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FROM SUPERPOSITIONS OF OSCILLATORS TO NONLINEAR DEFORMATIONS OF osp(1/2; R)

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Abstract

We superpose one oscillator of bosonic type and another one of fermionic type, both of different angular frequencies, and show that this superposition is invariant under a (nonlinear) deformation of the Lie superalgebra $osp(1/2; R)$. We also construct the unitary irreducible representations (positive discrete series) of this deformed structure.

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1. Introduction

As is well known, linear symmetries - associated with vector fields generating Lie algebras - play a prominent role in Physics. However, it has been recently realized that nonlinear symmetries have also to be taken care of, in gauge field theories for example [1] or in other physically interesting fields such as inverse scattering problems, conformal field theories,... [2]. The concept of Lie algebras appears to be too restrictive to describe such nonlinear symmetries and has to give way to deformed Lie algebras, in a general manner. For instance, if one considers the superposition of two independent bosonic oscillators of respective angular frequencies ω_1 and ω_2 corresponding to the Hamiltonian

$$H = \omega_1\,(a_1^\dagger a_1 + \frac{1}{2}) + \omega_2\,(a_2^\dagger a_2 + \frac{1}{2}), \tag{1}$$

one can verify that this Hamiltonian commutes, when $\omega_1 = \omega_2$, with the operators

$$J_3 = \frac{1}{2}(a_2^\dagger a_2 - a_1^\dagger a_1),\ J_+ = a_1 a_2^\dagger,\ J_- = a_1^\dagger a_2 \tag{2}$$

generating the Lie algebra $su(2, C)$ associated with the relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3. \tag{3}$$

One usually refers to (2) as the Schwinger boson realization of $su(2, C)$. If $\omega_1 \neq \omega_2$, say $\omega_1 = \frac{1}{2}$ and $\omega_2 = 1$, the Hamiltonian (1) is invariant under the operators [3]

$$\tilde{J}_3 = \frac{1}{3}(a_2^\dagger a_2 - a_1^\dagger a_1),\ \tilde{J}_+ = \frac{1}{\sqrt{3}}a_1^2 a_2^\dagger,\ \tilde{J}_- = -\frac{1}{\sqrt{3}}(a_1^\dagger)^2 a_2 \tag{4}$$

leading to a (nonlinear) deformation of $su(2, C)$

$$[\tilde{J}_3, \tilde{J}_\pm] = \pm\tilde{J}_\pm, \quad [\tilde{J}_+, \tilde{J}_-] = 4\tilde{J}_3^2 + c, \quad c = constant. \tag{5}$$

known as the $W_3^{(2)}$-algebra [4].

Another extension of Lie algebras consists in a Z_2-grading of these structures by introducing, besides the usual bosonic operators, odd powers of fermionic ones as typically developed in supersymmetric quantum mechanics for example [5]. Coming back on the pedagogical example of the harmonic oscillator, such a generalization leads, instead of (1), to the Hamiltonian

$$H_S = \omega_1 \, (a^\dagger a + \frac{1}{2}) + \omega_2 \, (b^\dagger b - \frac{1}{2}) \tag{6}$$

combining bosonic (a and $a^\dagger$) and fermionic (b and $b^\dagger$) annihilation and creation operators. Indeed, it is well known [6] that this operator (6) is invariant under the Lie superalgebra $osp(2/2) \uplus sh\,(2/2)$ ($\uplus$ meaning here the semi-direct sum) iff $\omega_1 = \omega_2$.

The purpose of this paper is then clear. We would like to combine these two possible generalizations of Lie algebras (the nonlinear deformation and the grading) and show through the simple Hamiltonian (6) that it is possible, when $\omega_1 \neq \omega_2$, to put in evidence a nonlinear deformation of $osp(2/2) \uplus sh\,(2/2)$ including a nonlinear deformation of the orthosymplectic superalgebra [7] $osp(1/2; R)$. The construction of this latest structure is presented in Section 2 by using the bosonic and fermionic creation and annihilation operators. We concentrate on its representations in Section 3 in the case $\omega_1 = \frac{1}{2}, \omega_2 = 1$. We conclude with some comments in Section 4.

2. A (nonlinear) deformed osp(1/2; R)

Let us take back the Hamiltonian (6) where we put $\omega_1 = \frac{1}{q}$ (q being a nonvanishing positive integer) and $\omega_2 = 1$ (without loss of generality due to the nilpotency of the fermionic operators). It is then easy to verify that this Hamiltonian H_S admits the following symmetry bosonic operators

$$H_B = \frac{1}{q}(a^\dagger a + \frac{1}{2}), \quad H_F = b^\dagger b - \frac{1}{2},$$

$$C_+ = (a^\dagger)^{2q}, \quad C_- = a^{2q},$$

$$P_+ = (a^\dagger)^q, \quad P_- = a^q$$

and the following fermionic ones

$$Q_+ = a^q b^\dagger, \quad Q_- = (a^\dagger)^q b,$$
$$S_+ = (a^\dagger)^q b^\dagger, \quad S_- = a^q b,$$
$$T_+ = b^\dagger, \quad T_- = b.$$

They evidently coincide with previous results [6] when $q = 1$.

It is a tedious but straightforward task to put in evidence the commutation and anticommutation relations satisfied by these operators and being typical of a deformation of $osp(2/2) \uplus sh\,(2/2)$ but, as mentioned earlier, we want here to concentrate on the deformation of $osp(1/2; R)$ only. We thus consider the linear combinations

$$K_0 = \frac{1}{2}H_B, \quad K_\pm = \frac{1}{2}C_\pm$$
$$F_\pm = \frac{1}{2}(Q_\mp + S_\pm) \tag{7}$$

leading to

$$[K_0, K_\pm] = \pm K_\pm, \tag{8}$$
$$[K_+, K_-] = -\frac{1}{4}(a^\dagger)^q a^q X_q - \frac{1}{4}(a^\dagger)^q X_q a^q - \frac{1}{4}a^q X_q (a^\dagger)^q - \frac{1}{4}X_q a^q (a^\dagger)^q, \tag{9}$$
$$[K_0, F_\pm] = \pm\frac{1}{2}F_\pm, \tag{10}$$
$$[K_\pm, F_\pm] = 0, \tag{11}$$
$$[K_+, F_-] = -\frac{1}{4}((a^\dagger)^q X_q + X_q (a^\dagger)^q)(b + b^\dagger), \quad [K_-, F_+] = -([K_+, F_-])^\dagger, \tag{12}$$
$$\{F_\pm, F_\pm\} = K_\pm, \tag{13}$$
$$\{F_+, F_-\} = \frac{1}{4}X_q + \frac{1}{2}(qH_B - \frac{1}{2})(qH_B - \frac{3}{2})\ldots(qH_B - q + \frac{1}{2}). \tag{14}$$

In these relations, we have put

$$X_q = [a^q, (a^\dagger)^q]$$

from which it is possible to prove the following recurrence relation

$$X_q = (N + q)X_{q-1} + (2q - 1)\frac{N!}{(N - q + 1)!} \tag{15}$$

where, as usual, the number operator N is defined by

$$N = a^\dagger a.$$

The relation (15) finally leads to

$$X_q = N!(N+q)! \sum_{j=1}^{q} \frac{(2q-2j+1)}{(N-q+j)!(N+q-j+1)!}.$$

If $q = 1$, X_q reduces to the identity and the relations (8)-(14) are the ones of $osp(1/2; R)$. If $q \neq 1$, we are in fact dealing with a nonlinear deformation of this Lie superalgebra characterized, as it can be verified, by the expected Jacobi identities. For example, if we concentrate on the first nontrivial value $q = 2$, we obtain

$$X_2 = 4N + 2$$

and the corresponding relations (9), (12) and (14) respectively write

$$[K_+, K_-] = -44K_0 - 256K_0^3, \tag{16}$$

$$[K_+, F_-] = -4(4K_0 - 1)F_+, \tag{17}$$

$$\{F_+, F_-\} = 8K_0^2 + \frac{3}{8}. \tag{18}$$

We propose now to construct the irreducible unitary representations (unirreps) of the deformed $osp(1/2; R)$ characterized by the relations (8), (10), (11), (13) and (16)-(18) in the particular case $q = 2$.

3. Unirreps of the deformed osp(1/2; R) for q = 2

Let us first notice that the set of the relations (8) and (16) defines a (nonlinear) deformation of the symplectic Lie algebra $sp(2; R)$ whose unirreps are well known [8]. The first step of our construction is thus to put in evidence the corresponding unirreps of the above set. In this aim, we limit ourselves to the analogues of the positive discrete series. This means that we consider the action of the bosonic operators as follows

$$K_0 \mid k,m >= \frac{m}{\gamma} \mid k,m >, \tag{19}$$

$$K_+ \mid k,m >= \sqrt{f(m)} \mid k,m+\gamma >, \tag{20}$$

$$K_- \mid k,m >= \sqrt{f(m-\gamma)} \mid k,m-\gamma >, \quad m=k,k+1,...;k>0. \tag{21}$$

In the above relations, the parameter γ (a strictly positive integer, due to the action of the ladder operators $K_\pm$, see (8)), has been introduced for generality by analogy with the (nonlinear) deformed $su(2,C)$ in which it has played an important role [9]. Moreover, the relation (21) ensures that K_- is the hermitian conjugate of K_+. In order to obey all the relations of the deformed $sp(2;R)$, we have to require

$$f(m) = f(m-\gamma) + 44\frac{m}{\gamma} + 256\frac{m^3}{\gamma^3}$$

which is satisfied if

$$f(m) = \frac{2}{\gamma^2}(m-k-n+\gamma)(m+k+n)(32\frac{k^2}{\gamma^2} + 32\frac{m^2}{\gamma^2} + 32\frac{n^2}{\gamma^2} + 64\frac{kn}{\gamma^2} - 32\frac{k}{\gamma} + 32\frac{m}{\gamma} - 32\frac{n}{\gamma} + 11)$$

with

$$m = k+n, k+n+\gamma, k+n+2\gamma, ..., \quad \gamma \neq 0,1,2,...,n,$$

n being a positive integer. The usual positive discrete series can be resolved from the start when $n=0$ and $\gamma=1$.

We now consider, as a second step, the actions of the fermionic operators $F_\pm$ on these states $\mid k,m >$. Remembering [10] that, in the case $q=1$, these operators $F_\pm$ connect two different unirreps of $sp(2;R)$, we propose here

$$F_+ \mid k,m >= \sqrt{g(m)} \mid \beta, m+\alpha >,$$

$$F_+ \mid \beta,m >= \sqrt{h(m)} \mid k, m-\alpha+\gamma >,$$

$$F_- \mid k,m >= \sqrt{h(m+\alpha-\gamma)} \mid \beta, m+\alpha-\gamma >, \tag{22}$$

$$F_- \mid \beta, m >= \sqrt{g\,(m-\alpha)} \mid k, m-\alpha >,$$

where the parameters α and β as well as the functions g and h have to be determined. In particular, we immediately notice that the relations (10) will be satisfied if $\alpha = \frac{1}{2}\gamma$. Moreover, after some tedious calculations in order to take account of the relations (11), (13), (17) and (18), we come to the conclusion that β is necessarily k while γ is equal to $8k$. Such results imply that

$$f(m) = \frac{1}{64k^4}(m+k+n)(m+7k-n)(n^2+8km+n^2-6kn+15k^2)$$

and, correspondingly,

$$g(m) = h(m) = \frac{1}{16k^2}\frac{(m+k+n)(m+7k-n)(n^2+8km+n^2-6kn+15k^2)}{(m+5k)(m+7k)}$$

with

$$m = k+n, 9k+n, ..., \quad n = 1, 2, ..., 8k-1. \tag{23}$$

The relation (18) is then ensured if

$$g(m) + g(m-4k) = \frac{m^2}{8k^2} + \frac{3}{8}$$

or, in other words,

$$n\,(n-2k)(n-4k)(n-6k) = 0$$

which is satisfied when $n = 0$ or when $k = \frac{1}{4}$ and $k = \frac{1}{2}$ according to the values (23).

In conclusion, the (positive discrete series) unirreps of the nonlinear deformed $osp(1/2; R)$ associated with $q = 2$ are given by

$$K_0 \mid k, m >= \frac{m}{8k} \mid k, m >, \tag{24}$$

$$K_+ \mid k, m >= \frac{1}{8k^2}\sqrt{(m+k)(m+3k)(m+5k)(m+7k)} \mid k, m+8k >, \tag{25}$$

$$K_- \mid k, m >= \frac{1}{8k^2}\sqrt{(m-k)(m-3k)(m-5k)(m-7k)} \mid k, m-8k >, \tag{26}$$

$$F_+ \mid k, m >= \frac{1}{4k}\sqrt{(m+k)(m+3k)} \mid k, m+4k >, \tag{27}$$

$$F_- \mid k, m >= \frac{1}{4k}\sqrt{(m-k)(m-3k)} \mid k, m-4k >, \tag{28}$$

with

$$m = k+n, 9k+n, ..., \quad n = 1, 2, ..., 8k-1 \text{ if } k = \frac{1}{4}, \frac{1}{2} \tag{29}$$

and

$$m = k, 9k, ... \text{ otherwise} \tag{30}$$

4. Comments and conclusion

We have shown that the superposition of two oscillators (one of bosonic type, the other one of fermionic type) of different angular frequencies is invariant under a (nonlinear) deformation of the Lie superalgebra $osp(1/2; R)$. We have concentrated our study on a specific case of this deformed structure in order to put in evidence its (positive discrete series) unirreps. A remarkable fact about these unirreps is that, by opposition to the undeformed context, the fermionic operators $F_\pm$ connect states of the same unirreps (cfr. $\beta = k$) of the (deformed) $sp(2; R)$. Moreover, for the values $k = \frac{1}{4}$ and $k = \frac{1}{2}$, all the states of the undeformed $osp(1/2; R)$ can be connected to the fundamental state $\mid k, k >$ by repeated actions of the ladder operators F_+ and K_+ while it is true for some of them only when k takes other values.

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