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Reciprocity theorems for holomorphic representations of some infinite-dimensional groups

Quelques théorèmes de réciprocité pour les représentations holomorphes irréductibles de certains groupes de dimension infinie

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Abstract. Let μ denote the *Gaussian measure* on $\mathbb{C}^{n\times k}$ defined by

$$d\mu(Z) = \pi^{-nk} \exp\left[-\operatorname{Tr}\left(ZZ^{\dagger}\right)\right] dZ,$$

where Tr denotes the trace function, $Z^{\dagger} = \bar{Z}^{T}$, and dZ denotes the Lebesgue measure on $\mathbb{C}^{n\times k}$. Let $\mathcal{F}_{n\times k}$ denote the Bargmann–Segal–Fock space of holomorphic entire functions on $\mathbb{C}^{n\times k}$ which are also square-integrable with respect to μ. Fix n and let $\mathcal{F}_{n\times\infty}$ denote the Hilbert-space completion of the *inductive limit* $\lim_{k\to\infty} \mathcal{F}_{n\times k}$. Let G_k and H_k be compact groups such that $H_k \subset G_k \subset \mathrm{GL}_k(\mathbb{C})$. Let G_∞ (resp. H_∞) denote the inductive limit $\bigcup_{k=1}^{\infty} G_k$ (resp. $\bigcup_{k=1}^{\infty} H_k$). Then the representation R_{G_∞} (resp. R_{H_∞}) of G_∞ (resp. H_∞), obtained by right translation on $\mathcal{F}_{n\times\infty}$, is a *holomorphic representation* of G_∞ (resp. H_∞) in the sense defined by Ol'shanskii. Then R_{G_∞} and R_{H_∞} give rise to the dual representations $R'_{G'_n}$ and $R'_{H'_n}$ of the *dual pairs* (G'_n, G_∞) and (H'_n, H_∞), respectively. The generalized *Bargmann-Segal-Fock space* $\mathcal{F}_{n\times\infty}$ can be considered as both a (G'_n, G_∞)-*dual module* and an (H'_n, H_∞)-dual module. It is shown that the following multiplicity-free decompositions of $\mathcal{F}_{n\times\infty}$ into *isotypic components* $\mathcal{F}_{n\times\infty} = \sum_{(\lambda)}\oplus \mathcal{I}^{(\lambda)}_{n\times\infty} = \sum_{(\mu)}\oplus \mathcal{I}^{(\mu)}_{n\times\infty}$ hold, where (λ) is a common *irreducible signature* of the pair (G'_n, G_∞) and (μ) a common irreducible signature of the pair (H'_n, H_∞), and $\mathcal{I}^{(\lambda)}_{n\times\infty}$ (resp. $\mathcal{I}^{(\mu)}_{n\times\infty}$) is both the isotypic component of the equivalence classes

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$(\lambda)_{G_\infty}$ (resp. $(\mu)_{H_\infty}$) and $(\lambda')_{G'_n}$ (resp. $(\mu')_{H'_n}$). A *reciprocity theorem,* giving the multiplicity of $(\mu)_{H_\infty}$ in the restriction to H_∞ of $(\lambda)_{G_\infty}$ in terms of the multiplicity of $(\lambda')_{G'_n}$ in the restriction to G'_n of $(\mu')_{H'_n}$, constitutes the main result of this paper. Several applications of this theorem to Physics are also discussed.

RÉSUMÉ. Soit μ la mesure de Gauss definie sur l'espace vectoriel $\mathbb{C}^{n\times k}$ par la formule

$$d\mu(Z) = \pi^{-nk}\exp\left[-\operatorname{Tr}\left(ZZ^\dagger\right)\right] dZ, \qquad z \in \mathbb{C}^{n\times k},$$

où l'on désigne par Tr la trace d'une matrice, $Z^\dagger = \bar{Z}^T$, et par dZ la mesure de Lebesgue sur $\mathbb{C}^{n\times k}$. Soit $\mathcal{F}_{n\times k}$ l'espace hilbertien de Bargmann–Segal–Fock des fonctions entières holomorphes $f\colon \mathbb{C}^{n\times k} \to \mathbb{C}$ telles que f soient de carré-integrable par rapport à la mesure μ. On fixe n et l'on désigne par $\mathcal{F}_{n\times\infty}$ le complété de la *limite inductive* par rapport à k des espaces $\mathcal{F}_{n\times k}$. Pour chaque k soient G_k et H_k deux groupes compacts tels que $H_k \subset G_k \subset \mathrm{GL}_k(\mathbb{C})$, et l'on suppose aussi que $H_{k-1} \subset H_k \subset H_{k+1} \subset \cdots$ et $G_{k-1} \subset G_k \subset G_{k+1} \subset \cdots$. Soit G_∞ (resp. H_∞) la limite inductive de la chaine $\{G_k\}$ (resp. $\{H_k\}$). Alors la représentation R_{G_∞} (resp. R_{H_∞}) de G_∞ (resp. H_∞), obtenue par translation à droite sur $\mathcal{F}_{n\times\infty}$, est *holomorphe* dans le sens de Ol'shanskii. Les représentations R_{G_∞} et R_{H_∞} donnent lieu aux représentations $R'_{G'_n}$ et $R'_{H'_n}$, respectivement, des *paires duales* (G'_n, G_∞) et (H'_n, H_∞). L'espace hilbertien generalisé de Bargmann–Segal–Fock $\mathcal{F}_{n\times\infty}$ peut être consideré en même temps comme un (G'_n, G_∞)-module et un (H'_n, H_∞)-module. On montre que l'on a les décompositions suivantes de $\mathcal{F}_{n\times\infty}$ en uniques *composantes isotypiques*

$$\mathcal{F}_{n\times\infty} = \sum_{(\lambda)} \oplus\, \mathcal{I}^{(\lambda)}_{n\times\infty} = \sum_{(\mu)} \oplus\, \mathcal{I}^{(\mu)}_{n\times\infty},$$

où (λ) est une *signature irréductible* commune de la paire (G'_n, G_∞) et (μ) celle de la paire (H'_n, H_∞), et où $\mathcal{I}^{(\lambda)}_{n\times\infty}$ (resp. $\mathcal{I}^{(\mu)}_{n\times\infty}$) est à la fois la composante isotypique de la classe d'équivalence de $(\lambda)_{G_\infty}$ (resp. $(\mu)_{H_\infty}$) et celle de $(\lambda')_{G'_n}$ (resp. $(\mu')_{H'_n}$). On donne une démonstration d'un *théorème de réciprocité,* donnant la multiplicité de $(\mu)_{H_\infty}$ dans la restriction à H_∞ de $(\lambda)_{G_\infty}$, en fonction de la multiplicité de $(\lambda')_{G'_n}$ dans la restriction à G'_n de $(\mu')_{H'_n}$. L'article se termine par une discussion de plusieurs applications en Physique du théorème précédant.

1 Introduction

In recent years there is great interest, both in Physics and in Mathematics, in the theory of unitary representations of infinite-dimensional groups and their Lie algebras (see, for example, [Ka1], and the literature cited therein). Starting with the seminal work of I. Segal in [Se] the representation theory of U(∞) and other classical infinite-dimensional groups was thoroughly investigated by Kirillov in [Ki], Stratila and Voiculescu in [S&V], Pickrell in [Pi], Ol'shanskii in [Ol1], Gelfand and Graev in [Ge&Gr], Kac in [Ka2], to cite just a few. A more complete list of references can be found in the comprehensive and important work of Ol'shanskii in [Ol2].

In [Ol2] Ol'shanski generalized Howe's theory of dual pairs to some infinite-dimensional dual pairs of groups. Recently in [TT1] and [TT2] we investigated the generalized Casimir invariants of these infinite-dimensional dual pairs. In [TT3] we gave a general reciprocity theorem for finite-dimensional dual pairs of groups which generalized our previous results in [KT1] and [LT1]. In this article we give a generalization of this reciprocity theorem to the case of dual pairs where one member is infinite-dimensional and the other is finite-dimensional, and discuss the general case where both members are infinite-dimensional. If Section 2 we will review the reciprocity theorem given in [TT3] which serves as the necessary background for the generalized theorem, and more importantly, discuss several interesting applications of this theorem. Section 3 deals with our main theorem, and the paper ends with a short conclusion in Section 4.

2 The Reciprocity Theorem for Finite-Dimensional Pairs of Groups and Its Applications

In [TT3] our reciprocity theorem can be applied to the more general context of dual representations but for this paper we shall restrict ourself to the case of the oscillator dual representations and where one of the members is a compact group.

Let $\mathbb{C}^{n\times k}$ denote the vector space of all $n\times k$ complex matrices. Let μ denote the Gaussian measure on $\mathbb{C}^{n\times k}$ defined by

$$d\mu(Z) = \pi^{-nk}\exp\left[-\operatorname{Tr}\left(ZZ^{\dagger}\right)\right] dZ, \qquad Z\in\mathbb{C}^{n\times k}, \tag{2.1}$$

where in Eq. (2.1) $Z^{\dagger}$ denotes the adjoint of the matrix Z and dZ denotes the Lebesgue measure on $\mathbb{C}^{n\times N}$. Let $\mathcal{F}_{n\times k}\equiv\mathcal{F}\left(\mathbb{C}^{n\times k}\right)$ denote the Bargmann–Segal–Fock space of all holomorphic entire functions on $\mathbb{C}^{n\times k}$ which are also square-integrable with respect to $d\mu$. Endowed with the inner product

$$(f\,|\,g) = \int_{\mathbb{C}^{n\times k}} f(Z)\,\overline{g(Z)}\,d\mu(Z)\,; \qquad f,g\in\mathcal{F}_{n\times k}, \tag{2.2}$$

$\mathcal{F}_{n\times k}$ has a Hilbert-space structure. It can be easily verified that the inner product

$$\langle f\,|\,g\rangle = f(D)\,\overline{g\left(\bar{Z}\right)}|_{Z=0} \tag{2.3}$$

where $f(D)$ denotes the formal power series obtained by replacing $Z_{\alpha j}$ by the partial derivative $\partial/\partial Z_{\alpha j}$ $(1\le\alpha\le n,\ 1\le j\le k)$. In fact if $(r)=(r_{11},\dots,r_{nk})$ is a multi-index of integers $r_{\alpha j}\ge 0$ let $Z^{(r)}\equiv Z_{11}^{r_{11}}\cdots Z_{nk}^{r_{nk}}$ and $(r)!=r_{11}!\cdots r_{nk}!$ then it is easy to verify that

$$\left(\frac{Z^{(r)}}{[(r)!]^{\frac12}}\,\middle|\,\frac{Z^{(r')}}{[(r')!]^{\frac12}}\right) = \left\langle\frac{Z^{(r)}}{[(r)!]^{\frac12}}\,\middle|\,\frac{Z^{(r')}}{[(r')!]^{\frac12}}\right\rangle = \delta_{(r)(r')}. \tag{2.4}$$

It follows immediately from Eq. (2.4) that $\left\{Z^{(r)}/[(r)!]^{\frac12}\right\}_{(r)}$ forms an orthonormal basis for $\mathcal{F}_{n\times k}$ when (r) ranges over all multi-indices; moreover $\mathcal{P}_{n\times k}\equiv\mathcal{P}\left(\mathbb{C}^{n\times k}\right)$, the subspace of all polynomial functions on $\mathbb{C}^{n\times k}$, is dense in $\mathcal{F}_{n\times k}$.

Let G and G' be two topological groups. Let R_G and $R'_{G'}$ be continuous unitary and completely (discretely) reducible representations of G and G' on $\mathcal{F}_{n\times k}$ such that R_G and $R'_{G'}$ commute. Then we have the following definition of *dual representations* (for the definition of dual representations in a more general context see [TT3]).

Definition 2.1. The representations R_G and $R'_{G'}$ are said to be *dual* if the $G' \times G$-module $\mathcal{F}_{n\times k}$ is decomposed into a multiplicity-free orthogonal direct sum of the form

$$\mathcal{F}_{n\times k} = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)}_{n\times k}, \tag{2.5}$$

where in Eq. (2.5) the label (λ) characterizes both an equivalence class of an irreducible unitary representation λ_G of G and an equivalence class of an irreducible representation $\lambda'_{G'}$, and $\mathcal{I}^{(\lambda)}_{n\times k}$ denotes the (λ)-*isotypic component,* i.e., the direct sum (not canonical) of all irreducible subrepresentations of R_G (resp. $R'_{G'}$) that belong to the equivalence class λ_G (resp. $\lambda'_{G'}$). Moreover the $G' \times G$-submodule $\mathcal{I}^{(\lambda)}_{n\times k}$ is irreducible for all *signatures* (λ); i.e., $\mathcal{I}^{(\lambda)}_{n\times k} \approx V^{(\lambda_G)} \hat{\otimes} W^{(\lambda'_{G'})}$, where $V^{(\lambda_G)}$ (resp. $W^{(\lambda'_{G'})}$) is an irreducible G-module of class (λ_G) (resp. G'-module of class $(\lambda'_{G'})$).

We refer to the decomposition (2.5) as the *canonical decomposition* of the $G' \times G$-module $\mathcal{F}_{n\times k}$.

In this context we have the following theorem which is a special case of Theorem 4.1 in [TT3].

Theorem 2.2. *Let G be a compact group. Let R_G and $R'_{G'}$ be given dual representations on $\mathcal{F}_{n\times k}$. Let H be a compact subgroup of G and let R_H be the representation of H on $\mathcal{F}_{n\times k}$ obtained by restricting R_G to H. If there exists a group $H' \supset G'$ and a representation $R'_{H'}$ on $\mathcal{F}_{n\times k}$ such that $R'_{H'}$ is dual to R_H and $R'_{G'}$ is the restriction of $R'_{H'}$ to the subgroup G' of H' then we have the following multiplicity-free decompositions of $\mathcal{F}_{n\times k}$ into isotypic components*

$$\mathcal{F}_{n\times k} = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)}_{n\times k} = \sum_{(\mu)} \oplus \mathcal{I}^{(\mu)}_{n\times k} \tag{2.6}$$

where (λ) is a common irreducible signature of the pair (G', G) and (μ) is a common irreducible signature of the pair (H', H).

If λ_G (resp. $\lambda'_{G'}$) denotes an irreducible unitary representation of class (λ) and μ_H (resp. $\mu'_{H'}$) denotes an irreducible unitary representation of class (μ) then the multiplicity $\dim[\mathrm{Hom}_H(\mu_H : \lambda_G|_H)]$ *of the irreducible representation μ_H in the restriction to H of the representation λ_G is equal to the multiplicity* $\dim[\mathrm{Hom}_{G'}(\lambda'_{G'} : \mu'_{H'}|_{G'})]$ *of the irreducible representation $\lambda'_{G'}$ in the restriction to G' of the representation $\mu'_{H'}$.*

Remarks. In many cases $\mathrm{Hom}_H(\mu : \lambda_G|_H)$ and $\mathrm{Hom}_G(\lambda'_G : \mu'_{H'}|_G)$ are shown to be isomorphic and can be explicitly constructed in terms of generalized Casimir operators as given in [KT2] and [LT2].

To illustrate this theorem we devote the rest of this section to some typical examples and discuss their generalization.

Examples 2.3. 1) Consider $\mathcal{F}_{1\times k}$ with $k \geq 2$; then $\mathcal{F}_{1\times k}$ is the classical Bargmann space first considered by V. Bargmann in [Ba]. Then $\mathcal{P}_{1\times k}$ is the algebra of all polynomial functions in k variables $(Z_1, \ldots, Z_k) = Z$. Let $G = \mathrm{U}(k)$ and $G' = \mathrm{U}(1)$; then the complexification of $\mathrm{U}(k)$ (resp. $\mathrm{U}(1)$) is $G_{\mathbb{C}} = \mathrm{GL}_k(\mathbb{C})$ (resp. $G'_{\mathbb{C}} = \mathrm{GL}_1(\mathbb{C})$). An element f of $\mathcal{F}_{1\times k}$ is of the form

$$f(Z) = \sum_{|(r)|=0}^{\infty} c_{(r)} Z^{(r)} \tag{2.7}$$

with $(r) = (r_1, \ldots, r_k)$, $|(r)| = r_1 + \cdots + r_k$, and $Z^{(r)} = Z_1^{r_1} \cdots Z_k^{r_k}$, $c_{(r)} \in \mathbb{C}$ such that $\sum_{|(r)|=0}^{\infty} |c_{(r)}|^2 (r)! < \infty$, where $(r)! = r_1! \cdots r_k!$. The system $\left\{ Z^{(r)} / [(r)!]^{\frac{1}{2}} \right\}$, where (r) ranges over all multi-indices, forms an orthonormal basis for $\mathcal{F}_{1\times k}$. $R_{G_{\mathbb{C}}}$ and R_G are defined by

$$\begin{cases} [R_{G_{\mathbb{C}}}(g) f](Z) = f(Zg), & g \in \mathrm{GL}_k(\mathbb{C}), \\ [R_G(u) f](Z) = f(Zu), & u \in \mathrm{U}(k). \end{cases} \tag{2.8}$$

$R'_{G'_{\mathbb{C}}}$ and $R'_{G'}$ are defined by

$$\begin{cases} \left[R'_{G'_{\mathbb{C}}}(g') f\right](Z) = f\left((g')^t Z\right), & g' \in \mathrm{GL}_1(\mathbb{C}), \\ [R'_{G'}(u') f](Z) = f\left((u')^t Z\right), & u' \in \mathrm{U}(1). \end{cases} \tag{2.9}$$

The infinitesimal action of $R_{G_{\mathbb{C}}}$ is given by

$$R_{ij} = Z_i \frac{\partial}{\partial Z_j}, \qquad 1 \leq i, j \leq k, \tag{2.10}$$

which form a basis for a Lie algebra isomorphic to $\mathrm{gl}_k(\mathbb{C})$.

The infinitesimal action of $R'_{G'_{\mathbb{C}}}$ is given by

$$L = \sum_{i=1}^{k} Z_i \frac{\partial}{\partial Z_i}, \tag{2.11}$$

which forms a basis for a Lie algebra isomorphic to $\mathrm{gl}_1(\mathbb{C})$. If $p, q \in \mathcal{P}_{1\times k}$ then from Eq. (2.1) of [TT4] we have

$$R_{G_{\mathbb{C}}}(g)\, p(D)\, R_{G_{\mathbb{C}}}(g^{-1}) = [R_{G_{\mathbb{C}}}(g^{\checkmark})\, p](D), \qquad g \in \mathrm{GL}_k(\mathbb{C}),\ g^{\checkmark} = (g^{-1})^t, \tag{2.12}$$

so that if $u \in \mathrm{U}(k)$ then

$$\begin{aligned}\langle R_G(u)\,p \,|\, R_G(u)\,q\rangle &= [R_G(u)\,p](D)\,\overline{(R_G(u)\,q)\,(\bar{Z})}\Big|_{Z=0} &(2.13)\\ &= R_G\left(u^{\checkmark}\right)p\,(D)\,R_G\left(u^t\right)R\,(\bar{u})\,\overline{q\,(\bar{Z})}\Big|_{Z=0}\\ &= p\,(D)\,R_G\left(u^t\bar{u}\right)q\left(\bar{Z}u^{\checkmark}\right)\Big|_{Z=0}\\ &= \langle p\,|\,q\rangle\,,\end{aligned}$$

since $u^t\bar{u} = 1$. A similar computation shows that

$$R'_{G'_{\mathbb{C}}}(g')\,p\,(D)\,R'_{G'_{\mathbb{C}}}\left((g')^{-1}\right) = \left[R\left((g')^{\checkmark}\right)\right](D)\,, \qquad g' \in \mathrm{GL}_1(\mathbb{C})\,, \tag{2.14}$$

so that if $u \in \mathrm{U}(1)$ then

$$\langle R'_{G'}(u')\,p \,|\, R'_{G'}(u')\,q\rangle = \langle p\,|\,q\rangle\,. \tag{2.15}$$

Note that all equations above from (2.12) to (2.15) remain valid if we replace $\mathbb{C}^{1\times k}$ by $\mathbb{C}^{n\times k}$ and $\mathrm{GL}_1(\mathbb{C})$ (resp. $\mathrm{U}(1)$) by $\mathrm{GL}_n(\mathbb{C})$ (resp. $\mathrm{U}(n)$).

It follows that R_G, $G = \mathrm{U}(k)$ (resp. $R'_{G'}$, $G' = \mathrm{U}(n)$) is a continuous unitary representation of G (resp. G') on $\mathcal{F}_{n\times k}$.

Let $\mathcal{P}^{(m)}_{1\times k}$ denote the subspace (of $\mathcal{F}_{1\times k}$) of all homogeneous polynomial functions of degree $m \geq 0$. Then by the Borel–Weil theorem (see, e.g., [TT4]) the restriction of $R_{G_{\mathbb{C}}}$ to $\mathcal{P}^{(m)}_{1\times k}$ is an irreducible subrepresentation of $R_{G_{\mathbb{C}}}$ with highest weight $\underbrace{(m,0,\ldots,0)}_{k}$ and highest weight vector cZ_1^m, $c \in \mathbb{C}^*$. In fact, by letting the infinitesimal operators R_{ij} act on $\mathcal{P}^{(m)}_{1\times k}$ one can easily show that $\mathcal{P}^{(m)}_{1\times k}$ is an irreducible subrepresentation of $R_{G_{\mathbb{C}}}$. By "Weyl's unitarian trick" the restriction of this irreducible subrepresentation to G gives an irreducible unitary representation of G.

Let $0 \neq p \in \mathcal{P}^{(m)}_{1\times k}$. Then $(R'_{G'}(g')\,p)(Z) = p\left((g')^t Z\right) = p(g'Z) = (g')^m p(Z)$ for all $g' \in \mathrm{GL}_1(\mathbb{C})$. So the one-dimensional subspace of $\mathcal{F}_{1\times k}$ spanned by p is an irreducible $G'_{\mathbb{C}}$-submodule with highest weight (m) and its restriction to G' is an irreducible unitary G'-submodule. In fact, Euler's formula implies that

$$Lp = mp, \qquad \text{for all } p \in \mathcal{P}^{(m)}_{1\times k}. \tag{2.16}$$

Thus the canonical decomposition of the $G' \times G$-module $\mathcal{F}_{1\times k}$ is simply

$$\mathcal{F}_{1\times k} = \sum_{m=0}^{\infty} \oplus\, \mathcal{P}^{(m)}_{1\times k}. \tag{2.17}$$

Let H denote the special orthogonal subgroup $\mathrm{SO}(k)$. Then $H_{\mathbb{C}} = \mathrm{SO}_k(\mathbb{C})$. Then the ring of all H (or $H_{\mathbb{C}}$)-invariant polynomials in $\mathcal{P}_{1\times k}$ is generated by the constants and

$p_0(Z) = \sum_{1\le i\le k} Z_i^2$. The ring of all H (or $H_{\mathbb{C}}$)-invariant differential operators with constant coefficients is generated by the constants and the Laplacian $\Delta = p_0(D) = \sum_{1\le i\le k} \partial^2/\partial Z_i^2$. To find the dual representation of R_H we follow the method given in [TT3] by setting

$$X^+ = \frac{1}{2}p_0, \qquad X^- = \frac{1}{2}p_0(D) = \frac{1}{2}\Delta, \qquad \text{and } E = \frac{k}{2} + L. \tag{2.18}$$

Then X^+ (resp. X^-) acts on $\mathcal{F}_{1\times k}$ as a *creation* (resp. *annihilation*) *operator* and E acts on $\mathcal{F}_{1\times k}$ as a number operator. In fact, if $p \in \mathcal{P}_{1\times k}^{(m)}$ then $X^+p = \frac{1}{2}p_0p$, $X^-p = \frac{1}{2}\Delta p$, and $Ep = ((k/2)+m)p$, so that X^+ *raises* $\mathcal{P}_{1\times k}^{(m)}$ to $\mathcal{P}_{1\times k}^{(m+2)}$, X^- *lowers* $\mathcal{P}_{1\times k}^{(m)}$ to $\mathcal{P}_{1\times k}^{(m-2)}$ and H *multiplies* (elementwise) $\mathcal{P}_{1\times k}^{(m)}$ by the *number* $(k/2)+m$. An easy computation shows that

$$[E, X^+] = 2X^+, \qquad [E, X^-] = -2X^-, \qquad [X^-, X^+] = E. \tag{2.19}$$

Eq. (2.19) gives a faithful representation of the Lie algebra $\mathrm{sl}_2(\mathbb{R})$. Thus the dual action of H is given by this representation. The integrated form of this Lie algebra representation is more subtle to describe: it is the *metaplectic representation of the two-sheeted covering group* $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ (or $\mathrm{Sp}_2(\mathbb{R})$), and this group is not a matrix group. Its concrete description can be obtained by applying the Bargmann–Segal transform which sends the Schrödinger representation of this group to its Fock representation $\mathcal{F}_{1\times k}$. However, for our purpose, its infinitesimal action (2.19) together with the action of its maximal compact group $G' = \mathrm{U}(1)$, which is particularly simple, will suffice. Indeed, it is easy to show that we have the following decomposition of $\mathcal{P}_{1\times k}^{(m)}$:

$$\mathcal{P}_{1\times k}^{(m)} = \sum_{i=0,\ldots,[m/2]} \oplus\, p_0^i \mathcal{H}_{1\times k}^{(m-2i)}, \tag{2.20}$$

where $[m/2]$ denotes the integral part of $m/2$, and $\mathcal{H}_{1\times k}^{(m-2i)}$ denotes the subspace of all harmonic homogeneous polynomials of degree $(m-2i)$, i.e., all functions $p \in \mathcal{P}_{1\times k}^{(m-2i)}$ such that $\Delta p = 0$. For an integer $r \ge 0$ then it can be easily shown that the restriction $R_H^{(r)}$ of R_H to $\mathcal{H}_{1\times k}^{(r)}$ is an irreducible representation of H with signature $\underbrace{(r, 0, \ldots, 0)}_{[k/2]}$ and highest weight vector

$$f^{(r)_H}(Z) = \begin{cases} (Z_1 + iZ_{s+1})^r, & \text{if } k = 2s, \\ (Z_1 + iZ_{s+2})^r, & \text{if } k = 2s+1, \qquad i = \sqrt{-1}. \end{cases} \tag{2.21}$$

For each integer $j \ge 0$, the restriction of R_H to the subspace $p_0^j \mathcal{H}^{(r)}$ is equivalent to $R_H^{(r)}$ since p_0^j is H-invariant. Set

$$\mathcal{I}_{1\times k}^{(r)} = \sum_{j=0}^{\infty} \oplus\, p_0^j \mathcal{H}_{1\times k}^{(r)}; \tag{2.22}$$

then $\mathcal{I}_{1\times k}^{(r)}$ is the $\underbrace{(r, 0, \ldots, 0)}_{k}$-isotypic component of $R_H^{(r)}$. From (2.20) and (2.22) we see that

$$\mathcal{F}_{1\times k} = \sum_{r=0}^{\infty} \oplus\, \mathcal{I}_{1\times k}^{(r)}. \tag{2.23}$$

Obviously, $R'_{H'}(u) = R'_G(u)$, $u \in G'$, leaves each one-dimensional subspace $cp_0^j h$, $c \in \mathbb{C}$, invariant, since $R'_G(u)\left(p_0^j h\right) = u^{r+2j}\left(p_0^j h\right)$ (alternatively, $E\left(p_0^j h\right) = ((k/2) + r + 2j)\left(p_0^j h\right)$), for all $h \in \mathcal{H}_{1\times k}^{(r)}$. Clearly, $X^+\left(p_0^j h\right) = \frac{1}{2} p_0^{(j+1)} h$, $h \in \mathcal{H}_{1\times k}^{(r)}$. Finally from the equation

$$X^-(p_0 f) = (k + 2s) f + \frac{1}{2} p_0 \bigtriangleup f \tag{2.24}$$

if f is a polynomial function of degree s, we deduce by induction on the integer $j \geq 1$ that

$$X^-\left(p_0^j h\right) = j\left(k + 2(r + j - 1)\right) p_0^{j-1} h, \qquad h \in \mathcal{H}_{1\times k}^{(r)}. \tag{2.25}$$

For each fixed $h \in \mathcal{H}_{1\times k}^{(r)}$ let Jh denote the subspace of $\mathcal{I}_{1\times k}^{(r)}$ spanned by the set

$$\left\{p_0^j h \mid j = 0, 1, 2, \dots\right\}.$$

Then it follows from the previous discussion that the subrepresentation of the Lie algebra $\mathrm{sl}_2(\mathbb{R})$ on Jh is irreducible, and thus the metaplectic subrepresentation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ on Jh is irreducible as well. As a $\mathrm{U}(1)$-module Jh is reducible, and for this special case each one-dimensional subspace $cp_0^j h$, $c \in \mathbb{C}$, is an irreducible submodule, and the lowest one is ch which has weight r (or $(k/2) + r$) since

$$R'_G(u)\, h = u^r h, \qquad u \in \mathrm{U}(1), \qquad \text{or} \qquad Eh = \left(\frac{k}{2} + r\right) h. \tag{2.26}$$

In general, if a *holomorphic discrete series* of a noncompact semisimple Lie group such as $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ considered as a K-module, where K is its maximal compact subgroup, decomposes into a discrete sum of irreducible submodules, each one of them can be characterized by a signature (highest weight, for example) and the one with the lowest highest weight (under the lexicographic ordering) is unique. This *lowest K-type highest weight* which corresponds to the *Harish Chandra's or Blattner's parameter,* can be used to label the given holomorphic discrete series. We shall call this label its *signature.* In our example, the holomorphic discrete series Jh of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ has signature r. If $\dim\left(\mathcal{H}_{1\times r}^{(r)}\right) = d$ (actually, $d = \binom{k+r-1}{r} - \binom{k+r-3}{r-2}$) $\mathcal{I}_{1\times k}^{(r)}$ is the r-isotypic component (of the metaplectic representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$) which contains d isomorphic copies of signature r.

Now let us verify Theorem 2.2 for this simple example. From Eq. (2.20) we have

$$\begin{aligned}\dim\left[\mathrm{Hom}_{\mathrm{SO}(k)}\left(\underbrace{(r,0,\dots,0)}_{[k/2]}{}_{\mathrm{SO}(k)} : \underbrace{(m,0,\dots,0)}_{k}{}_{\mathrm{U}(k)}\Big|_{\mathrm{SO}(k)}\right)\right]& \\ = \begin{cases} 1, & \text{if } r = m - 2i \text{ for } i = 0, \dots, [m/2], \\ 0, & \text{otherwise,}\end{cases}&\end{aligned} \tag{2.27}$$

and from Eq. (2.22) and Eq. (2.26) we have

$$\dim\left[\mathrm{Hom}_{\mathrm{U}(1)}\left(m_{\mathrm{U}(1)} : r_{\widetilde{\mathrm{SL}_2(\mathbb{R})}}\Big|_{\mathrm{U}(1)}\right)\right] = \begin{cases} 1, & \text{if } 2j + r = m, \\ 0, & \text{otherwise,}\end{cases} \tag{2.28}$$

which are obviously identical.

For arbitrary n such that $n \leq k$ Eq. (2.7) remains valid with $(r) = (r_{11}, \dots, r_{nk})$ and $Z^{(r)} = Z_{11}^{r_{11}} \cdots Z_{nk}^{r_{nk}}$. Eq. (2.8), (2.12), (2.13), (2.14) remain valid. Eq. (2.10) is replaced by

$$R_{ij} = \sum_{\alpha=1}^{n} Z_{\alpha i} \frac{\partial}{\partial Z_{\alpha j}}, \qquad 1 \leq i, j \leq k. \tag{2.10$'$}$$

Eq. (2.11) is replaced by

$$L_{\alpha\beta} = \sum_{i=1}^{k} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}}, \qquad 1 \leq \alpha, \beta \leq n. \tag{2.11$'$}$$

Let B_n' denote the lower triangular Borel subgroup of $G_{\mathbb{C}}' = \mathrm{GL}_n(\mathbb{C})$, let (λ) be an n-tuple of integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, let $\lambda \colon B_n' \to \mathbb{C}^*$ be the holomorphic character defined on B_n' by

$$\lambda(b') = (b_{11}')^{\lambda_1} \cdots (b_{nn}')^{\lambda_n} \qquad \text{if } b' = \begin{bmatrix} b_{11}' & & 0 \\ & \ddots & \\ * & & b_{nn}' \end{bmatrix} \text{ belongs to } B_n'.$$

Let $\mathcal{P}_{n\times k}^{(\lambda)}$ denote the subspace of all polynomial functions on $\mathbb{C}^{n\times k}$ which also satisfy the covariant condition

$$f(b'Z) = \lambda(b') f(Z), \qquad (b', Z) \in B_n' \times \mathbb{C}^{n\times k}. \tag{2.29}$$

Let R_λ denote the representation of G obtained by right translation on $\mathcal{P}_{n\times k}^{(\lambda)}$. Then by the Borel–Weil theorem (see, e.g., [TT4, Theorem 1.5]) R_λ is irreducible with highest weight (λ) and highest weight vector

$$cf_\lambda(Z) = c\Delta_1^{\lambda_1-\lambda_2}(Z)\,\Delta_2^{\lambda_2-\lambda_3}(Z)\cdots\Delta^{\lambda_n}(Z), \qquad c \in \mathbb{C}^*, \tag{2.30}$$

where in Eq. (2.30) $\Delta_i(Z)$ denotes the i^{th} principal minor of Z.

Similarly let B_k^t denote the upper triangular Borel subgroup of $G_{\mathbb{C}} = \mathrm{GL}_k(\mathbb{C})$ and let $\lambda' \colon B_k^t \to \mathbb{C}^*$ be the holomorphic character defined on B_k^t by

$$\lambda'(b) = b_{11}^{\lambda_1} \cdots b_{nn}^{\lambda_n} \qquad \text{if } b = \begin{bmatrix} b_{11} & & & & * \\ & \ddots & & & \\ & & b_{nn} & & \\ & & & \ddots & \\ 0 & & & & b_{kk} \end{bmatrix} \text{ belongs to } B_k^t.$$

Let $\mathcal{P}_{n\times k}^{(\lambda')}$ denote the subspace of all polynomial functions on $\mathbb{C}^{n\times k}$ which also satisfy the covariant condition

$$f(Zb) = \lambda'(b) f(Z), \qquad (b, Z) = B_k^t \times \mathbb{C}^{n\times k}. \tag{2.31}$$

Let $R'_{\lambda'}$ denote the representation of G' on $\mathcal{P}_{n\times k}^{(\lambda')}$ defined by

$$[R'_{\lambda'}(g') f](Z) = f\left((g')^t Z\right), \qquad g' \in G'. \tag{2.32}$$

Then $R'_{\lambda'}$ is irreducible with highest weight (λ') and with the same highest weight vector given by Eq. (2.30). By Weyl's unitarian trick the restriction of R_λ (resp. $R'_{\lambda'}$) to $G = \mathrm{U}(k)$ (resp. $G' = \mathrm{U}(n)$) remains irreducible with the same signature.

Let $\mathcal{I}_{n\times k}^{(\lambda)}$ denote the $G'_{\mathbb{C}} \times G_{\mathbb{C}}$ (or $G' \times G$)-cyclic module in $\mathcal{F}_{n\times k}$ generated by the highest vector f_λ given by Eq. (2.29); then by Theorem 3, p. 150, of [Ze], $\mathcal{I}_{n\times k}^{(\lambda)}$ is irreducible with highest weight (λ', λ). For the sake of simplicity we say that the $G'_{\mathbb{C}} \times G_{\mathbb{C}}$-module $\mathcal{I}_{n\times k}^{(\lambda)}$ has *signature* (λ). To prove that $\mathcal{I}_{n\times k}^{(\lambda)} \approx \mathcal{P}_{n\times k}^{(\lambda')} \hat{\otimes} \mathcal{P}_{n\times k}^{(\lambda)}$ we define a map $\Phi\colon \mathcal{P}_{n\times k}^{(\lambda')} \hat{\otimes} \mathcal{P}_{n\times k}^{(\lambda)} \to \mathcal{I}_{n\times k}^{(\lambda)}$ as follows:

Let $f' \otimes f \in \mathcal{P}_{n\times k}^{(\lambda')} \hat{\otimes} \mathcal{P}_{n\times k}^{(\lambda)}$. Then f' and f can be represented in the following form:

$$f' = \sum_{i\in I'} c'_i R'_{\lambda'}(g'_i) f_\lambda, \qquad f = \sum_{j\in I} c_j R_\lambda(g_j) f_\lambda, \tag{2.33}$$

where in Eq. (2.33) $c'_i, c_j \in \mathbb{C}$, $g'_i \in G'_{\mathbb{C}}$, $g_j \in G_{\mathbb{C}}$, and I' and I are two finite index sets. Set $\Phi(f' \otimes f) = \sum_{i\in I', j\in I} c'_i c_j T(g'_i, g_j) f_\lambda$, where $\left[T(g'_i, g_j) f_\lambda\right](Z) = f\left((g'_i)^t Z g_j\right)$. Since

$$R'_{\lambda'}(g') f' = \sum_{i\in I} c'_i R'_{\lambda'}(g' g'_i) f_\lambda$$

and

$$R_\lambda(g) f = \sum_{j\in I} c_j R_\lambda(g g_j) f_\lambda$$

it follows that

$$\begin{aligned} \Phi\left[\left(R'_{\lambda'}(g') \otimes R_\lambda(g)\right)(f' \otimes f)\right] &= \sum_{i\in I, j\in T} c'_i c_j T\left(g' g'_i, g g_j\right) f_\lambda \\ &= T(g', g)\, \Phi(f' \otimes f) \end{aligned}$$

for all $g' \in G'_{\mathbb{C}}$ and $g \in G_{\mathbb{C}}$. This means that Φ is an intertwining operator and by Schur's lemma Φ is either 0 or an isomorphism. Since

$$\Phi(f_\lambda \otimes f_\lambda) = f_\lambda$$

it follows that Φ is an isomorphism. Since $\mathcal{P}_{n\times k}$ is dense in $\mathcal{F}_{n\times k}$ Theorem 3 (p. 150) of [Ze] (see also [KT1]) implies that we have the Hilbert sum $\mathcal{F}_{n\times k} = \sum_{(\lambda)} \oplus \mathcal{I}_{n\times k}^{(\lambda)}$ for the pair $(\mathrm{U}(n), \mathrm{U}(k))$.

Now suppose $k > 2n$ and set $H = \mathrm{SO}(k)$, $H_{\mathbb{C}} = \mathrm{SO}_k(\mathbb{C})$. Let $J_{n\times k}$ denote the ring of all H (or $H_{\mathbb{C}}$)-invariant polynomials in $\mathcal{P}_{n\times k}$. Then $J_{n\times k}$ is generated by the constants and the $n(n+1)/2$ algebraically independent polynomials

$$p_{\alpha\beta}(Z) = \sum_{i=1}^{k} Z_{\alpha i} Z_{\beta i}, \qquad 1 \leq \alpha \leq \beta \leq n. \tag{2.34}$$

It follows that the ring of all H (or $H_{\mathbb{C}}$)-invariant differential operators with constant coefficients is generated by the constants and the Laplacians

$$\triangle_{\alpha\beta} = p_{\alpha\beta}(D) = \sum_{i=1}^{k} \frac{\partial^2}{\partial Z_{\alpha i} \partial Z_{\beta i}}, \qquad 1 \leq \alpha \leq \beta \leq n. \tag{2.35}$$

The infinitesimal action of $R'_{G_{\mathbb{C}}}$ is generated by

$$L_{\alpha\beta} = \sum_{i=1}^{k} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}}, \qquad 1 \leq \alpha, \beta \leq n. \tag{2.36}$$

Set $P_{\alpha\beta} = -p_{\alpha\beta}$, $E_{\alpha\beta} = L_{\alpha\beta} + \frac{1}{2}k\delta_{\alpha\beta}$, and $D_{\alpha\beta} = \triangle_{\alpha\beta}$; then it follows from [KLT] (see Eq. (3.3)) that $\{E_{\alpha\beta}, P_{\alpha\beta}, D_{\alpha\beta}\}$ defines a faithful representation of $\mathrm{sp}_{2n}(\mathbb{R})$ on $\mathcal{F}_{n\times k}$. By construction this representation is dual to the infinitesimal action of R_H. The global action $R'_{H'}$ is a unitary metaplectic representation of $\widetilde{\mathrm{Sp}_{2n}(\mathbb{R})}$, the two-sheeted covering of $\mathrm{Sp}_{2n}(\mathbb{R})$ (see [KLT] for details). As in the case of the pair $(\mathrm{U}(n), \mathrm{U}(k))$ the common highest weight vector (for $R'_{H'}$ the lowest K'-type highest weight vector) of signature $(\mu) = (\mu_1, \ldots, \mu_n)$ with $\mu_1 \geq \cdots \geq \mu_n \geq 0$ and $\mu_i \in \mathbb{N}$, $1 \leq i \leq n$, of the pair $\left(\widetilde{\mathrm{Sp}_{2n}(\mathbb{R})}, \mathrm{SO}(k)\right)$ is

$$f_\mu(Z) = \Delta_1^{\mu_1-\mu_2}(Zq)\, \Delta_2^{\mu_2-\mu_3}(Zq) \cdots \Delta_n^{\mu_n}(Zq), \tag{2.37}$$

where the $k \times k$ matrix q is given by

$$\frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \mathbb{1}_\nu & \mathbb{1}_\nu \\ \hline i\mathbb{1}_\nu & -i\mathbb{1}_\nu \end{array}\right] \qquad \text{if } k = 2\nu,$$

and

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1}_\nu & 0 & \mathbb{1}_\nu \\ 0 & \sqrt{2} & 0 \\ i\mathbb{1}_\nu & 0 & -i\mathbb{1}_\nu \end{bmatrix} \qquad \text{if } k = 2\nu + 1,$$

and where $\mathbb{1}_\nu$ is the unit matrix of order ν.

An element p of $\mathcal{P}_{n\times k}$ is called *H-harmonic* if $\Delta_{\alpha\beta}p = 0$ for all $\alpha,\beta = 1,\ldots,n$. Let $\mathcal{H}_{n\times k}$ denote the subspace of all H-harmonic polynomial functions of $\mathcal{P}_{n\times k}$ and let $\mathcal{H}_{n\times k}(\mu)$ denote the subspace of all elements h of $\mathcal{H}_{n\times k}$ which also satisfy the covariant condition

$$h(b'Z) = (b'_{11})^{\mu_1}\cdots(b'_{nn})^{\mu_n}\,h(Z), \qquad \forall\, b' \in B'_n. \tag{2.38}$$

Then according to Theorem 3.1 of [TT4], the representation R_H of H which is obtained by right translations on $\mathcal{H}_{n\times k}(\mu)$ is irreducible with signature (μ).

The infinitesimal action of R_H is given by

$$R^H_{ij} = \sum_{\alpha=1,\ldots,n}\left(Z_{\alpha i}\frac{\partial}{\partial Z_{\alpha j}} - Z_{\alpha j}\frac{\partial}{\partial Z_{\alpha i}}\right), \qquad 1 \le i < j \le k. \tag{2.39}$$

From [KLT] the dual infinitesimal action of R_H is given by the system $\{E_{\alpha\beta}, P_{\alpha\beta}, D_{\alpha\beta}\}$ which satisfies the commutation relations

$$\begin{cases} [E_{\alpha\beta}, E_{\mu\nu}] = \delta_{\beta\mu}E_{\alpha\nu} - \delta_{\alpha\nu}E_{\mu\beta} \\ [E_{\alpha\beta}, P_{\mu\nu}] = \delta_{\beta\mu}P_{\alpha\nu} + \delta_{\beta\nu}P_{\alpha\mu} \\ [E_{\alpha\beta}, D_{\mu\nu}] = -\delta_{\alpha\mu}D_{\beta\nu} - \delta_{\alpha\nu}D_{\beta\mu} \\ [P_{\alpha\beta}, D_{\mu\nu}] = \delta_{\alpha\mu}E_{\nu\beta} + \delta_{\alpha\nu}E_{\mu\beta} + \delta_{\beta\mu}E_{\nu\alpha} + \delta_{\beta\nu}E_{\mu\alpha} \\ [P_{\alpha\beta}, P_{\mu\nu}] = [D_{\alpha\beta}, D_{\mu\nu}] = 0 \\ P_{\alpha\beta},\ P_{\beta\alpha},\ D_{\alpha\beta} = D_{\beta\alpha} \\ P^\dagger_{\alpha\beta} = D_{\alpha\beta}, \qquad D^\dagger_{\alpha\beta} = P_{\alpha\beta}, \qquad E^\dagger_{\alpha\beta} = E_{\beta\alpha}, \\ \qquad \text{for all } \alpha,\beta,\mu,\nu = 1,\ldots,n. \end{cases} \tag{2.40}$$

By Corollary 3.11 of [TT4] the μ-isotypic component in $\mathcal{H}_{n\times k}$ consists of d_μ copies isomorphic to $\mathcal{H}_{n\times k}(\mu)$, where d_μ is the degree of an irreducible representation of $G' = \mathrm{U}(n)$ of signature $(\mu_1,\ldots,\mu_n)$. Since from Eq. (2.40) and the fact that f_μ is H-harmonic

$$\begin{aligned} D_{\mu\nu}E_{\alpha\beta}f_\mu &= [D_{\mu\nu}, E_{\alpha\beta}]\,f_\mu + E_{\alpha\beta}D_{\mu\nu}f_\mu \\ &= \delta_{\alpha\mu}D_{\beta\nu}f_\mu + \delta_{\alpha\nu}D_{\beta\mu}f_\mu \\ &= 0, \end{aligned}$$

it follows that $E_{\alpha\beta}f_\mu$ is H-harmonic for every $\alpha,\beta = 1,\ldots,n$. Since $\left[E_{\alpha\beta}, R^H_{ij}\right] = 0$ for all $\alpha,\beta = 1,\ldots,n$ and $i,j = 1,\ldots,k$ it follows that $E_{\alpha\beta}\colon \mathcal{H}_{n\times k}(\mu) \to \mathcal{H}_{n\times k}$ are intertwining operators, and thus are either 0 or isomorphisms. It follows that the $\mathfrak{g}'$-module generated by the cyclic vector f_μ is irreducible with signature $(\mu_1,\ldots,\mu_n)$. In fact, from Eq. (3.14) of [TT4] this space is a G'-module. Let $G'f_\mu$ denote this G'-module; then by construction $G'f_\mu \subset \mathcal{H}_{n\times k}$.

If $h \in G'f_\mu$ then from Eq. (2.40) we have

$$D_{\mu\nu}P_{\alpha\beta}h = [D_{\mu\nu}, P_{\alpha\beta}]\, h + P_{\alpha\beta}D_{\mu\nu}h$$

$$= -\left(\delta_{\alpha\mu}E_{\nu\beta} + \delta_{\alpha\nu}E_{\mu\beta} + \delta_{\beta\mu}E_{\nu\alpha} + \delta_{\beta\nu}E_{\mu\nu}\right) h,$$

and therefore $D_{\mu\nu}P_{\alpha\beta}h$ belongs to $G'f_\mu$. It follows that $J_{n\times k}G'f_\mu$ is an irreducible $\mathrm{sp}_{2n}(\mathbb{R})$-module with signature (μ). Let $\mathcal{H}'_{n\times k}(\mu)$ denote this module and let $\mathcal{I}^{(\mu)}_{n\times k}$ be the $H'\times H$-cyclic module generated by f_μ; then a proof similar to the case $\mathcal{I}^{(\lambda)}_{n\times k}$ shows that $\mathcal{H}'_{n\times k}(\mu)\,\hat{\otimes}\,\mathcal{H}_{n\times k}(\mu)$ is isomorphic to $\mathcal{I}^{(\mu)}_{n\times k}$. By the "separation of variables theorem" 2.5 of [TT4] and from the fact that $\mathcal{P}_{n\times k}$ is dense in $\mathcal{F}_{n\times k}$ it follows that the orthogonal direct sum decomposition $\mathcal{F}_{n\times k} = \sum\limits_{(\mu)}\oplus\mathcal{I}^{(\mu)}_{n\times k}$ holds. Therefore the reciprocity theorem 2.2 holds for these pairs (G', G) and (H', H) as well.

2) Let $k = 2l$ and consider again the dual pair $(G' = \mathrm{U}(n), G = \mathrm{U}(k))$. Let $H = \mathrm{Sp}(k)$; then $H_{\mathbb{C}} = \mathrm{Sp}_k(\mathbb{C})$. If $l \geq n \geq 2$ then the theory of symplectic harmonic polynomials in [TT5] implies that the dual representation to the representation R_H on $\mathcal{F}_{n\times k}$ is a representation of the group $\mathrm{SO}^*(2n) = H'$ whose infinitesimal action is given by Eq. (4.2) of [KLT]. Using Theorem 2.1 of [TT5] and the "separation of variables theorem" for this case we can show similarly that $\mathcal{F}_{n\times k} = \sum\limits_{(\mu)}\oplus\mathcal{I}^{(\mu)}_{n\times k}$ for this dual pair $(\mathrm{SO}^*(2n), \mathrm{Sp}(k))$. Thus the reciprocity theorem 2.2 holds again for these pairs (G', G) and (H', H).

3) The case of the dual pairs

$$(G' = \mathrm{U}(p)\times\mathrm{U}(q), G = \mathrm{U}(k)\times\mathrm{U}(k))$$

and

$$(H' = \mathrm{U}(p,q), H = \mathrm{U}(k))$$

can be treated in a similar fashion using the results of [TT6] and the infinitesimal action of H' on $\mathcal{F}_{n\times k}$ is given by Eq. (6.4) of [TT3]. However, its generalization to the case $H = \mathrm{U}(\infty)$ in Section 3 is quite delicate and requires a quite different embedding that we shall describe in detail below.

Let p and q be positive integers such that $p + q = n$. Let k be an integer such that $k \geq 2\max(p,q)$. Let (λ) be a q-tuple of integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q \geq 0$. Let R_λ denote the representation of $\mathrm{GL}_k(\mathbb{C})$ (or $\mathrm{U}(k)$) defined on $\mathcal{P}^{(\lambda)}_{q\times k}$ given by Eq. (2.29) and (2.30) with n replaced by q. We define the *contragredient* (or *dual*) *representation* of R_λ as follows.

Let s_r denote the $r \times r$ matrix with ones along the reverse diagonal and zero elsewhere:

$$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

If $W \in \mathbb{C}^{q\times k}$ let $\tilde{W} = s_q W s_k$. Thus $\tilde{W}$ is of the form

$$\tilde{W} = \begin{bmatrix} W_{q,k} & \cdots & W_{q,1} \\ \vdots & & \vdots \\ W_{1,k} & \cdots & W_{1,1} \end{bmatrix}. \tag{2.41}$$

Let $\mathcal{P}_{q\times k}^{(\lambda\ )}$ denote the subspace of all polynomial functions in $\tilde{W}$ which also satisfy the covariant condition

$$f\left(\tilde{b}'\tilde{W}\right) = \lambda(b') f\left(\tilde{W}\right) \tag{2.42}$$

for all $b' \in B_q'$, where B_q' is the lower triangular Borel subgroup of $\mathrm{GL}_q(\mathbb{C})$, and $\tilde{b}' = s_q b' s_q$.

Define the representation R_λ of $\mathrm{GL}_k(\mathbb{C})$ (or $\mathrm{U}(k)$) on $\mathcal{P}_{q\times k}^{(\lambda\ )}$ by

$$[R_\lambda\ (g) f]\left(\tilde{W}\right) = f\left(\tilde{W} s_k g s_k\right), \qquad g \in \mathrm{GL}_k(\mathbb{C}). \tag{2.43}$$

Then R_λ is irreducible with signature $\underbrace{(0,\ldots,0,-\lambda_q,-\lambda_{q-1},\ldots,-\lambda_1)}_{k}$ and *lowest weight vector*

$$cf_\lambda\ \left(\tilde{W}\right) = \Delta_1^{\lambda_1-\lambda_2}(\tilde{w})\,\Delta_2^{\lambda_2-\lambda_3}(\tilde{w})\cdots\Delta^{\lambda_q}(\tilde{w}), \qquad c \in \mathbb{C}^*, \tag{2.44}$$

of weight $(-\lambda_1,-\lambda_2,\ldots,-\lambda_q,0,\ldots,0)$.

Let $\mathcal{P}_{q\times k}^{(\lambda\ )'}$ denote the subspace of all polynomial functions in $\tilde{W}$ which also satisfy the covariant condition

$$f\left(\tilde{W}\tilde{b}\right) = \lambda(b) f\left(\tilde{W}\right), \tag{2.45}$$

where $\tilde{b} = s_k b s_k$, $b \in B_k^t$ (it follows that $\tilde{b}$ is a lower triangular matrix of the form $\tilde{b} = \begin{pmatrix} b_{kk} & & 0 \\ & \ddots & \\ * & & b_{11} \end{pmatrix}$). Let $R'_{(\lambda\ )'}$ denote the representation of $\mathrm{GL}_q(\mathbb{C})$ (or of $G' = \mathrm{U}(q)$) on $\mathcal{P}_{q\times k}^{(\lambda\ )'}$ defined by

$$\left[R'_{(\lambda\ )'}(g') f\right]\left(\tilde{W}\right) = f\left(s_q (g')^{-1} s_q \tilde{W}\right), \qquad g' \in \mathrm{GL}_q(\mathbb{C}). \tag{2.46}$$

Then $R'_{(\lambda\ )'}$ is irreducible with highest weight $\left(\lambda^{\checkmark}\right)'$ and with *lowest weight vector* given by cf_λ , $c \in \mathbb{C}^*$, of weight $(-\lambda_1,-\lambda_2,\ldots,-\lambda_q)$.

As in the case $(\lambda')\otimes(\lambda)$ it can be shown that $\mathcal{P}_{q\times k}^{(\lambda\ )'}\hat{\otimes}\mathcal{P}_{q\times k}^{(\lambda\ )}$ is isomorphic to $\mathcal{I}_{q\times k}^{(\lambda\ )}$ and we have the Hilbert sum decomposition $\mathcal{F}_{q\times k}=\sum_{(\lambda\ )}\oplus\mathcal{I}_{q\times k}^{(\lambda\ )}$ for the pair $(\mathrm{U}(q),\mathrm{U}(k))$.

Now let $G=\mathrm{U}(k)\times\mathrm{U}(k)$ act on $\mathcal{F}_{n\times k}$ via the outer tensor product

$$\left[R_{\mathrm{U}(k)}\hat{\otimes}R_{\mathrm{U}(k)}\right](g_1,g_2)\,f\left(\begin{bmatrix}Z\\ \hdashline \tilde{W}\end{bmatrix}\right)=f\left(\begin{bmatrix}Zg_1\\ \hdashline \tilde{W}s_kg_2^{\checkmark}s_k\end{bmatrix}\right),\tag{2.47}$$

where $Z\in\mathbb{C}^{p\times k}$, $\tilde{W}\in\mathbb{C}^{q\times k}$, $p+q=n$, $g_1,g_2\in\mathrm{U}(k)$. Then $G'=\mathrm{U}(p)\times\mathrm{U}(q)$ acts on $\mathcal{F}_{n\times k}$ via the outer tensor product

$$\left[R'_{\mathrm{U}(p)}\hat{\otimes}R'_{\mathrm{U}(q)}\right](g_1',g_2')\,f\left(\begin{bmatrix}Z\\ \hdashline \tilde{W}\end{bmatrix}\right)=f\left(\begin{bmatrix}(g_1')^tZ\\ \hdashline s_q(g_2')^{-1}s_q\tilde{W}\end{bmatrix}\right),\tag{2.48}$$

where $(g_1',g_2')\in\mathrm{U}(p)\times\mathrm{U}(q)$.

It follows that we have the isotypic decomposition for the dual pairs (G',G)

$$\mathcal{F}_{n\times k}=\sum_{(\nu)\otimes(\lambda\ )}\oplus\mathcal{I}_{n\times k}^{(\nu)\otimes(\lambda\ )},\tag{2.49}$$

where $\mathcal{I}_{n\times k}^{(\nu)\otimes(\lambda\ )}$ is isomorphic to $\mathcal{I}_{p\times k}^{(\nu)}\otimes\mathcal{I}_{q\times k}^{(\lambda\ )}$.

Let $H=\{(g,g):g\in\mathrm{U}(k)\}$; then H is isomorphic to $\mathrm{U}(k)$ and H acts on $\mathcal{F}_{n\times q}$ via the inner (or Kronecker) tensor product $R_H=R_{\mathrm{U}(k)}\otimes R_{\mathrm{U}(k)}$. Let $J_{n\times k}$ denote the ring of all H (or $H_{\mathbb{C}}\approx\mathrm{GL}_k(\mathbb{C})$)-invariant polynomials in $\mathcal{P}_{n\times k}$. Then from [TT6] and [TT3] $J_{n\times k}$ is generated by the constants and the $p\times q$ algebraically independent polynomials

$$p_{\alpha\beta}\left(\begin{bmatrix}Z\\ \hdashline \tilde{W}\end{bmatrix}\right)=\left(Zs_k\tilde{W}^t\right)_{\alpha\beta}=\sum_{i=1}^{k}Z_{\alpha i}W_{\beta i},\qquad 1\leq\alpha\leq p,\ 1\leq\beta\leq q.\tag{2.50}$$

It follows that the ring of all H or $(H_{\mathbb{C}})$-invariant differential operators with constant coefficients is generated by the constants and the Laplacians

$$\triangle_{\alpha\beta}=p_{\alpha\beta}(D)=\sum_{i=1}^{k}\frac{\partial^2}{\partial Z_{\alpha i}\partial W_{\beta i}},\qquad 1\leq\alpha\leq p,\ 1\leq\beta\leq q.\tag{2.51}$$

Together with the infinitesimal action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathcal{F}_{n\times k}$ the $p_{\alpha\beta}$'s and $\triangle_{\alpha\beta}$'s generate a Lie algebra isomorphic to $\mathrm{su}(p,q)$ with commutation relations given by Eq. (6.4) in [TT3]. The global action of this infinitesimal action defines a representation $R'_{H'}$ of $H'=\mathrm{SU}(p,q)$ on $\mathcal{F}_{n\times k}$ which is dual to the representation R_H.

An element p of $\mathcal{P}_{n\times k}$ is called *H-harmonic* if $\triangle_{\alpha\beta}p=0$ for all $\alpha=1,\ldots,p$, and $\beta=1,\ldots,q$. Let $\mathcal{H}_{n\times k}$ denote the subspace of all H-harmonic polynomial functions of $\mathcal{P}_{n\times k}$ and let $\mathcal{H}_{n\times k}(\mu)$ denote the subspace of $\mathcal{H}_{n\times k}$ generated by the elements $f\in\mathcal{P}_{p\times k}^{(\nu)}\otimes\mathcal{P}_{q\times k}^{(\lambda\ )}$

which also satisfy the condition $\Delta_{\alpha\beta} = 0$, $1 \le \alpha \le p$, $1 \le \beta \le q$. Let $R_H^{(\mu)}$, $\mu = (\nu) \otimes (\lambda^{\checkmark})$, denote the representation of H on $\mathcal{H}_{n\times k}(\mu)$ defined by

$$\left[R_H^{(\mu)}(g) f\right]\left(\begin{bmatrix} Z \\ \cdots \\ \tilde{W} \end{bmatrix}\right) = f\left(\begin{bmatrix} Zg \\ \cdots\cdots \\ \tilde{W} s_k g^{\checkmark} s_k \end{bmatrix}\right), \tag{2.52}$$

for all $g \in H$. Then Theorem 5.2 of [TT3] implies that:

The representation $R_H^{(\mu)}$ of $H \approx \mathrm{U}(k)$ on $\mathcal{H}_{n\times k}(\mu)$ is an irreducible unitary representation of class (μ) which has signature

$$(\mu) = (\underbrace{\nu_1, \ldots, \nu_p, 0, \ldots, 0, -\lambda_q, \ldots, -\lambda_1}_{k}), \tag{2.53}$$

where in Eq. (2.53) ν_α, $1 \le \alpha \le p$, and λ_β, $1 \le \beta \le q$, are integers such that $\nu_1 \ge \cdots \ge \nu_p \ge 0$ and $\lambda_1 \ge \cdots \ge \lambda_q \ge 0$. Let $f_\mu\left(\begin{bmatrix} Z \\ \cdots \\ \tilde{W} \end{bmatrix}\right) = f_\nu(Z) f_\lambda\left(\tilde{W}\right)$, where f_ν is given by Eq. (2.30) with ν replacing λ and f_λ is given by Eq. (2.44). Let $\mathcal{I}_{n\times k}^{(\mu)}$ be the $H' \times H$-cyclic module generated by f_μ; then a proof similar to the previous cases shows that $\mathcal{H}'_{n\times k}(\mu) \hat{\otimes} \mathcal{H}_{n\times k}(\mu)$ is isomorphic to $\mathcal{I}_{n\times k}^{(\mu)}$. By the "separation of variables theorem" 1.5 of [TT6] and Theorem 5.1 of [TT3] it follows that the orthogonal direct sum decomposition $\mathcal{F}_{n\times k} = \sum_{(\mu)} \oplus \mathcal{I}_{n\times k}^{(\mu)}$ holds. Therefore the reciprocity theorem 2.2 also holds for these pairs (G', G) and (H', H).

4) This example is a generalization of the previous example. Consider r copies of one of the following groups: $\mathrm{U}(k)$, $\mathrm{SO}(k)$, or $\mathrm{Sp}(k)$, with k even for the last, and let each of them act on a Bargmann–Segal–Fock space $\mathcal{F}_{p_i\times k}$, $1 \le i \le r$, by right translations. Let $p_1 + p_2 + \cdots + p_r = n$, and let G denote the direct product of r copies of each type of group. In the case of $\mathrm{U}(k)$ we allow the r^{th} copy to act on $\mathcal{F}_{p_2\times k}$ either directly or contragrediently; for the other cases it is not necessary to consider the contragredient representations since they are identical to the direct representations.

On each $\mathcal{F}_{p_i\times k}$ for the $\mathrm{U}(k)$ action we have the dual action of $\mathrm{U}(p_i)$ by left translations, and with possibly the dual (left) contragredient representation in the case $i = r$. For $\mathrm{SO}(k)$ we have the metaplectic representation of $\widetilde{\mathrm{Sp}_{2p_i}}(\mathbb{R})$, and for $\mathrm{Sp}(k)$ we have the corresponding representation of $\mathrm{SO}^*(2p_i)$. Let G' denote the dual group of G thus obtained. Let H denote the diagonal subgroup of G; then in the case of $\mathrm{U}(k)$ an element of H is of the form $(\underbrace{u, u, \ldots, u}_{r})$ or $(\underbrace{u, \ldots, u}_{r-1}, \bar{u})$, $u \in \mathrm{U}(k)$, and in other cases an element of H is of the form $(\underbrace{u, u, \ldots, u}_{r})$, $u \in \mathrm{SO}(k)$ or $u \in \mathrm{Sp}(k)$. Let H' denote the dual group of H thus obtained. Then H' is isomorphic in each case to $\mathrm{U}(n)$, $\widetilde{\mathrm{Sp}_{2n}}(\mathbb{R})$, or $\mathrm{SO}^*(2n)$. As in previous examples it is straightforward to verify that the reciprocity theorem 2.2 holds for these pairs (G', G) and (H', H).

3 Reciprocity Theorems for Finite-Infinite Dimensional Dual Pairs of Groups

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space with a fixed basis

$$\{e_1, e_2, \ldots, e_k, \ldots\}.$$

Let $\mathrm{GL}_k(\mathbb{C})$ denote the group of all invertible bounded linear operators on $\mathcal{H}$ which leave the vectors e_n, $n > k$, fixed. We define $\mathrm{GL}_\infty(\mathbb{C})$ as the *inductive limit* of the ascending chain of subgroups

$$\mathrm{GL}_1(\mathbb{C}) \subset \cdots \subset \mathrm{GL}_k(\mathbb{C}) \subset \cdots.$$

Thus

$$\mathrm{GL}_\infty(\mathbb{C}) = \{A = (a_{ij}),\ i, j \in \mathbb{N} \mid A \text{ is invertible} \\ \text{and all but a finite number of } a_{ij} - \delta_{ij} \text{ are } 0\}.$$

If for each k we have a Lie subgroup G_k of $\mathrm{GL}_k(\mathbb{C})$ such that G_k is naturally embedded in G_{k+1}, $k = 1, \ldots, n, \ldots$, then we can define the *inductive limit* $G_\infty = \varinjlim G_k = \bigcup_{k=1}^{\infty} G_k$. For example, $\mathrm{U}(\infty) = \{u \in \mathrm{GL}_\infty(\mathbb{C}) : u^* = u^{-1}\}$, and thus $\mathrm{U}(\infty)$ is the inductive limit of the groups U_k of all unitary operators of $\mathcal{H}$ which leave the vectors e_n, $n > k$, fixed.

Following Ol'shanskii we call a unitary representation of G_∞ *tame* if it is continuous in the group topology in which the ascending chain of subgroups of type $\left\{\begin{pmatrix} 1_k & 0 \\ 0 & * \end{pmatrix}\right\}$, $k = 1, 2, 3, \ldots$, constitutes a fundamental system of neighborhoods of the identity 1_∞. Assume that for each k a continuous unitary representation $(R_k, \mathcal{H}_k)$ is given and an isomorphic embedding $i_{k+1}^k \colon \mathcal{H}_k \to \mathcal{H}_{k+1}$ commuting with the action of G_k (i.e., $i_{k+1}^k \circ R_k(g) = R_{k+1}(g) \circ i_{k+1}^k$) is given. For $j \le k$ define the *connecting map* $\varphi_{jk} \colon G_j \times \mathcal{H}_j \to G_k \times \mathcal{H}_k$ by

$$\varphi_{jk}(g_j, x_j) = (g_k, x_k), \qquad (g_j, x_j) \in G_j \times \mathcal{H}_j, \tag{3.1}$$

where in Eq. (3.1) g_k (resp. x_k) denotes the natural embedding of g_j (resp. x_j) in G_k (resp. $\mathcal{H}_k$). Then obviously the diagram

$$\begin{array}{ccc} G_j \times \mathcal{H}_j & \xrightarrow{R_j} & \mathcal{H}_j \\ \varphi_{jk}\big\downarrow & & \big\downarrow i_k^j = i_k^{k-1} \circ \cdots \circ i_{j+2}^{j+1} \circ i_{j+1}^j \\ G_k \times \mathcal{H}_k & \xrightarrow{R_k} & \mathcal{H}_k \end{array} \tag{3.2}$$

is commutative. Let $\mathcal{H}_\infty$ denote the Hilbert-space completion of $\bigcup_{k=1}^{\infty} \mathcal{H}_k$ and define a representation R_∞ of G_∞ on $\mathcal{H}_k$ by

$$R_\infty(g)\, x = R_k(g)\, x \qquad \text{if } g \in G_k \text{ and } x \in \mathcal{H}_k. \tag{3.3}$$

Then obviously R_∞ is a unique continuous unitary representation of G_∞ on $\bigcup_{k=1}^{\infty} \mathcal{H}_k$ which can be extended to a unique continuous unitary representation of G_∞ on $\mathcal{H}_\infty$. Let φ_k denote the canonical map of $(G_k, \mathcal{H}_k)$ into $(G_\infty, \mathcal{H}_\infty)$ and i_k denote the canonical map of $\mathcal{H}_k$ into $\mathcal{H}_\infty$; then obviously the diagram

$$\begin{array}{ccc} G_k \times \mathcal{H}_k & \xrightarrow{R_k} & \mathcal{H}_k \\ \varphi_k \big\downarrow & & \big\downarrow i_k \\ G_\infty \times \mathcal{H}_\infty & \xrightarrow{R_\infty} & \mathcal{H}_\infty \end{array} \tag{3.4}$$

is commutative.

The following theorem, which is well-known when i_{k+1}^k is an isometric embedding (see, e.g., [Ol2]), is crucial for what follows.

Theorem 3.1. *If the representations $(R_k, \mathcal{H}_k)$ are all irreducible then the inductive limit representation $(R_\infty, \mathcal{H}_\infty)$ is also irreducible.*

Proof. Let A be a bounded operator on $\mathcal{H}_\infty$ which belongs to the commutant of the algebra of operators generated by the set $\{R_\infty(g), g \in G_\infty\}$. Since $\bigcup_{k=1}^{\infty} \mathcal{H}_k$ is dense in $\mathcal{H}_\infty$ and all the linear operators involved are continuous we can without loss of generality consider them as operating on $\bigcup_{k=1}^{\infty} \mathcal{H}_k$ and satisfying $A\left(i_l^k(x)\right) = Ax$ for $k \le l$ and for all $x \in \mathcal{H}_k$. Let P_k denote the projection of $\bigcup_{k=1}^{\infty} \mathcal{H}_k$ onto $\mathcal{H}_k$. Let A_k denote the restriction of A to $\mathcal{H}_k$; then A_k is a bounded linear operator of $\mathcal{H}_k$ into $\bigcup_{n=1}^{\infty} \mathcal{H}_n$. It follows immediately that $P_k A_k \colon \mathcal{H}_k \to \mathcal{H}_k$ is a bounded linear operator on $\mathcal{H}_k$. Let $x \in \mathcal{H}_k$ and suppose $A_k x = Ax$ belongs to $\mathcal{H}_l$. If $l \le k$ we may use the isomorphic embedding $i_k^l = i_k^{k-1} \circ \cdots \circ i_{l+1}^l \colon \mathcal{H}_l \to \mathcal{H}_k$ to identify Ax with an element of $\mathcal{H}_k$ so that $P_k A_k x = A_k x = Ax$, and thus

$$R_k(g_k) P_k A_k x = R_\infty(g_k) Ax = AR_\infty(g_k) x = P_k R_k(g_k) x, \qquad \forall\, g_k \in G_k.$$

If $l > k$ then use i_l^k to identify $\mathcal{H}_k$ with a subspace of $\mathcal{H}_l$. Write $Ax = y + z$ where y belongs to the identified subspace of $\mathcal{H}_k$ and z belongs to its orthogonal complement in $\mathcal{H}_l$. Since all representations are unitary and for $g_k \in G_k$ we have $i_l^k \circ R_k(g_k) = R_l(g_k) \circ i_l^k$ it follows that

$$P_k R_\infty(g_k) A_k x = P_k R_k(g_k) y = R_\infty(g_k) P_k A x.$$

By assumption $R_\infty(g_k) Ax = AR_\infty(g_k) x$, therefore

$$\begin{aligned} R_k(g_k) P_k A_k x &= R_\infty(g_k) P_k A x \\ &= P_k R_\infty(g_k) A_k x = P_k A_k R_\infty(g_k) x = P_k A_k R_k(g_k) x. \end{aligned}$$

Since this relation holds for all $x \in \mathcal{H}_k$ and $g_k \in G_k$ it follows that $P_k A_k$ belongs to the commutant of the algebra of operators on $\mathcal{H}_k$ generated by the set $\{R_k(g_k), g_k \in G_k\}$. Schur's lemma for operator algebras (see, e.g., [Di, Proposition 2.3.1, p. 39]) implies that $P_k A_k = \lambda_k I_k$, where λ_k is a scalar depending on k and I_k is the identity operator on $\mathcal{H}_k$. Now A is a map of inductive limit sets such that $P_k A_k \colon \mathcal{H}_k \to \mathcal{H}_k$, and it follows from the definition of an inductive limit map that $\lambda_k = \lambda_l$ for sufficiently large k, l with $k < l$. Indeed, if $x \in \mathcal{H}_k$ and $A_k x = Ax \in \mathcal{H}_j$ with $j \leq k$ then $P_k A_k x = i_k^j (Ax) = \lambda_k x$. For $l > k$ we then have

$$\begin{aligned}\lambda_l i_l^k (x) = P_l A_l \left(i_l^k (x)\right) = P_l A \left(i_l^k (x)\right) = P_l Ax \\ = i_l^j (Ax) = i_l^k \left(i_k^j (Ax)\right) = i_l^k \left(P_k A_k (x)\right) = \lambda_k i_l^k (x).\end{aligned}$$

On the other hand, if $Ax \in \mathcal{H}_j$ with $j > k$ then for all $l \geq j$ we have

$$\begin{aligned}P_l A_l \left(i_l^k (x)\right) = P_l A \left(i_l^k (x)\right) = P_l Ax = P_l A_j \left(i_j^k (x)\right) \\ = P_l P_j A_j \left(i_j^k (x)\right) = P_l \left(\lambda_j i_j^k (x)\right) = i_l^j \left(\lambda_j i_j^k (x)\right) = \lambda_j i_l^j \left(i_j^k (x)\right) = \lambda_j i_l^k (x).\end{aligned}$$

Since $P_l A_l \left(i_l^k (x)\right) = \lambda_l i_l^k (x)$, we must have $\lambda_j = \lambda_l$ for all $l \geq j$. This implies that $A = \lambda I_\infty$ where $\lambda \in \mathbb{C}$ is a constant and I_∞ is the identity on $\mathcal{H}_\infty$. By the same Schur's lemma quoted above the representation R_∞ on $\mathcal{H}_\infty$ must be irreducible. $\square$

Now fix n and consider the chain of Hilbert spaces $\mathcal{F}_{n\times k}$ from Section 2 with $k > 2n$. Let (G'_n, G_k) denote a dual pair of groups with dual representations (R'_n, R_k) acting on $\mathcal{F}_{n\times k}$ as in Theorem 2.2. Then we have the chain of embedded subgroups $G_k \subset G_{k+1} \subset \cdots$; for example, $\mathrm{U}(k)$ is naturally embedded in $\mathrm{U}(k+1)$ via the embedding $u \to \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $u \in \mathrm{U}(k)$. Therefore we can define the inductive limit $G_\infty = \varinjlim G_k = \bigcup_{k>2n}^{\infty} G_k$. We also have an isometric embedding $i_{k+1}^k \colon \mathcal{F}_{n\times k} \to \mathcal{F}_{n\times(k+1)}$ such that

$$i_{k+1}^k \circ R_k (g) = R_{k+1} (g) \circ i_{k+1}^k.$$

To see this we take the case $n = 1$: then an element f of $\mathcal{F}_{n\times k}$ is a function of $Z = (Z_1, \ldots, Z_k)$ of the form given by Eq. (2.7), and the verification of the equation above is straightforward. Let $\mathcal{F}_{n\times\infty}$ denote the Hilbert-space completion on $\bigcup_{k>2n}^{\infty} \mathcal{F}_{n\times k}$. Then it is clear that the inductive limit representation R_∞ of G_∞ on $\mathcal{F}_{n\times\infty}$ is tame and satisfies the relations (3.2), (3.3), and (3.4).

If G_k is a compact group then every irreducible unitary representation of G_k is of the form $(\rho_{\lambda_k}, V_{\lambda_k})$ with highest weight $(\lambda_k) = (m_1, m_2, \ldots, m_i, \ldots)$, where $m_1, m_2, \ldots$ are nonnegative integers satisfying $m_1 \geq m_2 \geq \cdots$ and the numbers m_i are equal to 0 for sufficiently large i. Consider the decomposition (2.5) of Definition 2.1 of the dual module $\mathcal{F}_{n\times k}$ into isotypic components

$$\mathcal{F}_{n\times k} = \sum_{(\lambda_k)} \oplus \mathcal{I}_{n\times k}^{(\lambda_k)}$$

where the signatures (λ_k) actually depend essentially on n, but since n is fixed, to alleviate the notation we just tacitly assume this dependence. Also for k sufficiently large if $(\lambda_k) = (m_1, \ldots, m_i, \ldots)$ then $(\lambda_{k+1}) = (m_1, \ldots, m_i, \ldots, \ldots)$ and we write succinctly $(\lambda_k) \subset (\lambda_{k+1})$.

For sufficiently large k we can exhibit an isomorphic embedding $i_{k+1}^k \colon \mathcal{I}_{n\times k}^{(\lambda_k)} \to \mathcal{I}_{n\times(k+1)}^{(\lambda_{k+1})}$. If H_k is a subgroup of G_k such that H'_n contains G'_n and (H'_n, H_k) forms a dual pair then the same process can be repeated for the chain $(H_n', H_k) \subset (H_n', H_{k+1}) \subset \cdots$. If G_k (or H_k) is of the type $\underbrace{\mathrm{U}(k) \times \cdots \times \mathrm{U}(k)}_{r}$ then each i_{k+1}^k is an isometric embedding; for other types of G_k (or H_k) the definition of i_{k+1}^k is more subtle. This can be examined case by case although the process is very tedious. To illustrate this we consider the case $\mathcal{F}_{1\times k}$ with $H_k = \mathrm{SO}(k)$ and $G_1 = \widetilde{\mathrm{Sp}_2}(\mathbb{R}) = \widetilde{\mathrm{SL}_2}(\mathbb{R})$. Then Eq. (2.22) and Eq. (2.23) imply that

$$\mathcal{F}_{1\times k} = \sum_{r=0}^{\infty} \oplus \mathcal{I}_{1\times k}^{(r)_k} \qquad \text{with} \qquad \mathcal{I}_{1\times k}^{(r)_k} = \sum_{j=0}^{\infty} \oplus\, p_{0,k}^j \mathcal{H}_{1\times k}^{(r)_k}$$

where $p_{0,k}(Z) = Z_1^2 + \cdots + Z_k^2$, $(r)_k = (\underbrace{r, 0, \ldots, 0}_{k})$, and $\mathcal{H}_{1\times k}^{(r)}$ are the subspace of all harmonic homogeneous polynomials of degree r. Obviously a harmonic homogeneous polynomial h of degree r in k variables can be considered as a harmonic homogeneous polynomial of r in $k+1$ variables. So we can define an isomorphic embedding $i_{k+1}^k : \mathcal{I}_{1\times k}^{(r)_k} \to \mathcal{I}_{1\times(k+1)}^{(r)_{k+1}}$ by sending $p_{0,k}^j h$ into $p_{0,(k+1)}^j h$, and clearly

$$\begin{aligned} R_H(u_k)\left(p_{0,(k+1)}^j h\right) &= p_{0,(k+1)}^j R_H(u_k)\, h = i_{k+1}^k p_{0,k}^j R_H(u_k)\, h \\ &= i_{k+1}^k \left(\left(R_H(u_k)\, p_{0,k}^j\right)\left(R_H(u_k)\, h\right)\right) = i_{k+1}^k \left(R_H(u_k)\left(p_{0,k}^j h\right)\right) \end{aligned}$$

for all $u_k \in H_k$. Thus, $R_H(u_k) \circ i_{k+1}^k = i_{k+1}^k \circ R_H(u_k)$ for all $u_k \in H_k$. It follows that i_{k+1}^k can be extended to the whole space $\mathcal{F}_{1\times k}$ and that $i_{k+1}^k(\mathcal{F}_{1\times k}) = \sum_{r=0}^{\infty} \oplus\, i_{k+1}^k \left(\mathcal{I}_{1\times k}^{(r)_k}\right)$ is an isomorphic embedding of $\mathcal{F}_{1\times k}$ into $\mathcal{F}_{1\times(k+1)}$. Also note in this very special case $(r)_k \subset (r)_{k+1}$ for all $k > 2$ and that no other signatures $(r)_{k+1}$ occur in $\mathcal{F}_{1\times(k+1)}$ without $(r)_k$ occurring in $\mathcal{F}_{1\times k}$; this fact is an exception and almost never happens in the general case (e.g., $n \geq 2$). By Theorem 3.1 the tensor product representations $R_{G_n'}^{(\lambda')} \otimes R_{G_\infty}^{(\lambda)}$ and $R_{H_n'}^{(\mu')} \otimes R_{H_\infty}^{(\mu)}$ of $G_n' \times G_\infty$ and $H_n' \times H_\infty$ on $\mathcal{I}_{n\times\infty}^{(\lambda)}$ and $\mathcal{I}_{n\times\infty}^{(\mu)}$, respectively, are irreducible with signature $(\lambda)_\infty$ and $(\mu)_\infty$, respectively, where if $(\lambda_k) = (m_1, m_2, \ldots, m_i, \ldots)$ then $(\lambda)_\infty = (\underbrace{m_1, m_2, \ldots, m_i, \ldots, 0}_{\infty}, \ldots, 0)$ and similarly for $(\mu)_\infty$. Note that as n is fixed, the group G_n' remains fixed; however, its representation $R_{G_n'}'$ on $\mathcal{F}_{n\times k}$ does depend on k, and should be written as $\left(R_{G_n'}'\right)_k$, and as $k \to \infty$, $\left(R_{G_n'}'\right)_\infty$ has to be considered as an inductive limit of representations, although for k sufficiently large all the representations $\left(R_{G_n'}^{(\lambda')}\right)_k$ are equivalent. The same observations apply to $\left(R_{H_n'}'\right)_k$ and $\left(R_{H_n'}^{(\mu')}\right)_k$. To illustrate this let us consider again the case $\mathrm{U}(1) \times \mathrm{U}(k)$ and $\widetilde{\mathrm{SL}_2}(\mathbb{R}) \times \mathrm{SO}(k)$ acting on $\mathcal{F}_{1\times k}$. Indeed, the infinitesimal action of $R_{G_{\mathbb{C}}'}'$ is given by Eq. (2.11) as $L_k = \sum_{i=1}^k Z_i \partial/\partial Z_i$ and $L_{k+1} = \sum_{i=1}^{k+1} Z_i \partial/\partial Z_i$, and for $p \in \mathcal{P}_{1\times k}^{(m)} \subset \mathcal{P}_{1\times(k+1)}^{(m)}$ Eq. (2.16) implies that

$$L_k p = L_{k+1} p = mp.$$

By Eq. (2.18) the infinitesimal actions of $R'_{H'_1}$ on $\mathcal{F}_{1\times k}$ and $\mathcal{F}_{1\times(k+1)}$ are given, respectively, by

$$\begin{cases} E_k = \dfrac{k}{2} + L_k, & X_k^+ = \dfrac{1}{2}\displaystyle\sum_{i=1}^{k} Z_i^2, \quad X_k^- = \dfrac{1}{2}\displaystyle\sum_{i=1}^{k}\frac{\partial^2}{\partial Z_i^2}, \quad \text{and} \\ E_{k+1} = \dfrac{k+1}{2} + L_{k+1}, & X_{k+1}^+ = \dfrac{1}{2}\displaystyle\sum_{i=1}^{k+1} Z_i^2, \quad X_{k+1}^- = \dfrac{1}{2}\displaystyle\sum_{i=1}^{k+1}\frac{\partial^2}{\partial Z_i^2}. \end{cases} \tag{3.5}$$

If $h_k \in \mathcal{H}_{1\times k}^{(r)}$ then Eqs. (2.24), (2.25) applied to $\{E_k, X_k^+, X_k^-\}$ show that $J_k h_k$ is an irreducible representation of $\mathrm{sl}_2(\mathbb{R})$ with signature (r). Similarly if $h_{k+1} \in \mathcal{H}_{1\times(k+1)}^{(r)}$ then $J_{k+1}h_{k+1}$ is also an irreducible representation of $\mathrm{sl}_2(\mathbb{R})$ with signature (r).

Let $\mathcal{F}_{n\times\infty}$ denote the Hilbert-space completion of $\bigcup_k \mathcal{F}_{n\times k}$; then $\mathcal{F}_{n\times\infty} = \varinjlim \mathcal{F}_{n\times k}$ is the inductive limit of the chain $\{\mathcal{F}_{n\times k}\}$.

After this necessary preparatory work we can now state and prove the main theorem of this paper.

Theorem 3.2. *Let G_∞ denote the inductive limit of a chain $G_k \subset G_{k+1} \subset \cdots$ of compact groups. Let R_{G_∞} and $R'_{(G'_n)_\infty}$ be given dual representations on $\mathcal{F}_{n\times\infty}$. Let H_∞ denote the inductive limit of a chain of compact subgroups $H_k \subset H_{k+1} \subset \cdots$ such that $H_k \subset G_k$ for all k. Let R_{H_∞} be the representation of H_∞ on $\mathcal{F}_{n\times\infty}$ obtained by restricting R_{G_∞} to H_∞. If there exists a group $H'_n \supset G'_n$ and a representation $R'_{(H'_n)_\infty}$ on $\mathcal{F}_{n\times\infty}$ such that $R'_{(H'_n)_\infty}$ is dual to R_{H_∞} and $R'_{(G'_n)_\infty}$ is the restriction of $R'_{(H'_n)_\infty}$ to the subgroup G'_n of H'_n then we have the following multiplicity-free decompositions of $\mathcal{F}_{n\times\infty}$ into isotypic components:*

$$\mathcal{F}_{n\times\infty} = \sum_{(\lambda)} \oplus \mathcal{I}_{n\times\infty}^{(\lambda)} = \sum_{(\mu)} \oplus \mathcal{I}_{n\times\infty}^{(\mu)} \tag{3.6}$$

where (λ) is a common irreducible signature of the pair (G'_n, G_∞) and (μ) is a common irreducible signature of the pair (H'_n, H_∞).

If λ_{G_∞} (resp. $\lambda'_{(G'_n)_\infty}$) denotes an irreducible unitary representation of class (λ) and μ_{H_∞} (resp. $\mu'_{(H'_n)_\infty}$) denotes an irreducible unitary representation of class (μ) then the multiplicity $\dim\left[\mathrm{Hom}_{H_\infty}\left(\mu_{H_\infty} : \lambda_{G_\infty}|_{H_\infty}\right)\right]$ *of the irreducible representation μ_{H_∞} in the restriction to H_∞ of the representation λ_{G_∞} is equal to the multiplicity* $\dim\left[\mathrm{Hom}_{G'_n}\left(\lambda'_{(G'_n)_\infty} : \mu'_{(H'_n)_\infty}\Big|_{G'_n}\right)\right]$ *of the irreducible representation $\lambda'_{(G'_n)_\infty}$ in the restriction to G'_n of the representation $\mu'_{(H'_n)_\infty}$.*

Proof. As remarked above, the dual (G'_n, G_∞)-module $\mathcal{I}_{n\times\infty}^{(\lambda)}$ is irreducible (by Theorem 3.1) with signature (λ), and isotypic components of different signatures are mutually orthogonal since their projections $\mathcal{I}_{n\times k}^{(\lambda)_k}$ are mutually orthogonal. Finally if a vector in $\mathcal{F}_{n\times\infty}$, which we may assume to belong to $\mathcal{F}_{n\times k}$ for some k, is orthogonal to $\mathcal{I}_{n\times\infty}^{(\lambda)}$ for all (λ), it must therefore be orthogonal to $\mathcal{I}_{n\times k}^{(\lambda)_k}$ for all $(\lambda)_k$, and hence must be the zero vector in $\mathcal{F}_{n\times k}$, and thus zero in $\mathcal{F}_{n\times\infty}$. A similar argument applies to the isotypic components $\mathcal{I}_{n\times\infty}^{(\mu)}$, and thus Eq. (3.6) holds.

Now fix (λ) and (μ). Then the restriction of R_{G_∞} to $\mathcal{I}_{n\times\infty}^{(\lambda)}$ decomposes into a (non-canonical) orthogonal direct sum of equivalent irreducible unitary representations of signature $(\lambda)_\infty$. A representative of this representation may be obtained by applying Theorem 3.1 to get the inductive limit $\left(G_\infty, R_{(\lambda)_\infty}\right)$ of the chain (G_k, R_{λ_k}); for example, when $G_k = \mathrm{U}\,(k)$, the representation R_{λ_k} is given by Eq. (2.29) on $\mathcal{P}_{n\times k}^{(\lambda)_k}$. Considered as a G'_n-module $\mathcal{I}_{n\times\infty}^{(\lambda)}$ decomposes into a (non-canonical) orthogonal direct sum of equivalent irreducible unitary representations of signature $(\lambda')_n$. A representative of this representation may be obtained by applying Theorem 3.1 to get the inductive limit $\left(G'_n, R'_{(\lambda'_n)_\infty}\right)$ (note that although G'_n is a stationary chain at n, the representations $R'_{(\lambda'_n)_k}$ depend on k even though they are all equivalent and belong to the class $(\lambda')_n$); for example, when $G_n = \mathrm{U}\,(n)$ the representation $R'_{\lambda'_n}$ is given by Eq. (2.32) on $\mathcal{P}_{n\times k}^{(\lambda')_n}$ which is defined by Eq. (2.31). By an analogous argument we infer that the same conclusions hold for (μ), $\mathcal{I}_{n\times\infty}^{(\mu)}$, $\left(H_\infty, R_{(\mu)_\infty}\right)$, $\left(H'_n, R'_{(\mu'_n)_\infty}\right)$.

Now consider the decomposition of the restriction to H_k of the representation R_{λ_k} of G_k. The multiplicity of $(\mu)_k$ in $(\lambda)_k|_{H_k}$ is the dimension of $\mathrm{Hom}_{H_k}\left(R_{\mu_k} : R_{\lambda_k}|_{H_k}\right)$, where $\mathrm{Hom}_{H_k}\left(R_{\mu_k} : R_{\lambda_k}|_{H_k}\right)$ is the vector space of linear homomorphisms intertwining R_{μ_k} and $R_{\lambda_k}|_{H_k}$. Since G_k and H_k are, by assumption, compact, this dimension is finite. If $T_k\colon \mathcal{H}_{\mu_k} \to \mathcal{H}_{\lambda_k}$ is an element of $\mathrm{Hom}_{H_k}\left(R_{\mu_k} : R_{\lambda_k}|_{H_k}\right)$, where $\mathcal{H}_{\mu_k}$ (resp. $\mathcal{H}_{\lambda_k}$) denotes the representation space of R_{μ_k} (resp. R_{λ_k}), then since $\mathcal{H}_{\mu_k} \subset \mathcal{I}_{n\times k}^{(\mu)_k}$ and $\mathcal{H}_{\lambda_k} \subset \mathcal{I}_{n\times k}^{(\lambda)_k}$ it follows that we have an inductive chain of homomorphisms $\{T_k\colon \mathcal{H}_{\mu_k} \to \mathcal{H}_{\lambda_k}\}$. Let $\mathcal{H}_{\mu_\infty}$ (resp. $\mathcal{H}_{\lambda_\infty}$) denote the inductive limit of $\mathcal{H}_{\mu_k}$ (resp. $\mathcal{H}_{\lambda_k}$); then there exists a unique homomorphism $T_\infty\colon \mathcal{H}_{\mu_\infty} \to \mathcal{H}_{\lambda_\infty}$ (see, for example, [Du, Theorem 2.5, p. 430], or [Ro, p. 44]). Again by Theorem 3.1, $R_{\lambda_\infty} = \varinjlim R_{\lambda_k}$ (resp. $R_{\mu_\infty} = \varinjlim R_{\mu_k}$) is irreducible with signature $(\lambda)_\infty$ (resp. $(\mu)_\infty$), and it is easy to show that T_∞ is an intertwining homomorphism. Conversely, all homomorphisms of inductive limits arise that way. Consequently, the chain $\mathrm{Hom}_{H_k}\left(R_{\mu_k} : R_{\lambda_k}|_{H_k}\right)$ induces the inductive limit $\mathrm{Hom}_{H_\infty}\left(R_{\mu_\infty} : R_{\lambda_\infty}|_{H_\infty}\right)$. Obviously for sufficiently large k, $\dim\left[\mathrm{Hom}_{H_k}\left(R_{\mu_k} : R_{\lambda_k}|_{H_k}\right)\right] = \dim\left[\mathrm{Hom}_{H_\infty}\left(R_{\mu_\infty} : R_{\lambda_\infty}|_{H_\infty}\right)\right]$. By duality, we obtain in the same way the inductive limit $\left[\mathrm{Hom}_{G'_n}\left(R'_{(\lambda'_n)_\infty} : R'_{(\mu'_n)_\infty}\Big|_{G'_n}\right)\right]$; actually this chain stabilizes for k sufficiently large. It follows from Theorem 2.2 (see also the proof of Theorem 4.1 in [TT3]) that $\dim\left[\mathrm{Hom}_{H_\infty}\left(\mu_{H_\infty} : \lambda_{G_\infty}|_{H_\infty}\right)\right] = \dim\left[\mathrm{Hom}_{G'_n}\left(\lambda'_{(G'_n)_\infty} : \mu'_{(H'_n)_\infty}\Big|_{G'_n}\right)\right]$. $\square$

As an example we again consider the case $\mathcal{F}_{1\times\infty}$ with $G_\infty = \mathrm{U}\,(\infty)$, $G'_1 = \mathrm{U}\,(1)$, $H_\infty = \mathrm{SO}\,(\infty)$, and $H'_1 = \widetilde{\mathrm{SL}_2\,(\mathbb{R})}$. Then from Eq. (2.17), $(\lambda)_k = (\underbrace{m, 0, \ldots, 0}_{k})$, $\lambda'_1 = (m)$, and $\mathcal{I}_{1\times k}^{(\lambda)_k} = \mathcal{P}_{1\times k}^{(m)}$. It follows that $(\lambda)_\infty = \left(m, 0, 0, \vec{0}\right)$ and $\mathcal{I}_{1\times\infty}^{(\lambda)_\infty} = \mathcal{P}_{1\times\infty}^{(m)_\infty}$, the vector space of all homogeneous polynomials of degree m in infinitely many variables Z_1, Z_2, etc. The infinitesimal action of $R'_{(\mathrm{U}(1))_k}$ is given by Eq. (2.10), $L_k = \sum_{i=1}^{k} Z_i \partial/\partial Z_i$, so the infinitesimal action $L_{(m)_\infty}$ is given the formal series $\sum_{i=1}^{\infty} Z_i \partial/\partial Z_i$. For $H_\infty = \mathrm{SO}(\infty)$ and $H_1' = \widetilde{\mathrm{SL}_2(\mathbb{R})}$ the actions are more delicate to describe. From Eq. (2.22), $(\mu)_k = \underbrace{(r, 0, \ldots, 0)}_{[k/2]}$, where r is an integer ≥ 0, and therefore $(\mu)_\infty = \left(r, 0, 0, \vec{0}\right)$. Let $\mathcal{H}_{1\times k}^{(r)_k}$ denote the space of all harmonic homogeneous polynomials of degree r in k variables $Z_1, \ldots, Z_k$ then from Eq. (2.22) $\mathcal{I}_{1\times k}^{(r)_k} = \sum_{j=0}^{\infty} \oplus\, p_{0,k}^j \mathcal{H}_{1\times k}^{(r)_k}$, where $p_{0,k}(Z) = \sum_{i=1}^{k} Z_i^2$. We define the actions $R_{\mathrm{SO}(\infty)}$ and $R'_{\left(\widetilde{\mathrm{SL}_2(\mathbb{R})}\right)_\infty}$ as follows:

Consider the algebras $(\mathrm{sl}_2(\mathbb{R}))_k$ with the bases $\{E_k, X_k^+, X_k^-\}$ given by Eq. (3.5); define the *projective* or *inverse limit* of the family $\left\{(\mathrm{sl}_2(\mathbb{R}))_k, \mathcal{I}_{1\times k}^{(r)_k}\right\}$ as follows: For each pair of indices l, k with $l \leq k$ a continuous homomorphism ϕ_l^k: $(\mathrm{sl}_2(\mathbb{R}))_k \to (\mathrm{sl}_2(\mathbb{R}))_l$ by sending E_k to E_l, X_k^+ to X_l^+, X_k^- to X^{-l}, and extends by linearity to $(\mathrm{sl}_2(\mathbb{R}))_k \to (\mathrm{sl}_2(\mathbb{R}))_l$. Clearly ϕ_l^k satisfies fhe following:

a) ϕ_k^k is the identity map for all k,

b) if $i \leq l \leq k$ then $\phi_i^k = \phi_i^l \circ \phi_l^k$.

The inverse limit of the system $\{\mathrm{sl}_2(\mathbb{R})_k\}$ is denoted by

$$\mathrm{sl}_2(\mathbb{R})_\infty = \varprojlim \mathrm{sl}_2(\mathbb{R})_k = \langle E_\infty, X_\infty^+, X_\infty^- \rangle,$$

$$\text{where } E_\infty = \frac{1}{2} 1_\infty + L_\infty,\ X_\infty^+ = \frac{1}{2}\sum_{i=1}^{\infty} Z_i^2,\ X_\infty^- = \frac{1}{2}\sum_{i=1}^{\infty} \frac{\partial}{\partial Z_i^2}. \tag{3.7}$$

Then $\{E_\infty, X_\infty^+, X_\infty^-\}$ acts on $\mathcal{F}_{1\times\infty}$ as follows: If $f \in \mathcal{F}_{1\times\infty}$ then we may assume that $f \in \mathcal{F}_{1\times k}$ for some k and

$$E_\infty f = E_k f, \qquad X_\infty^+ f = X_k^+ f \qquad \text{and} \qquad X_\infty^- f = X_k^- f. \tag{3.8}$$

If $\mathcal{H}_{1\times\infty}^{(r)_\infty}$ denotes the subspace (of $\mathcal{P}_{1\times\infty}^{(r)_\infty}$) of all harmonic homogeneous polynomials of infinitely many variables Z_1, Z_2, etc. (i.e., $h \in \mathcal{H}_{1\times\infty}$ if and only if $h \in \mathcal{P}_{1\times\infty}^{(r)_\infty}$ and $X_\infty^- h = 0$) then

$$\mathcal{I}_{1\times\infty}^{(r)_\infty} = \sum_{j=0}^{\infty} \oplus \left(2X_\infty^+\right)^j \mathcal{H}_{1\times\infty}^{(r)_\infty}, \tag{3.9}$$

where in Eq. (3.9) $2X_\infty^+ = (p_0)_\infty = \sum_{i=1}^{\infty} Z_i^2$. Note that $\mathcal{H}_{1\times\infty}^{(r)_\infty}$ corresponds to the inductive limit of the chain $\left\{\mathcal{H}_{1\times k}^{(r)_k}\right\}$. Let $R_{\mathrm{SO}(\infty)}^{(r)_\infty}$ denote the inductive limit representation of the chain $R^{(r)_l}_{\mathrm{SO}(k)}$; then $R^{(r)_\infty}_{\mathrm{SO}(\infty)}$ together with Eq. (3.8) describes completely the action of the dual pair $\left(\widetilde{\mathrm{SL}_2(\mathbb{R})}, \mathrm{SO}(\infty)\right)$ on the isotypic component $\mathcal{I}^{(r)_\infty}_{1\times\infty}$ and thus we have the isotypic decompositions for the dual pairs $(\mathrm{U}(1), \mathrm{U}(\infty))$ and $\left(\widetilde{\mathrm{SL}_2(\mathbb{R})}, \mathrm{SO}(\infty)\right)$,

$$\mathcal{F}_{1\times\infty} = \sum_{m=0}^{\infty} \oplus \mathcal{I}^{(m)_\infty}_{1\times\infty} = \sum_{r=0}^{\infty} \oplus \mathcal{I}^{(r)_\infty}_{1\times\infty},$$

and thus Theorem 3.2 is verified for this example.

Since the next two examples are very important by their applications to Physics we shall state them as corollaries to Theorem 3.2.

Corollary 3.3. *Let G_∞ denote the direct product of r copies of H_∞ where $H_\infty = \mathrm{U}(\infty)$, $\mathrm{SO}(\infty)$, or $\mathrm{Sp}(\infty)$. If G_∞ acts as the exterior tensor product representation $V^{(\lambda_1)_\infty} \otimes \cdots \otimes V^{(\lambda_r)_\infty}$, where each $V^{(\lambda_i)_\infty}$, $1 \le i \le r$, is an irreducible unitary H_∞-module, then H_∞ acts as the inner (or Kronecker) tensor product representation on $V^{(\lambda_1)_\infty} \hat{\otimes} \cdots \hat{\otimes} V^{(\lambda_r)_\infty}$. If λ_{G_∞} denotes an irreducible unitary representation of class $(\lambda_1)_{G_\infty} \otimes \cdots \otimes (\lambda_r)_{G_\infty}$ and μ_{H_∞} denotes an irreducible unitary representation of class $(\mu)_{H_\infty}$ then the multiplicity $\dim\left[\mathrm{Hom}_{H_\infty}\left(\mu_{H_\infty} : \lambda_{G_\infty}|_{H_\infty}\right)\right]$ of the representation $(\mu)_{H_\infty}$ in the inner tensor product $(\lambda_1)_\infty \hat{\otimes} \cdots \hat{\otimes} (\lambda_r)_\infty$ is equal to the multiplicity of $(\mu)_{H_k}$ in the inner tensor product $(\lambda_1)_k \hat{\otimes} \cdots \hat{\otimes} (\lambda_r)_k$ for sufficiently large k.*

Proof. If $(\lambda_i)_\infty = \left(\lambda_i^1, \lambda_i^2, \ldots, \lambda_i^j, \ldots\right)$ where λ_i^j are integers such that $\lambda_i^1 \ge \lambda_i^2 \ge \cdots$ and $\lambda_i^j = 0$ for all but a finite number of j, let n denote the total number of all nonzero entries λ_i^j, $1 \le i \le r$; then $V^{(\lambda_1)_\infty} \otimes \cdots \otimes V^{(\lambda_r)_\infty}$ can be realized as a subspace of the Bargmann–Segal–Fock space $\mathcal{F}_{n\times\infty}$. From Theorem 3.2 it follows that $V^{(\lambda_1)_\infty} \otimes \cdots \otimes V^{(\lambda_r)_\infty}$ belongs to the isotypic component $\mathcal{I}^{(\lambda)_{G_\infty}}_{n\times\infty}$ of $\mathcal{F}_{n\times\infty}$, thus $V^{(\lambda_1)_\infty} \otimes \cdots \otimes V^{(\lambda_r)_\infty}$ is the inductive limit of the chain $\left\{V^{(\lambda_1)_k} \otimes \cdots \otimes V^{(\lambda_r)_k}\right\}$. If μ_{H_∞} is an irreducible unitary representation of class $(\mu)_{H_\infty}$ then by Theorem 3.2

$$\dim\left[\mathrm{Hom}_{H_\infty}\left(\mu_{H_\infty} : \lambda_{G_\infty}|_{H_\infty}\right)\right] = \dim\left[\mathrm{Hom}_{G'_n}\left(\lambda'_{(G'_n)_\infty} : \mu'_{(H'_n)_\infty}\Big|_{G'_n}\right)\right],$$

where $\lambda'_{(G'_n)_\infty}$ (resp. $\mu'_{(H'_n)_\infty}$) is the representation of G'_n (resp. H_n) dual to λ_{G_∞}(resp. μ_{H_∞}). For sufficiently large k every μ_{H_∞} is the inductive limit of a chain μ_{H_k} and for such a k Theorem 2.2 implies that

$$\begin{aligned}\dim\left[\mathrm{Hom}_{H_k}\left(\mu_{H_k} : \lambda_{G_k}|_{H_k}\right)\right] &= \dim\left[\mathrm{Hom}_{G'_n}\left(\lambda'_{(G'_n)_k} : \mu'_{(H'_n)_k}\Big|_{G'_n}\right)\right] \\ &= \dim\left[\mathrm{Hom}_{G'_n}\left(\lambda'_{(G'_n)_\infty} : \mu'_{(H'_n)_\infty}\Big|_{G'_n}\right)\right],\end{aligned}$$

and this achieves the proof of Corollary 3.3. □

Remark. The reason that this corollary only holds for sufficiently large k can be seen in the following example. Let $G_k = \underbrace{\mathrm{U}(k) \times \cdots \times \mathrm{U}(k)}_{4 \text{ times}}$ and $H_k = \mathrm{U}(k)$ and consider the tensor product $\underbrace{(1,0,\ldots,0)}_{k} \otimes \underbrace{(2,0,\ldots,0)}_{k} \otimes \underbrace{(2,0,\ldots,0)}_{k} \otimes \underbrace{(3,0,\ldots,0)}_{k}$; then for $k = 2$ we have the spectral decomposition

$$(1,0) \otimes (2,0) \otimes (2,0) \otimes (3,0) = (8,0) + 3\,(7,1) + 5\,(6,2) + 5\,(5,3) + 2\,(4,4)\,,$$

for $k = 3$ we have

$$\begin{aligned}(1,0,0) \otimes (2,0,0) \otimes (2,0,0) \otimes (3,0,0) \\ = (8,0,0) + 3\,(7,1,0) + 5\,(6,2,0) + 5\,(5,3,0) + 2\,(4,4,0) \\ + 3\,(6,1,1) + 6\,(5,2,1) + 5\,(4,3,1) + 3\,(4,2,2) + 2\,(3,3,2)\,,\end{aligned}$$

for $k \geq 4$ we have

$$\begin{aligned}\underbrace{(1,0,\ldots,0)}_{k} \otimes \underbrace{(2,0,\ldots,0)}_{k} \otimes \underbrace{(2,0,\ldots,0)}_{k} \otimes \underbrace{(3,0,\ldots,0)}_{k} \\ = (8,0,\ldots,0) + 3\,(7,1,0,\ldots,0) + 5\,(6,2,0,\ldots,0) + 5\,(5,3,0,\ldots,0) \\ + 2\,(4,4,0,\ldots,0) + 3\,(6,1,1,0,\ldots,0) + 6\,(5,2,1,0,\ldots,0) + 5\,(4,3,1,0,\ldots,0) \\ + 3\,(4,2,2,0,\ldots,0) + 2\,(3,3,2,0,\ldots,0) + (5,1,1,1,0,\ldots,0) \\ + 2\,(4,2,1,1,0,\ldots,0) + (3,3,1,1,0,\ldots,0) + (3,2,2,1,0,\ldots,0)\,.\end{aligned}$$

Thus we can see that the spectral decomposition of $\left(1,\vec{0}\right)_\infty \otimes \left(2,\vec{0}\right)_\infty \otimes \left(2,\vec{0}\right)_\infty \otimes \left(3,\vec{0}\right)_\infty$ is the same as that of order k for $k \geq 4$, with infinitely many zeroes at the end of each signature.

Note also that this corollary applied to the tensor product $\underbrace{\left(1,\vec{0}\right)_\infty \otimes \cdots \otimes \left(1,\vec{0}\right)_\infty}_{r \text{ times}}$ together with the Schur–Weyl Duality Theorem for $\mathrm{U}(r)$ implies the generalized Schur–Weyl Duality Theorem proved by Kirillov for $\mathrm{U}(\infty)$ in [Ki].

Corollary 3.4. *Let $V^{(\lambda_1)_\infty}, \ldots, V^{(\lambda_r)_\infty}$ and $V^{(\mu)_\infty}$ be irreducible unitary representation of H_∞. Let $V^{(\mu^{\checkmark})_\infty}$ be the representation (of H_∞) contragredient to $V^{(\mu)_\infty}$. Let I^∞ denote the equivalence class of the identity representation of H_∞. Then the multiplicity of $(\mu)_\infty$ in the tensor product $(\lambda_1)_\infty \hat{\otimes} \cdots \hat{\otimes} (\lambda_r)_\infty$ is equal to the multiplicity of I^∞ in the tensor product $(\lambda_1)_\infty \hat{\otimes} \cdots \hat{\otimes} (\lambda_r)_\infty \hat{\otimes} (\mu^{\checkmark})_\infty$.*

Proof. To prove this corollary we apply Corollary 3.3 to $G_\infty = \underbrace{H_\infty \times \cdots \times H_\infty}_{r}$ and $G_k = \underbrace{H_k \times \cdots \times H_k}_{r}$, then apply Theorem 3.2 to $G_\infty = \underbrace{H_\infty \times \cdots \times H_\infty}_{r} \times H_\infty^{\checkmark}$ and $G_k = \underbrace{H_k \times \cdots \times H_k}_{r} \otimes H_k^{\checkmark}$, and finally apply Theorem 2.1 of [KT3] to obtain the desired result at order k. The main difficulty resides with the definition of the identity representation on $V^{(\lambda_1)_\infty} \hat{\otimes} \cdots \hat{\otimes} V^{(\lambda_r)_\infty} \hat{\otimes} V^{(\mu\ )_\infty}$, which we will construct below.

For each k let I^k denote the identity representation of H_k on $V^{(\lambda_1)_k} \hat{\otimes} \cdots \hat{\otimes} V^{(\lambda_r)_k} \hat{\otimes} V^{(\mu\ )_k}$ This means that if I^k occurs with multiplicity d in $V^{(\lambda_1)_k} \hat{\otimes} \cdots \hat{\otimes} V^{(\lambda_r)_k} \hat{\otimes} V^{(\mu\ )_k}$ then there exist d nonzero vectors $f_{i,k}$, $i = 1, \ldots, d$, such that $R_{H_k}(u) f_{i,k} = f_{i,k}$ for all $u \in H_k$. By construction each $f_{i,k}$ is a polynomial function in $\mathcal{F}_{n\times k}$ for some n. Thus $f_{i,k}$ is an H_k-invariant polynomial in $\mathcal{F}_{n\times k}$. If $J_{i,k}$ denotes the one-dimensional subspace spanned by $f_{i,k}$, then for sufficiently large k and for each fixed $i = 1, \ldots, d$ we have a chain of irreducible unitary representations $\left\{H_k, I^k, J_{i,k}\right\}_k$. We can define the isomorphism $\psi^k_{k+1} \colon J_{i,k} \to J_{i,k+1}$ by $\psi^k_{k+1}(cf_{i,k}) = cf_{i,k+1}$, $c \in \mathbb{C}$; then obviously

$$\psi^k_{k+1}\left(R_{H_k}(u) f_{i,k}\right) = R_{H_{k+1}}(u) f_{i,k+1} = R_{H_{k+1}}(u)\, \psi^k_{k+1}(f_{i,k}),$$

for all $u \in H_k$. Also for all k, l, m with $k \le l \le m$ we have $\psi^k_m = \psi^l_m \circ \psi^k_l$. Thus we can define the inductive limit representation $\{H_\infty, I^\infty, J_{i,\infty}\}$, where the action of H_∞ on $J_{i,\infty}$ is defined as follows:

Let $u \in H_\infty$; then $u \in H_k$ for some k. If $f \in J_{i,l}$ for some l then

$$R_{H_\infty}(u) f_l = R_{H_k}(u)\, \psi^l_k f \qquad \text{for } l < k,$$

and

$$R_{H_\infty}(u) f_l = R_{H_k}(u)\, \psi^k_l f \qquad \text{for } k \le l.$$

Then it follows from Theorem 3.1 that $\{H_\infty, I^\infty, J_{i,\infty}\}$ is irreducible with signature $\left(\vec{0}\right)_\infty$. The only problem with this approach is that the isomorphism embedding ψ^k_{k+1} is not the isomorphic embedding $i^k_{k+1} \colon \mathcal{F}_{n\times k} \to \mathcal{F}_{n\times(k+1)}$. To circumvent this difficulty we define the *inverse* or *projective limit* of the family $\left\{H_k, I^k, J_k\right\}$ where J_k denotes the subspace of all H_k-invariants in $V^{(\lambda_1)_k} \hat{\otimes} \cdots \hat{\otimes} V^{(\lambda_r)_k} \hat{\otimes} V^{(\mu\ )_k}$, as follows: For each pair of indices l, k with $l \le k$ define a continuous homomorphism $\phi^k_l \colon J_k \to J_l$ such that

i) ϕ^k_k is the identity map on J_k,

ii) if $i \le l \le k$ then $\phi^k_i = \phi^l_i \circ \phi^k_l$.

Here we can take ϕ^k_l as the *truncation* homomorphism, i.e., ϕ^k_l is defined on the generators $f_{i,k}$ by

$$\phi^k_l(f_{i,k}) = f_{i,l}.$$

The *projective limit* of the system $\left\{H_k, J_k, \phi^k_l\right\}$ is then formally defined by

$$J_{\infty_\leftarrow} := \varprojlim J_k = \left\{(f_k) \in \prod_k J_k : f_l = \phi^k_l(f_k),\ \forall\, l \le k\right\}.$$

Let $\pi_k : J_{\infty_\leftarrow} \to J_k$ denote the projection of $J_{\infty_\leftarrow}$ onto J_k. Let $I^{\infty_\leftarrow}$ denote the representation of H_∞ on $J_{\infty_\leftarrow}$; then $\pi_k\left(I^{\infty_\leftarrow} f\right) = \pi_k(f)$. Recall that if $\mathcal{P}_{n\times k}$ denotes the subspace of all polynomial functions on $\mathbb{C}^{n\times k}$ the $\mathcal{P}_{n\times k}$ is dense in $\mathcal{F}_{n\times k}$. Let $\mathcal{P}_{n\times\infty} = \bigcup_{k=1}^{\infty} \mathcal{P}_{n\times k}$ denote the inductive limit of $\mathcal{P}_{n\times k}$; then clearly $\mathcal{P}_{n\times\infty}$ is dense in $\mathcal{F}_{n\times\infty}$. Let $\mathcal{P}^*_{n\times\infty}$ (resp. $\mathcal{F}^*_{n\times\infty}$) denote the *dual* or *adjoint* space of $\mathcal{P}_{n\times\infty}$ (resp. $\mathcal{F}_{n\times\infty}$). Then since $\mathcal{P}_{n\times\infty}$ is dense in $\mathcal{F}_{n\times\infty}$, $\mathcal{F}^*_{n\times k}$ is dense in $\mathcal{P}^*_{n\times\infty}$. By the Riesz representation theorem for Hilbert spaces, every element $f^* \in \mathcal{F}^*_{n\times\infty}$ is of the form $\langle \cdot \,|\, f\rangle$ for some $f \in \mathcal{F}_{n\times\infty}$, and the map $f^* \to f$ is an anti-linear (or conjugate-linear) isomorphism. Thus we can identify $\mathcal{F}^*_\infty$ with $\mathcal{F}_\infty$ and obtain the *rigged Hilbert space* as the triple $\mathcal{P}_{n\times\infty} \subset \mathcal{F}_{n\times\infty} \subset \mathcal{P}^*_{n\times\infty}$ (see [G&V] for the definition of rigged Hilbert spaces). However, generally an element of $J_{\infty_\leftarrow}$ does not belong to $\mathcal{P}^*_{n\times\infty}$, but can still be considered as a linear functional (not necessarily continuous) on $\mathcal{P}_{n\times\infty}$, and furthermore, in this context the identity representation $I^{\infty_\leftarrow}$ will respect the isomorphic embedding $i^k_{k+1} : \mathcal{F}_{n\times k} \to \mathcal{F}_{n\times(k+1)}$. □

4 Conclusion

We have studied thoroughly several reciprocity theorems for some dual pairs of groups (G'_n, G_∞) and (H'_n, H_∞), where G_∞ is the inductive limit of a chain $\{G_k\}$ of compact groups, H_∞ is the inductive limit of a chain $\{H_k\}$ such that for each k, H_k is a compact subgroup of G_k, and $G'_n \subset H'_n$ are finite-dimensional Lie groups. These theorems show, in particular, that the multiplicity of an irreducible unitary representation of H_∞ with signature $(\mu)_{H_\infty}$ in the restriction to H_∞ of an irreducible unitary representation of G_∞ with signature $(\lambda)_{G_\infty}$ is always finite. This is extremely important in the problem of spectral decompositions of tensor products of irreducible unitary representations of inductive limits of compact classical groups. This type of problems arises naturally in Physics (cf. [K&R]), and in [H&T] tensor product decompositions of tame representations of $\mathrm{U}(\infty)$ are investigated. In [Ol2] Ol'shanskii generalized Howe's theory of dual pairs to some infinite-dimensional dual pairs of groups. This is the right context to generalize the reciprocity theorem 3.2 for the infinite-dimensional dual pairs (G'_∞, G_∞) and (H'_∞, H_∞) which will be part of our work in a forthcoming publication.

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