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Artikel: Some notes on Lidstone's and other approximations to temporary life annuities when the force of mortality is (1 + k) x+t

Autor: Åkerberg, Bengt

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Some notes on Lidstone's and other approximations to temporary life annuities when the force of mortality is $(1+k)\,\mu_{x+t}$

By *Bengt Åkerberg*,
Actuary, Life Assurance Company Skandia-Nordstjernan Ltd. Stockholm

Suppose that the force of mortality is $(1+k)\,\mu_{x+t}$, then we have for an endowment assurance

$$\frac{d}{dt}\,{}_tV^{(k)}_{x\overline{n}|} = \delta\,{}_tV^{(k)}_{x\overline{n}|} + p^{(k)}_{x\overline{n}|} - (1+k)\,\mu_{x+t}\,(1-{}_tV^{(k)}_{x\overline{n}|})\,.$$

After multiplying both sides with $\frac{D_{x+t}}{D_x}$ we get

$$d\left(\frac{D_{x+t}}{D_x}\,{}_tV^{(k)}_{x\overline{n}|}\right) = p^{(k)}_{x\overline{n}|}\,\frac{D_{x+t}}{D_x}\,dt - \mu_{x+t}\,\frac{D_{x+t}}{D_x}\,dt - k\,\mu_{x+t}\,\frac{D_{x+t}}{D_x}\,(1-{}_tV^{(k)}_{x\overline{n}|})\,dt$$

and after integrating from 0 to n

$$\frac{D_{x+n}}{D_x} = \left(\frac{1}{\bar{a}^{(k)}_{x\overline{n}|}} - \delta\right)\bar{a}_{x\overline{n}|} - \left(1 - \delta\,\bar{a}_{x\overline{n}|} - \frac{D_{x+n}}{D_x}\right) - k\int_0^n \mu_{x+t}\,\frac{D_{x+t}}{D_x}\,(1-{}_tV^{(k)}_{x\overline{n}|})\,dt.$$

Hence

$$\frac{1}{\bar{a}^{(k)}_{x\overline{n}|}} = \frac{1}{\bar{a}_{x\overline{n}|}} + \frac{k}{\bar{a}_{x\overline{n}|}}\int_0^n \mu_{x+t}\,\frac{D_{x+t}}{D_x}\,(1-{}_tV^{(k)}_{x\overline{n}|})\,dt\,. \qquad (1)$$

This formula turns out to be the key formula, from which a group of approximations can be deduced fairly easily.

If we replace the expression $1-{}_tV^{(k)}_{x\overline{n}|} = \frac{\bar{a}^{(k)}_{x+t\,\overline{n-t}|}}{\bar{a}^{(k)}_{x\overline{n}|}}$ by $\frac{\bar{a}_{\overline{n-t}|}}{\bar{a}_{\overline{n}|}}$

we obtain

$$\frac{1}{\bar{a}^{(k)}_{x n|}} \sim \frac{1}{\bar{a}_{x\overline{n}|}} + \frac{k}{\bar{a}_{x n|}} \cdot \frac{\bar{a}_{n|} - \bar{a}_{x\overline{n}|}}{\bar{a}_{\overline{n}|}}$$

$$\sim \frac{1+k}{\bar{a}_{x\overline{n}|}} - \frac{k}{\bar{a}_{\overline{n}|}} \tag{2}$$

and for $k = 1$, Lidstone's formula

$$\frac{1}{\bar{a}_{xx\overline{n}|}} \sim \frac{2}{\bar{a}_{x n|}} - \frac{1}{\bar{a}_{\overline{n}|}}. \tag{2 a}$$

Supposing $\dfrac{\bar{a}^{(k)}_{x+t\,\overline{n-t}|}}{\bar{a}^{(k)}_{x n|}} \sim \dfrac{\bar{a}_{\overline{n-t}|}}{\bar{a}_{x\overline{n}|}}$ we have, by (1)

$$\frac{1}{\bar{a}^{k}_{x n|}} \sim \frac{1}{\bar{a}_{x\overline{n}|}} + \frac{k}{\bar{a}_{x n|}} \cdot \frac{\bar{a}_{\overline{n}|} - \bar{a}_{x\overline{n}|}}{\bar{a}_{x\overline{n}|}}$$

$$\sim \frac{1-k}{\bar{a}_{x\overline{n}|}} + k\,\frac{\bar{a}_{\overline{n}|}}{(\bar{a}_{x\overline{n}|})^2} \tag{3}$$

and for $k = 1$

$$\bar{a}_{xx\overline{n}|} \sim \frac{(\bar{a}_{x\overline{n}|})^2}{\bar{a}_{\overline{n}|}}. \tag{3 a}$$

And finally if $\dfrac{\bar{a}^{(k)}_{x+t\,\overline{n-t}|}}{\bar{a}^{(k)}_{x n|}}$ is replaced by $\dfrac{\bar{a}_{\overline{n-t}|}}{\bar{a}^{(k)}_{x n|}}$ we obtain

$$\frac{1}{\bar{a}^{(k)}_{x n|}} \sim \frac{1}{\bar{a}_{x\overline{n}|}} + \frac{k}{\bar{a}_{x n|}} \cdot \frac{\bar{a}_{\overline{n}|} - \bar{a}_{x\overline{n}|}}{\bar{a}^{(k)}_{x n|}}.$$

Hence

$$a^{(k)}_{x n|} \sim (1+k)\,\bar{a}_{x n|} - k\,\bar{a}_{\overline{n}|} \tag{4}$$

and

$$\bar{a}_{xx\overline{n}|} \sim 2\,\bar{a}_{x n|} - \bar{a}_{\overline{n}|}. \tag{4 a}$$

The formulas (2 a), (3 a) and (4 a) may also be written

(2 a) $$\bar{a}_{x\overline{n}|} \sim \frac{2}{\dfrac{1}{\bar{a}_{xx\overline{n}|}} + \dfrac{1}{\bar{a}_{\overline{n}|}}}$$ (the harmonical mean),

(3 a) $$\bar{a}_{x\overline{n}|} \sim \sqrt{\bar{a}_{xx\overline{n}|}\,\bar{a}_{\overline{n}|}}$$ (the geometrical mean),

(4 a) $$\bar{a}_{x\overline{n}|} \sim \frac{\bar{a}_{xx\overline{n}|} + \bar{a}_{\overline{n}|}}{2}$$ (the arithmetical mean).

Denoting now $\frac{\bar{a}_{\overline{n}|}}{\bar{a}_{\overline{xn}|}}$ by $1+\lambda$ $(\lambda > 0)$ it is easily seen that by

$$\text{(2 a)} \qquad \bar{a}_{\overline{xxn}|} \sim \frac{1}{1+2\lambda}\,\bar{a}_{\overline{n}|},$$

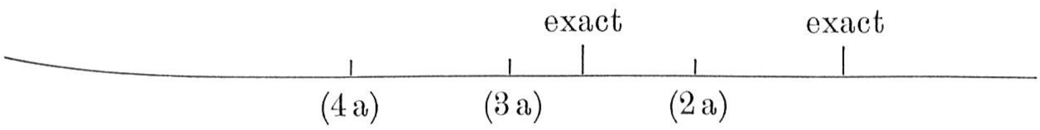
$$\text{(3 a)} \qquad \bar{a}_{\overline{xxn}|} \sim \frac{1}{(1+\lambda)^2}\,\bar{a}_{\overline{n}|},$$

$$\text{(4 a)} \qquad \bar{a}_{\overline{xxn}|} \sim \frac{1-\lambda}{1+\lambda}\,\bar{a}_{\overline{n}|}.$$

Since $$\frac{1-\lambda}{1+\lambda}\,\bar{a}_{\overline{n}|} < \frac{1}{(1+\lambda)^2}\,\bar{a}_{\overline{n}|} < \frac{1}{1+2\lambda}\,\bar{a}_{\overline{n}|}$$

it follows that the values for $\bar{a}_{\overline{xxn}|}$ by (2a), (3a) and (4a) are decreasing.

We represent those values on a line



and shall prove that the exact value lies either between (3a) and (2a) or, when n is chosen sufficiently large, to the right of (2a).

It is immediately seen that the following temporary life annuity

$$\bar{a}_{\overline{xn}|} + \bar{a}_{\overline{xn}|} - \bar{a}_{\overline{xxn}|} < \bar{a}_{\overline{n}|}$$

hence the exact value $$\bar{a}_{\overline{xxn}|} > 2\,\bar{a}_{\overline{xn}|} - \bar{a}_{\overline{n}|}.$$

From Schwarz' inequality

$$\int_0^n (f(t))^2\,dt \int_0^n (\varphi(t))^2\,dt \geqq \left[\int_0^n f(t)\,\varphi(t)\,dt\right]^2$$

it follows when $$f(t) = \frac{l_{x+t}}{l_x}\,e^{-\frac{\delta t}{2}}$$

and $$\varphi(t) = e^{-\frac{\delta t}{2}}$$

that the exact value $$\bar{a}_{\overline{xxn}|} \geqq \frac{(\bar{a}_{\overline{xn}|})^2}{\bar{a}_{\overline{n}|}}.$$

Turning now to the study of the formula (2).
From the formula (1) it is seen that if

$$\frac{\bar{a}^{(k)}_{x+t\,\overline{n-t}|}}{\bar{a}^{(k)}_{x\overline{n}|}} > \frac{\bar{a}_{\overline{n-t}|}}{\bar{a}_{\overline{n}|}}$$

during the whole interval $(0,n)$ then

$$\frac{1}{\bar{a}^{(k)}_{x\overline{n}|}} > \frac{1+k}{\bar{a}_{x\overline{n}|}} - \frac{k}{\bar{a}_{\overline{n}|}}. \qquad (5)$$

Hence
$$\frac{1}{\bar{a}_{xx\overline{n}|}} > \frac{2}{\bar{a}_{x\overline{n}|}} - \frac{1}{\bar{a}_{\overline{n}|}}$$

and the exact value $\bar{a}_{xx\overline{n}|}$ lies between (3 a) and (2 a).

First we shall prove that the inequality (5) holds if $\mu_{x+t}\,\bar{a}_{\overline{n-t}|}$ never increases during the whole interval $(0,n)$.

Thereafter we shall–under the assumption that $\mu_{x+t} = \alpha + \beta\, e^{\gamma(x+t)}$ –investigate if possible some simple criterions can be found to decide when $\mu_{x+t}\,\bar{a}_{\overline{n-t}|}$ never increases.

The premium to be paid by a constant yearly amount during the whole period of insurance for the annuity $\bar{a}_{\overline{n}|} - \bar{a}_{x\overline{n}|}$ we denote by

$$\pi_{x\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - \bar{a}_{x\overline{n}|}}{\bar{a}_{x\overline{n}|}}.$$

From the differential equation

$$\frac{d\,{}_tV_{x\overline{n}|}}{dt} = (\delta + \mu_{x+t})\,{}_tV_{x\overline{n}|} + \pi_{x\overline{n}|} - \mu_{x+t}\,\bar{a}_{\overline{n-t}|}$$

it follows that $\pi_{x\overline{n}|} < \mu_x\,\bar{a}_{\overline{n}|}$ when $\mu_{x+t}\,\bar{a}_{\overline{n-t}|}$ never increases. For otherwise $\frac{d}{dt}\,({}_tV_{x\overline{n}|}) > 0$ and ${}_tV_{x\overline{n}|} > 0$ when $t > 0$ contrary the fact that ${}_tV_{x\overline{n}|} = 0$ when $t = n$.

$\mu_{x+t}\,\bar{a}_{\overline{n-t}|}$ never increasing in the interval $(0,n)$ we thus have

$$\pi_{x+t\,\overline{n-t}|} < \mu_{x+t}\,\bar{a}_{\overline{n-t}|}$$

or
$$\frac{1}{\bar{a}_{x+t\,\overline{n-t}|}} - \frac{1}{\bar{a}_{\overline{n-t}|}} < \mu_{x+t}.$$

Now $$\frac{d}{dt}\left(\frac{\bar{a}_{\overline{n-t|}}}{\bar{a}_{x+t\,\overline{n-t|}}}\right) = \frac{\bar{a}_{\overline{n-t|}}}{\bar{a}_{x+t\,\overline{n-t|}}}\left[\frac{1}{\bar{a}_{x+t\,\overline{n-t|}}} - \frac{1}{\bar{a}_{\overline{n-t|}}} - \mu_{x+t}\right]$$

so that $\dfrac{\bar{a}_{\overline{n-t|}}}{\bar{a}_{x+t\,\overline{n-t|}}}$ never increases.

Hence
$$\frac{\bar{a}_{\overline{n|}}}{\bar{a}_{x\overline{n|}}} > \frac{\bar{a}_{\overline{n-t|}}}{\bar{a}_{x+t\,\overline{n-t|}}}$$

or
$$\frac{\bar{a}_{x+t\,\overline{n-t|}}}{\bar{a}_{x\overline{n|}}} > \frac{\bar{a}_{\overline{n-t|}}}{\bar{a}_{\overline{n|}}}.$$

The inequality (5) thus holds for $k = 0$ and since $(1+k)\,\mu_{x+t}\,\bar{a}_{\overline{n-t|}}$ never increases it holds generally when $k \neq 0$.

We shall now enter upon the discussion about some sufficient conditions to decide when $\mu_{x+t}\,\bar{a}_{\overline{n-t|}}$ never increases in the whole interval $(0,n)$ on the assumption that $\mu_{x+t} = \alpha + \beta\,e^{\gamma(x+t)}$.

It is convenient to introduce the notation

$$\varrho_{x+t\,\overline{n-t|}} = \mu_{x+t}\,\bar{a}_{\overline{n-t|}}$$

and from
$$\frac{d}{dt}\,\varrho_{x+t\,\overline{n-t|}} = \mu_{x+t}\left[\gamma\,\bar{a}_{\overline{n-t|}} - e^{-\delta(n-t)}\right] - \alpha\,\gamma\,\bar{a}_{\overline{n-t|}}$$

it is immediately seen that

$$\frac{d}{dt}\,\varrho_{x+t\,\overline{n-t|}} \leqq 0$$

in the interval $0 \leqq t \leqq n$ and irrespective of x certainly if

$$\gamma\,\bar{a}_{\overline{n|}} - e^{-\delta n} \leqq 0.$$

When
$$n \leqq \frac{1}{\delta}\log\frac{\gamma+\delta}{\gamma}$$

we thus find that $\varrho_{x+t\,\overline{n-t|}}$ never increases.

When
$$n > \frac{1}{\delta}\log\frac{\gamma+\delta}{\gamma}$$

it is seen that
$$\frac{d}{dt}\,\varrho_{x+t\,\overline{n-t|}} \leqq 0$$

wherever

$$\mu_{x+t} \leqq \frac{\alpha \gamma \bar{a}_{\overline{n-t|}}}{\gamma \bar{a}_{\overline{n-t|}} - \bar{e}^{-\delta(n-t)}},$$

which can also be written (using the Makeham expression for μ_{x+t})

$$x + n \leqq \frac{1}{\gamma} \log \left[\frac{\alpha \delta}{\beta} \cdot \frac{e^{(\gamma-\delta)(n-t)}}{\gamma - (\gamma+\delta) e^{-\delta(n-t)}} \right]. \qquad (6)$$

Now, in practice, we always have $\gamma - \delta > 0$. Thus the right-hand side of (6) considered as a function of $(n-t)$, tends to $+\infty$ when $(n-t)$ tends to $\frac{1}{\delta} \log \frac{\gamma+\delta}{\gamma}$ and when $(n-t)$ tends to $+\infty$.

Its derivative is found to be

$$\frac{\gamma-\delta}{\gamma} \cdot \frac{e^{\delta(n-t)} - \frac{\gamma+\delta}{\gamma-\delta}}{e^{\delta(n-t)} - \frac{\gamma+\delta}{\gamma}}$$

and it is thus seen that the function never increases when $(n-t)$ increases from $\frac{1}{\delta} \log \frac{\gamma+\delta}{\gamma}$ to a value $\frac{1}{\delta} \log \frac{\gamma+\delta}{\gamma-\delta}$ and never decreasing when $(n-t) > \frac{1}{\delta} \log \frac{\gamma+\delta}{\gamma-\delta}$.

Consequently, the function

$$\frac{1}{\gamma} \log \left[\frac{\alpha \delta}{\beta} \cdot \frac{e^{(\gamma-\delta)(n-t)}}{\gamma - (\gamma+\delta) e^{-\delta(n-t)}} \right]$$

has a minimum when

$$n - t = \frac{1}{\delta} \log \frac{\gamma+\delta}{\gamma-\delta}$$

and inserting this value, we find the minimum to be

$$\frac{1}{\gamma} \log \left[\frac{\alpha}{\beta} \left(\frac{\gamma+\delta}{\gamma-\delta} \right)^{\frac{\gamma-\delta}{\delta}} \right].$$

The results reached can be summarized as follows. $\varrho_{x+t\,\overline{n-t}|}$ never increases in the interval $(0,n)$ when

$$n \leqq \frac{1}{\delta} \log \frac{\gamma+\delta}{\gamma}$$

irrespective of x or when

$$x+n \leqq \frac{1}{\gamma} \log \left[\frac{\alpha}{\beta} \left(\frac{\gamma+\delta}{\gamma-\delta} \right)^{\frac{\gamma-\delta}{\delta}} \right]$$

even if the condition for n is not satisfied.

For instance, it can be mentioned that with the system of assumptions at present adopted by the Swedish life insurance companies, i. e. a rate of interest of 2,25 % and the Swedish mortality table D 37 with a loading of 2,8 ‰ on the interest and mortality, we have

$$n \leqq 9{,}4\,,$$

$$x+n \leqq 64{,}0\,.$$

Finally we shall prove that if n is chosen sufficiently large, the exact value $\bar{a}_{xx\overline{n}|} >$ the value by Lidstone's formula (2 a).

From (1) it follows that

$$\frac{1}{\bar{a}_{xx\overline{n}|}} < \frac{1}{\bar{a}_{x\overline{n}|}} + \frac{1}{\bar{a}_{x\overline{n}|}} \int_0^n \frac{D_{x+t}}{D_x} \mu_{x+t}\, dt$$

or

$$\frac{1}{\bar{a}_{xx\overline{n}|}} < \frac{1}{\bar{a}_{x\overline{n}|}} + \frac{1}{\bar{a}_{x\overline{n}|}} \left[1 - \delta\, \bar{a}_{x\overline{n}|} - \frac{D_{x+n}}{D_x} \right]$$

and for $n = +\infty$

$$\frac{1}{\bar{a}_{xx}} < \frac{2}{\bar{a}_x} - \delta\,.$$

From Lidstone's formula we obtain when $n = +\infty$

$$\frac{1}{\bar{a}_{xx}} = \frac{2}{\bar{a}_x} - \delta$$

hence the desired result.