties of the Runge-Lenz vector are similar to those of the conformal one. If next we take the *momentum conjugate* of (3.8), we obtain the following tensor in Killing-Yano form

$$\tilde{A}_i = \frac{1}{8}\varepsilon_{ijk}\{(f^2 - 2)\tilde{f}_{jk} - 2f_{jk}(f \cdot \tilde{f})\}. \quad (3.9)$$

This tensor can be viewed as a symmetry generator like A_i , but in the momentum space. Then we have obtained the following formula for the Killing-Yano tensors in terms of the Runge-Lenz vectors and the conformal operator in the space and momentum subspaces.

$$f_{jk} = -\varepsilon_{ijk}(2\tilde{A}_i + C_i) \quad , \quad \tilde{f}_{jk} = -\varepsilon_{ijk}(2A_i + \tilde{C}_i). \quad (3.10)$$

In this way the toroid dipole tensor (3.6) can be directly related to $SO(4,2)$ symmetry generators in the phase-space :

$$T_i = (2\tilde{A}_i + C_i)D. \quad (3.11)$$

When we move to the next rank multipolar tensors we encounter the charge octupole tensor

$$Q_{ijk} = \frac{1}{4}\left\{\varepsilon_{imn}(\delta_{jk}f^2 + 2f_{jl}f_{lk}) - \frac{1}{5}f^2(\varepsilon_{imn}\delta_{jk} + \varepsilon_{jmn}\delta_{ik} + \varepsilon_{kmn}\delta_{ij})\right\}, \quad (3.12)$$

the magnetic quadrupole tensor

$$\mu_{ij} = \frac{1}{3}(x_iL_j + x_jL_i) = -\frac{1}{3}\{f_{ik}f_{kl}\tilde{f}_{lj} + f_{jk}f_{kl}\tilde{f}_{li}\}, \quad (3.13)$$

and the toroid quadrupole tensor

$$T_{ij} = \left(f_{im}f_{mj} - \frac{1}{4}\delta_{ij}f^2\right)(f \cdot \tilde{f}) - \frac{5}{2}f_{im}\tilde{f}_{mj}f^2. \quad (3.14)$$

Using again (3.10) we write the last tensor in terms of dynamical symmetry generators, in the position subspace of $Rot(3)$ and $SO(4,2)$, for the purely transversal gauge mentioned above

$$T_{ij} = Q_{ij}D = \frac{1}{8}\left(f_{im}f_{mj} - \frac{1}{3}\delta_{ij}f^2\right)(f \cdot \tilde{f}). \quad (3.15)$$

4 Concluding remarks

In this paper the Killing-Yano symmetry was generalized to the phase space. We have introduced Killing-Yano tensors in the momentum space in order to relate the geometrical objects f and \tilde{f} with the dynamics. On the phase space constructed in terms of Killing-Yano tensors f and \tilde{f} we have new supersymmetries in the case of geodesic motion of a spinning particle. On the other hand in the case of the n -dimensional flat space all Killing-Yano tensors are Nambu tensors. In this case we have constructed a symplectic structure to the space of Killing-Yano tensors and the geometrical signification of these tensors was clarified.

We found that on an even-dimensional Riemannian manifold every non-degenerate covariant constant Killing-Yano tensor of order two is a symplectic structure . We showed that it is possible to relate the toroid dipole and quadrupole tensors to $SO(4,2)$ and $Rot(3)$ generators acting in the phase-space. This pattern is followed also by the toroid and magnetic tensors with higher multipolarity. Similar multipolar tensors occur in the theory of continuous media [23, 24] and can be related to dynamical symmetry generators in the full phase-space using a geometrical representation valid in flat and curved spaces as we showed above. In this way we associate a geometrical meaning to such physical observables of the continua.

Finding all *Kähler* manifolds which have Killing-Yano tensors is an interesting problem and it requires further investigations [25].

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