Some elementary researches in the mathematics of life insurance

Autor(en): **Sumitsuji, Osamu**

Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries**

Band (Jahr): **59 (1959)**

PDF erstellt am: **02.06.2024**

Persistenter Link: https://doi.org/10.5169/seals-966817

Some Elementary Researches in the Mathematics of Life Insurance

By Osamu Sumitsuji, Osaka (Japan)

Summary

In this paper approximation formulae are derived for the present value of annuity, the net premium and the mathematical reserve. They may be helpful to estimate the monetary functions in absence of complete commutation columns as well as the effect of variations in the underlying mortality or the interest. They are also of interest for actuarial studies in the domain of substandard risks.

I. Approximation Formulas

1. The equated time of Payments.

Let the various sums be $S_0, S_1, \ldots, S_{n-1}$ due at the end of $n_0, n_1, \ldots, n_{n-1}$ years respectively, and n the equated time. Then

$$v^n \sum_0^{n-1} S_r = \sum_0^{n-1} v^{n_r} S_r,$$

hence

$$n = \frac{1}{\delta}\left(\ln \sum S_r - \ln \sum S_r v^{n_r}\right) =$$

$$= \frac{\sum n_r S_r}{\sum S_r} - \frac{\delta}{2}\left\{\frac{\sum n_r^2 S_r}{\sum S_r} - \left(\frac{\sum n_r S_r}{\sum S_r}\right)^2\right\} + \text{terms involving higher power of } \delta,$$

where $\delta = \ln(1+i)$ and i is the ordinary interest.

Hence, as a first approximation

$$n \sim \frac{\sum n_r S_r}{\sum S_r}, \quad \left(n < \frac{\sum n_r S_r}{\sum S_r} \text{ is well known}\right)$$

and as a second approximation

$$n \sim \frac{\sum n_r S_r}{\sum S_r} - \frac{\delta}{2}\left\{\frac{\sum n_r^2 S_r}{\sum S_r} - \left(\frac{\sum n_r S_r}{\sum S_r}\right)^2\right\}. \tag{1}$$

In this formula, if we put $S_0 = S_1 = \ldots = S_{n-1} = 1$ and $n_0 = 0$, $n_1 = 1$, $n_2 = 2$, $\ldots$, $n_{n-1} = n-1$,

clearly we have
$$\frac{\sum n_r S_r}{\sum S_r} = \frac{n-1}{2},$$
$$\frac{\sum n_r^2 S_r}{\sum S_r} - \left(\frac{\sum n_r S_r}{\sum S_r}\right)^2 = \frac{n^2-1}{12},$$

therefore we obtain
$$\ddot{a}_{\overline{n}|} = n v^{\frac{n-1}{2} - \frac{n^2-1}{24}\delta + \delta^2(\ldots) - \ldots},$$
$$\ddot{a}_{\overline{n}|} \sim n v^{\frac{n-1}{2} - \frac{n^2-1}{24}\delta}. \tag{2}$$

Example: $i = 3\%$, $\ddot{a}_{\overline{n}|} = 20{,}1885$ true value,
$\ddot{a}_{\overline{n}|} \sim 20{,}1935$ by formula (2).

Similarly $\ddot{a}_{x:\overline{n}|}$ and $A_{x:\overline{n}|}$ are expressed as follows:

$$\ddot{a}_{x:\overline{n}|} = \frac{\sum_0^{n-1} l_{x+t}}{l_x} v^E, \tag{3}$$

$$\text{with } E = \frac{\sum_1^{n-1} t\, l_{x+t}}{\sum_0^{n-1} l_{x+t}} - \frac{\delta}{2}\left\{\frac{\sum_1^{n-1} t^2 l_{x+t}}{\sum_0^{n-1} l_{x+t}} - \left(\frac{\sum_1^{n-1} t\, l_{x+t}}{\sum_0^{n-1} l_{x+t}}\right)^2\right\} + \delta^2(\ldots),$$

$$\ddot{a}_{x:\overline{n}|} \sim \frac{\sum_0^{n-1} l_{x+t}}{l_x} v^E, \tag{4}$$

$$\text{with } E = \frac{\sum_1^{n-1} t\, l_{x+t}}{\sum_0^{n-1} l_{x+t}} - \frac{\delta}{2}\left\{\frac{\sum_1^{n-1} t^2 l_{x+t}}{\sum_0^{n-1} l_{x+t}} - \left(\frac{\sum_1^{n-1} t\, l_{x+t}}{\sum_0^{n-1} l_{x+t}}\right)^2\right\},$$

$$A_{x:\overline{n}|} = v^E, \tag{5}$$

$$\text{with } E = \frac{\sum_0^{n-1} l_{x+t}}{l_x} - \frac{\delta}{2}\left\{\frac{\sum_0^{n-1} (2t+1)\, l_{x+t}}{l_x} - \left(\frac{\sum_0^{x-1} l_{x+t}}{l_x}\right)^2\right\} + \delta^2(\ldots),$$

$$A_{x:\overline{n}|} \sim v^E, \tag{6}$$

$$\text{with } E = \frac{\sum_0^{n-1} l_{x+t}}{l_x} - \frac{\delta}{2}\left\{\frac{\sum_0^{n-1}(2t+1)\,l_{x+t}}{l_x} - \left(\frac{\sum_0^{n-1} l_{x+t}}{l_x}\right)^2\right\}.$$

When n is large, approximations (4) and (6) are not so good, but in case of estimation of $A'_{x:\overline{n}|}$ or $\ddot{a}'_{x:\overline{n}|}$ at new interest rate i', when the value of $A_{x:\overline{n}|}$ or $\ddot{a}_{x:\overline{n}|}$ at interest i is known, they are useful. The procedure is as fellows: From

$$A_{x:\overline{n}|} = v^m$$

we obtain the exact value of m. Then we write

$$a = \frac{\sum_0^{n-1} l_{x+t}}{l_x} - \frac{\delta}{2}\left\{\frac{\sum_0^{n-1}(2t+1)\,l_{x+t}}{l_x} - \left(\frac{\sum_0^{n-1} l_{x+t}}{l_x}\right)^2\right\} - m,$$

and we have

$$A'_{x:\overline{n}|} \sim v'^E$$

$$\text{with } E = \frac{\sum_0^{n-1} l_{x+t}}{l_x} - \frac{\delta'}{2}\left\{\frac{\sum_0^{n-1}(2t+1)\,l_{x+t}}{l_x} - \left(\frac{\sum_0^{n-1} l_{x+t}}{l_x}\right)^2\right\} - \left(\frac{\delta'}{\delta}\right)^2 a.$$

This result is excellent in usual.

Examples: C.S.O. Table, $i = 2{,}5\,\%$, $i' = 3\,\%$.

a)

$$A_{30:\overline{30}|} = 0{,}5170,$$

$$\frac{\sum_0^{29} l_{30+t}}{l_{30}} = 27{,}250,$$

$$\frac{\sum_0^{29}(2t+1)\,l_{30+t}}{l_{30}} - \left(\frac{\sum_0^{29} l_{30+t}}{l_{30}}\right)^2 = 38{,}182,$$

$$a = 0{,}0644,$$

$$\text{hence } A'_{30:\overline{30}|} \sim v'^{\,27{,}250 - \frac{\delta'}{2}\,38{,}182 - \left(\frac{\delta'}{\delta}\right)^2 0{,}0644} = 0{,}4556,$$

$$\text{true value of } A'_{30:\overline{30}|} = 0{,}4556.$$

b)
$$A_{30} = 0{,}4138\,,$$
$$\frac{\sum\limits_{0} l_{30+t}}{l_{30}} = 38{,}242\,,$$
$$\frac{\sum\limits_{0} (2t+1)\, l_{30+t}}{l_{30}} - \left(\frac{\sum\limits_{0} l_{30+t}}{l_{30}}\right)^2 = 192{,}036\,,$$
$$a = 0{,}1374\,,$$
hence
$$A'_{30} \sim v'^{\,38{,}242 - \frac{\delta'}{2}\,192{,}036 - \left(\frac{\delta'}{\delta}\right)^2 0{,}1374} = 0{,}3532\,,$$
$$\text{true value of } A'_{30} = 0{,}3532\,.$$

Now we introduce new notations $q_{x:\overline{n}|}$, $\bar{q}_{x:\overline{n}|}$ and $\bar{\bar{q}}_{x:\overline{n}|}$ which denote as

$$\frac{\sum\limits_{0}^{n-1} l_{x+t}}{l_x} = \ddot{e}_{x:\overline{n}|} = n\left(1 - \frac{n-1}{2}\, q_{x:\overline{n}|}\right),$$
$$\frac{\sum\limits_{1}^{n-1} t\, l_{x+t}}{l_x} = \frac{n\,(n-1)}{2}\left(1 - \frac{2n-1}{3}\, \bar{q}_{x:\overline{n}|}\right),$$
$$\frac{\sum\limits_{1}^{n-1} t^2\, l_{x+t}}{l_x} = \frac{n\,(n-1)\,(2n-1)}{6}\left(1 - \frac{3n\,(n-1)}{2\,(2n-1)}\, \bar{\bar{q}}_{x:\overline{n}|}\right).$$

Ordinarily $q_{x:\overline{n}|} < \bar{q}_{x:\overline{n}|} < \bar{\bar{q}}_{x:\overline{n}|}$. But when x and n are not large, we may assume $q_{x:\overline{n}|} \sim \bar{q}_{x:\overline{n}|} \sim \bar{\bar{q}}_{x:\overline{n}|}$. In such a case we may write approximately

$$\frac{\sum\limits_{1}^{n-1} t\, l_{x+t}}{\sum\limits_{0}^{n-1} l_{x+t}} \sim \frac{n-1}{2}\left(1 - \frac{n+1}{6}\, q_{x:\overline{n}|}\right), \qquad \frac{\sum\limits_{1}^{n-1} t^2 l_{x+t}}{\sum\limits_{0}^{n-1} l_{x+t}} - \left(\frac{\sum\limits_{1}^{n-1} t\, l_{x+t}}{\sum\limits_{0}^{n-1} l_{x+t}}\right)^2 \sim \frac{n^2-1}{12}. \tag{7}$$

$$\frac{\sum\limits_{0}^{n-1} (2t+1)\, l_{x+t}}{l_x} - \left(\frac{\sum\limits_{0}^{n-1} l_{x+t}}{l_x}\right)^2 \sim \frac{n^2}{12}\, n\, q_{x:\overline{n}|}\,(4 - 3n\, q_{x:\overline{n}|}) \sim \tag{8}$$
$$\sim \frac{(n - \ddot{e}_{x:\overline{n}|})\,(3\,\ddot{e}_{x:\overline{n}|} - n)}{3}\,.$$

The value of $q_{x:\overline{n}|}$ is easily obtained from the mortality table $\sum l_x$ column. Here we show some of them: C.S.O. Table, $n = 10$ and 30,

$$q_{10:\overline{10}|} = 0{,}0020, \qquad q_{10:\overline{30}|} = 0{,}0025,$$
$$q_{30:\overline{10}|} = 0{,}0041, \qquad q_{30:\overline{30}|} = 0{,}0063,$$
$$q_{50:\overline{10}|} = 0{,}0146, \qquad q_{50:\overline{30}|} = 0{,}0215.$$

2. The approximations to $\dfrac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}$ and $\dfrac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}}$.

Here we denote

$$(Ia)_{x:\overline{n-1}|} = -\frac{d}{d\delta}\,\ddot{a}_{x:\overline{n}|} = \frac{S_{x+1} - S_{x+n} - (n-1)\,N_{x+n}}{D_x},$$

$$(IA)_{x:\overline{n}|} = -\frac{d}{d\delta}\,A_{x:\overline{n}|} = \frac{R_x - R_{x+n} - n\,M_{x+n} + n\,D_{x+n}}{D_x}.$$

Now we begin with $\dfrac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}}$ where

$$(Ia)_{\overline{n-1}|} = -\frac{d}{d\delta}\,\ddot{a}_{\overline{n}|} = v + 2v^2 + 3v^3 + \ldots + (n-1)\,v^{n-1}.$$

a) Differentiate (2) with respect to δ, and we have

$$-\frac{d}{d\delta}\,\ddot{a}_{\overline{n}|} = \left(\frac{n-1}{2} - \frac{n^2-1}{12}\,\delta + 3\,\delta^2\,(\ldots)\right)\ddot{a}_{\overline{n}|},$$

hence we may write ordinarily

$$\frac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}} \sim \frac{n-1}{2}\left(1 - \frac{n+1}{6}\,i\right).$$

b) Carring "the equated time of Payments" method directly to $(Ia)_{\overline{n-1}|}$ and $\ddot{a}_{\overline{n}|}$ we have ordinarily

$$(Ia)_{\overline{n-1}|} = \frac{n\,(n-1)}{2}\,v^{\frac{2n-1}{3} - \frac{(n-1)(n-2)}{36}\,\delta + \delta^2\,(\ldots)},$$

$$\ddot{a}_{\overline{n}|} = n\,v^{\frac{n-1}{2} - \frac{n^2-1}{24}\,\delta + \delta^2\,(\ldots)},$$

therefore

$$\frac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}} \sim \frac{n-1}{2}\,v^{\frac{n+1}{6} + \frac{(n+1)^2}{72}\,\delta} \sim \frac{n-1}{2}\left(1 - \frac{n+1}{6}\,i\right).$$

In the same way, differentiating (3) and (5) with respect to δ and using (7) and (8) respectively we have, when x and n are not large

$$\frac{(Ia)_{x:\overline{n-1|}}}{\ddot{a}_{x:\overline{n|}}} \sim \frac{n-1}{2}\left(1-\frac{n+1}{6}\left(i+q_{x:\overline{n|}}\right)\right), \tag{9}$$

$$\begin{aligned}\frac{(IA)_{x:\overline{n|}}}{A_{x:\overline{n|}}} &\sim n\left(1-\frac{n-1}{2}q_{x:\overline{n|}}\right)-\frac{n^2}{12}\delta n\, q_{x:\overline{n|}}\left(4-3n\, q_{x:\overline{n|}}\right) \sim \\ &\sim \ddot{e}_{x:\overline{n|}}-\frac{\left(n-\ddot{e}_{x:\overline{n|}}\right)\left(3\ddot{e}_{x:\overline{n|}}-n\right)}{3}\,i.\end{aligned} \tag{10}$$

When x and n are not large, another approximation to $\dfrac{(Ia)_{x:\overline{n-1|}}}{\ddot{a}_{x:\overline{n|}}}$ is obtained as follows:

$$\frac{(Ia)_{x:\overline{n-1|}}}{\ddot{a}_{x:\overline{n|}}} = \frac{D_{x+1}}{D_x}\left(1-{}_1V_{x:\overline{n|}}\right)+\frac{D_{x+2}}{D_x}\left(1-{}_2V_{x:\overline{n|}}\right)+\ldots.$$

Here we may assume

$$1-{}_tV_{x:\overline{n|}} \sim 1-\frac{t}{n}\left(1-\frac{n-t}{2}\right)i,$$

and assume though roughly

$$\frac{(I^2a)_{x:\overline{n-1|}}}{(Ia)_{x:\overline{n-1|}}} \sim \frac{2}{3}(n-1) \tag{11}$$

where

$$(I^2a)_{x:\overline{n-1|}} = \frac{d^2}{d\delta^2}\ddot{a}_{x:\overline{n|}} \sim \frac{n(n-1)(2n-1)}{6}\left(1-\frac{3}{4}n\,q_{x:\overline{n|}}\right)v^{\frac{3(n-1)}{4}},$$

(using the "equated time of Payments" method).

Hence we have

$$\frac{(Ia)_{x:\overline{n-1|}}}{\ddot{a}_{x:\overline{n|}}} \sim \ddot{a}_{x:\overline{n|}}-1-\frac{\ddot{a}_{x:\overline{n|}}}{n}+\frac{i}{6}(Ia)_{x:\overline{n-1|}},$$

therefore

$$\frac{(Ia)_{x:\overline{n-1|}}}{\ddot{a}_{x:\overline{n|}}} \sim \frac{\ddot{a}_{x:\overline{n|}}-1}{1+\dfrac{\ddot{a}_{x:\overline{n|}}}{n}-\dfrac{i}{6}\ddot{a}_{x:\overline{n|}}}. \tag{12}$$

Examples: C.S.O. Table, $i = 2\frac{1}{2}\%$.

$$\frac{(Ia)_{30:\overline{29|}}}{\ddot{a}_{30:\overline{30|}}} = 12{,}048,$$

$$\text{by formula (9)} = 12{,}166,$$

$$\text{by formula (12)} = 11{,}919,$$

$$\frac{(IA)_{30:\overline{30|}}}{A_{30:\overline{30|}}} = 26{,}111,$$

$$\text{by formula (10)} = 26{,}055.$$

3. Interest rate and Net Level Premium.

Here $P_{x:\overline{n|}}$ and $P'_{x:\overline{n|}}$ represents Net Level Premium at interest rate i respectively i', and

$$\Delta P_{x:\overline{n|}} = P'_{x:\overline{n|}} - P_{x:\overline{n|}}.$$

a) Clearly we have

$$\frac{d}{d\delta} P_{x:\overline{n|}} = \frac{d}{d\delta} \frac{A_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}} = -\frac{A_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}} \left(\frac{(IA)_{x:\overline{n|}}}{A_{x:\overline{n|}}} - \frac{(Ia)_{x:\overline{n-1|}}}{\ddot{a}_{x:\overline{n|}}} \right).$$

Carrying (4), (6), (7), (8), (9) and (10) to the right hand side of this relation, we have ordinarily

$$-\frac{d}{d\delta} P_{x:\overline{n|}} \sim \frac{n+1}{2n} v^{\frac{n+1}{2}} (1 + \xi(i)),$$

where

$$\xi(i) = \frac{n i}{24}(4 - n i) + \left(1 - \frac{\mathring{e}_{x:\overline{n|}}}{n}\right)\left(1 - \frac{\mathring{e}_{x:\overline{n|}}}{n} - \frac{4 + n i}{6}\right).$$

Therefore when $i' - i$ is small, we may write

$$\Delta P_{x:\overline{n|}} \sim \frac{1}{n}\left(v'^{\frac{n+1}{2}} - v^{\frac{n+1}{2}}\right)\left(1 + \xi\left(\frac{i+i'}{2}\right)\right), \tag{13}$$

and when $\xi\left(\frac{i+i'}{2}\right)$ is small, simplifying

$$\Delta P_{x:\overline{n|}} \sim \frac{1}{n}\left(v'^{\frac{n+1}{2}} - v^{\frac{n+1}{2}}\right). \tag{14}$$

Examples: C. S. O. Table, $i = 2{,}5\,\%$, $i' = 3\,\%$.

		true value	by formula (13)	by formula (14)
$n = 10$	$-\Delta P_{10:\overline{10|}}$	$= 0{,}00239$	0,00240	
	$-\Delta P_{30:\overline{10|}}$	$= 0{,}00237$	0,00238	0,00231
	$-\Delta P_{50:\overline{10|}}$	$= 0{,}00232$	0,00231	
$n = 20$	$-\Delta P_{25:\overline{20|}}$	$= 0{,}00201$	0,00202	
	$-\Delta P_{40:\overline{20|}}$	$= 0{,}00196$	0,00195	0,00192
	$-\Delta P_{50:\overline{20|}}$	$= 0{,}00181$	0,00183	
$n = 30$	$-\Delta P_{10:\overline{30|}}$	$= 0{,}00178$	0,00178	
	$-\Delta P_{30:\overline{30|}}$	$= 0{,}00172$	0,00172	0,00165
	$-\Delta P_{50:\overline{30|}}$	$= 0{,}00158$	0,00157	

For the m payment n year endowment, in the same way as (14), we may write

$$\Delta\, {}_mP_{x:\overline{n|}} \sim \frac{1}{m}\left(v'^{\frac{2n-m+1}{2}} - v^{\frac{2n-m+1}{2}}\right).$$

b) Another approximation formula to $\Delta P_{x:\overline{n|}}$ is obtained as follows

$$P_{x:\overline{n|}} = \frac{1}{\ddot{a}_{x:\overline{n|}}} - d, \qquad \frac{d}{d\delta}\, P_{x:\overline{n|}} = -v + \frac{(Ia)_{x:\overline{n-1|}}}{(\ddot{a}_{x:\overline{n|}})^2}.$$

Hence, when $\Delta i = i' - i$ is small, we have

$$-\Delta P_{x:\overline{n|}} \sim v\,\Delta i\left(v - \frac{(Ia)_{x:\overline{n-1|}}}{(\ddot{a}_{x:\overline{n|}})^2}\right). \tag{15}$$

On the other hand, clearly (see Appendix)

$$-\Delta P_{x:\overline{n|}} = \frac{\sum\limits_{0}^{n-1} \left({}_tV_{x:\overline{n|}} + P_{x:\overline{n|}} + \Delta\, {}_tV_{x:\overline{n|}} + \Delta P_{x:\overline{n|}}\right) D_{x+t}\, v}{N_x - N_{x+n}}\,\Delta i, \tag{16}$$

where $\Delta\, {}_tV_{x:\overline{n|}} = {}_tV'_{x:\overline{n|}} - {}_tV_{x:\overline{n|}}$, ${}_tV'_{x:\overline{n|}}$ with interest rate i', and

$$\frac{d}{d\delta} P_{x:\overline{n|}} = \frac{\sum\limits_{0}^{n-1} \left({}_tV_{x:\overline{n|}} + P_{x:\overline{n|}}\right) D_{x+t}}{N_x - N_{x+n}}. \tag{17}$$

From (16) and (17) we see the error committed in using (15) is

$$\frac{\sum_{0}^{n-1}(\Delta\,{}_tV_{x:\overline{n}|}+\Delta P_{x:\overline{n}|})\,D_{x+t}\,v}{N_x-N_{x+n}}\,\Delta i.$$

To evaluate this error, from (17) we may write

$$\frac{\sum_{0}^{n-1}({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})\,D_{x+t}}{D_x}=-\ddot{a}_{x:\overline{n}|}\left(\frac{d}{d\delta}P_{x:\overline{n}|}\right).$$

Differentiate this relation with respect to δ, and write

$$\frac{d^2}{d\delta^2}\ddot{a}_{x:\overline{n}|}=\ddot{a}^{(2)}_{x:\overline{n}|},\quad \frac{d}{d\delta}\ddot{a}_{x:\overline{n}|}=\ddot{a}^{(1)}_{x:\overline{n}|},$$

for the sake of brevity, we have roughly

$$\frac{\sum_{0}^{n-1}\left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|}+\frac{d}{d\delta}P_{x:\overline{n}|}\right)D_{x+t}}{D_x}=$$

$$=\frac{1}{2}\frac{\ddot{a}^{(2)}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}-\left(\frac{\ddot{a}^{(1)}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}\right)^2-\frac{1}{2}\frac{\ddot{a}^{(1)}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}-v\,\ddot{a}_{x:\overline{n}|}\sim\ddot{a}_{x:\overline{n}|}\left(\frac{n}{12}+\frac{1}{2}\right),$$

(from (9) and (11), assuming that x and n are not large).

Hence we obtain

$$-\Delta P_{x:\overline{n}|}\sim\left(v-\frac{(Ia)_{x:\overline{n-1}|}}{(\ddot{a}_{x:\overline{n}|})^2}\right)v\,\Delta i-\left(\frac{n}{12}+\frac{1}{2}\right)v^2\,(\Delta i)^2. \tag{18}$$

For ordinary life, en passant, we may write

$$-\Delta P_x\sim\left(v-\frac{(Ia)_x}{(\ddot{a}_x)^2}\right)v\,\Delta i-\left(\frac{\ddot{e}_x}{12}+\frac{1}{2}\right)v^2\,(\Delta i)^2, \tag{19}$$

where $\ddot{e}_x=\dfrac{1}{l_x}\sum_{0}l_{x+t}$.

Examples: C.S.O. Table, $i = 2{,}5\,\%$, $i = 3\,\%$.

$$-\Delta P_{30:\overline{30}|} = 0{,}00172,$$

$$\left(v - \frac{(Ia)_{30:\overline{29}|}}{(\ddot{a}_{30:\overline{30}|})^2}\right) v\,\Delta i = 0{,}00179,$$

$$\left(\frac{30}{12} + \frac{1}{2}\right) v^2\,(\Delta i)^2 = 0{,}00007,$$

$$\text{hence by formula (18)} = 0{,}00172,$$

$$-\Delta P_{30} = 0{,}00131,$$

$$\left(v - \frac{(Ia)_{30}}{(\ddot{a}_{30})^2}\right) v\,\Delta i = 0{,}00138,$$

$$\left(\frac{\mathring{e}_{30}}{12} + \frac{1}{2}\right) v^2\,(\Delta i)^2 = 0{,}00009,$$

$$\text{hence by formula (19)} = 0{,}00129.$$

c) From the mean value theorem, we have following approximations:

$$\Delta P_{x:\overline{n}|} \sim \Delta P_{\overline{n}|} \frac{\frac{d}{d\delta} P_{x:\overline{n}|}}{\frac{d}{d\delta} P_{\overline{n}|}}, \qquad \text{where } P_{\overline{n}|} = \frac{v^n}{\ddot{a}_{\overline{n}|}},$$

$$\Delta P_{x:\overline{n}|} \sim \Delta\left(\frac{1}{n} v^{\frac{n+1}{2}}\right) \frac{\frac{d}{d\delta} P_{x:\overline{n}|}}{\frac{n+1}{2n} v^{\frac{n+1}{2}}}, \quad \text{where } \frac{d}{d\delta} P_{x:\overline{n}|} = v - \frac{(Ia)_{x:\overline{n-1}|}}{(\ddot{a}_{x:\overline{n}|})^2},$$

(see Appendix).

Now we notice the following relations:

$$\Delta P_{x:\overline{n}|} = \left(\frac{d}{d\delta} P_{x:\overline{n}|}\right) v\,\Delta i + \frac{1}{2}\left(\frac{d^2}{d\delta^2} P_{x:\overline{n}|} - \frac{d}{d\delta} P_{x:\overline{n}|}\right) v^2\,(\Delta i)^2 + \ldots$$

and

$$\frac{\sum_0^{n-1}\left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|} + \frac{d}{d\delta} P_{x:\overline{n}|}\right) D_{x+t}}{N_x - N_{x+n}} = \frac{1}{2}\left(\frac{d^2}{d\delta^2} P_{x:\overline{n}|} - \frac{d}{d\delta} P_{x:\overline{n}|}\right).$$

4. Interest rate and Premium Reserve.

Differentiate the following well-known relation with respect to δ

$${}_tV_{x:\overline{n}|} = P_{x:\overline{n}|} \frac{\sum\limits_0^{t-1} D_{x+t}}{D_{x+t}} - \frac{\sum\limits_0^{t-1} C_{x+t}}{D_{x+t}},$$

and we have

$$\frac{d}{d\delta}\, {}_tV_{x:\overline{n}|} = \left(\frac{d}{d\delta} P_{x:\overline{n}|}\right) \frac{\sum\limits_0^{t-1} D_{x+t}}{D_{x+t}} + \frac{\sum\limits_0^{t-1} ({}_tV_{x:\overline{n}|} + P_{x:\overline{n}|})\, D_{x+t}}{D_{x+t}}.$$

Hence

$$\frac{d}{d\delta}\, {}_tV_{x:\overline{n}|} = \left(\frac{d}{d\delta} P_{x:\overline{n}|}\right) \ddot{a}_{x:\overline{t}|} \frac{D_x}{D_{x+t}} \left(1 - \frac{\ddot{a}_{x:\overline{n}|} \sum\limits_0^{t-1} ({}_tV_{x:\overline{n}|} + P_{x:\overline{n}|})\, D_{x+t}}{\ddot{a}_{x:\overline{t}|} \sum\limits_0^{n-1} ({}_tV_{x:\overline{n}|} + P_{x:\overline{n}|})\, D_{x+t}}\right).$$

Therefore when x and n are not large we may write roughly

$$\frac{d}{d\delta}\, {}_tV_{x:\overline{n}|} \sim \left(\frac{d}{d\delta} P_{x:\overline{n}|}\right) \ddot{a}_{x:\overline{t}|} \frac{D_x}{D_{x+t}} \left(1 - \frac{t}{n}\right), \tag{20}$$

Hence, when Δi and t are small, we have

$$\Delta\, {}_tV_{x:\overline{n}|} \sim \Delta P_{x:\overline{n}|}\, \ddot{a}_{x:\overline{t}|} \frac{D_x}{D_{x+t}} \left(1 - \frac{t}{n}\right), \tag{21}$$

or more simplifying

$$\Delta\, {}_tV_{x:\overline{n}|} \sim \Delta P_{x:\overline{n}|}\, t \left(1 - \frac{t}{n}\right). \tag{22}$$

Example: C. S. O. Table, $i = 2{,}5\,\%$, $i' = 3\,\%$.

$$-\Delta\, {}_{10}V_{30:\overline{30}|} = 0{,}01336,$$

by formula (21) $= 0{,}01349$.

A better approximation is obtained as follows. The following relation is well-known:

$$\Delta\, {}_tV_{x:\overline{n}|} = \left(\frac{d}{d\delta}\, {}_tV_{x:\overline{n}|}\right) v\, \Delta i + \frac{1}{2}\left(\frac{d^2}{d\delta^2}\, {}_tV_{x:\overline{n}|} - \frac{d}{d\delta} P_{x:\overline{n}|}\right) v^2 (\Delta i)^2 + \ldots .$$

The coefficient of $v^2(\Delta i)^2$ of the right hand side is estimated approximately as follows:

$$\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|} = -\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}\left(\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} - \frac{(Ia)_{x+t:\overline{n-t-1}|}}{\ddot{a}_{x+t:\overline{n-t}|}}\right) \sim -\frac{t}{2n}(n-t)$$

(from (9)),

$$\frac{d^2}{d\delta^2}\,{}_tV_{x:\overline{n}|} = -\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}\left(\frac{\ddot{a}^{(2)}_{x+t:\overline{n-t}|}}{\ddot{a}_{x+t:\overline{n-t}|}} - 2\frac{\ddot{a}^{(1)}_{x+t:\overline{n-t}|}}{\ddot{a}_{x+t:\overline{n-t}|}}\frac{\ddot{a}^{(1)}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + 2\left(\frac{\ddot{a}^{(1)}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}\right)^2 - \frac{\ddot{a}^{(2)}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}\right) \sim$$

$$\sim \frac{t(n-t)}{6n}\left\{n - 2t + \frac{t(2n-t)}{2}\,i\left(1 - \frac{2n-t}{12}\,i\right)\right\}$$

(from (9) and (11)).

Therefore we have

$$\Delta\,{}_tV_{x:\overline{n}|} \sim \left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|}\right) v\,\Delta i +$$

$$+ \frac{1}{2}\,\frac{t(n-t)}{6n}\left\{n - 2t + 3 + \frac{t(2n-t)}{2}\,i\left(1 - \frac{2n-t}{12}\,i\right)\right\} v^2(\Delta i)^2. \qquad (23)$$

Example: C.S.O. Table, $i = 2{,}5\,\%$, $i' = 3\,\%$.

$$-\Delta\,{}_{10}V_{30:\overline{30}|} = 0{,}01336\,,$$

$$\left(-\frac{d}{d\delta}\,{}_{10}V_{30:\overline{30}|}\right) v\,\Delta i = 0{,}01360\,,$$

by formula (23) $= 0{,}01336$.

c) Approximation, derived from the mean value theorem. Clearly we have

$$\Delta\,{}_tV_{x:\overline{n}|} \sim \Delta\,{}_tV_{\overline{n}|}\,\frac{\dfrac{d}{d\delta}\,{}_tV_{x:\overline{n}|}}{\dfrac{d}{d\delta}\,{}_tV_{\overline{n}|}} = \Delta\,{}_tV_{\overline{n}|}\,\frac{\dfrac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}\left(\dfrac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} - \dfrac{(Ia)_{x+t:\overline{n-t-1}|}}{\ddot{a}_{x+t:\overline{n-t}|}}\right)}{\dfrac{\ddot{a}_{\overline{n-t}|}}{\ddot{a}_{\overline{n}|}}\left(\dfrac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}} - \dfrac{(Ia)_{\overline{n-t-1}|}}{\ddot{a}_{\overline{n-t}|}}\right)}.$$

5. Mortality rate and Net Level Premium.

Changes in mortality assume many different forms, but here we deal only two forms $q'_x = (1+\beta)\,q_x$ and $q'_x = q_x + c$, where β and c are positive constants.

First of all, we derive an approximation formula which is used in our study. If we define $q'_{x:\overline{n}|}$ in the same way as $q_{x:\overline{n}|}$, clearly we have, when $q'_x = (1+\beta)\, q_x$:

$$n\left(1-\frac{n-1}{2}\, q'_{x:\overline{n}|}\right) = 1+(1-(1+\beta)\, q_x)+(1-(1+\beta)\, q_x)\,(1-(1+\beta)\, q_{x+1})+\ldots =$$

$$= 1+p_x+p_x\, p_{x+1}+\ldots-\beta\{q_x+(q_x+q_{x+1})+\ldots-2\sum qq+\ldots\}+$$
$$+\beta^2\{\sum qq-3\sum qqq+\ldots\}-\beta^3\{\sum qqq-\ldots\}+\ldots.$$

On the other hand, from the definition of $q_{x:\overline{n}|}$,

$$(n-1)\, q_x+(n-2)\, q_{x+1}+\ldots = \frac{n\,(n-1)}{2}\, q_{x:\overline{n}|}+\sum qq\ \ldots.$$

Hence, when q and β are small, omitting $\sum qqq$ and higher products of q, we may have

$$n\left(1-\frac{n-1}{2}\, q'_{x:\overline{n}|}\right) \sim n\left(1-\frac{n-1}{2}\, q_{x:\overline{n}|}\right)-\beta\left(\frac{n\,(n-1)}{2}\, q_{x:\overline{n}|}-\sum qq\right)+\beta^2\sum qq.$$

Assuming that $n\left(1-\dfrac{n-1}{2}\, q_{x:\overline{n}|}\right) = \dfrac{1-(1-q)^n}{q}$,

now we assume though roughly

$$q \sim q_{x:\overline{n}|}\left(1+\frac{n-2}{3}\, q_{x:\overline{n}|}\right)$$

and

$$\sum qq \sim \frac{n\,(n-1)\,(n-2)}{6}\, q^2_{x:\overline{n}|}\left(1+\frac{2}{3}\,(n-2)\, q_{x:\overline{n}|}\right).$$

Then we may have, when β is small

$$q'_{x:\overline{n}|} \sim (1+\beta)\, q_{x:\overline{n}|}\left\{1-\beta\,\frac{n-2}{3}\, q_{x:\overline{n}|}\left(1+\frac{2}{3}\,(n-2)\, q_{x:\overline{n}|}\right)\right\}. \quad (24)$$

Example: C.S.O. Table, $\beta = 100\,\%$.

$$q_{30:\overline{30}|} = 0{,}0063,$$
$$q'_{30:\overline{00}|} = 0{,}01174,$$
by formula (24) $= 0{,}01177$.

When $q'_{x:\overline{n}|}$ at 200% mortality rate is known, of course, "Interpolation" method is useful to estimate the value of $q'_{x:\overline{n}|}$ at mortality rate $q'_x = (1+\beta)\,q_x$. Therefore, it is desirable to append the living number column of 200% mortality rate, to the ordinary mortality table.

(i) $$q'_x = (1+\beta)\,q_x.$$

When $q'_x = q_x + \Delta q_x$, we have clearly (see Appendix)

$$P'_{x:\overline{n}|} - P_{x:\overline{n}|} = \Delta P_{x:\overline{n}|} = \frac{\sum_{0}^{n-1} D_{x+t}\, v\, \Delta q_{x+t}\,(1 - {}_{t+1}V'_{x:\overline{n}|})}{N_x - N_{x+n}}, \tag{25}$$

where $P'_{x:\overline{n}|}$ is the Net Level Premium and ${}_{t+1}V'_{x:\overline{n}|}$ the Premium Reserve at mortality rate q'_x. Therefore when $\Delta q_x = \beta q_x$:

$$\Delta P_{x:\overline{n}|} \sim \beta \frac{\sum_{0}^{n-1} C_{x+t}\,(1 - {}_{t+1}V_{x:\overline{n}|})}{N_x - N_{x+n}}, \tag{26}$$

(assuming that β is small).

From (26) some approximation formulas to $\Delta P_{x:\overline{n}|}$ may be derived.

a) When C_{x+t} (t runs over from 0 to $n-1$) is about constant we may write

$$\Delta P_{x:\overline{n}|} \sim \frac{1}{2}\beta\, P'_{x:\overline{n}|} \sim \frac{1}{2}\beta \frac{l_x - l_{x+n}}{n\, l_x}.$$

(applying the "equated time of payments" method to $P'_{x:\overline{n}|}$).

b) When n is not large, we may assume

$${}_{t+1}V_{x:\overline{n}|} \sim \frac{t+1}{n},$$

hence we have

$$\Delta P_{x:\overline{n}|} \sim \beta \frac{1}{N_x - N_{x+n}} \left(M_x - \frac{R_x - R_{x+n}}{n} \right).$$

c) Here we derive another formula. Clearly

$$\Delta P_{x:\overline{n}|} = \frac{1}{\ddot{a}'_{x:\overline{n}|}} - \frac{1}{\ddot{a}_{x:\overline{n}|}} = \frac{\ddot{a}_{x:\overline{n}|} - \ddot{a}'_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}\,\ddot{a}'_{x:\overline{n}|}},$$

where $\ddot{a}'_{x:\overline{n}|}$ is calculated on the basis of the mortality $q'_x = (1+\beta)\,q_x$.

If we assume $\quad \ddot{a}_{x:\overline{n}|}-\ddot{a}'_{x:\overline{n}|}=\beta z_{x:\overline{n}|}(\ddot{a}_{\overline{n}|}-\ddot{a}_{x:\overline{n}|})$,

then we have

$$\Delta P_{x:\overline{n}|}=\beta(P_{x:\overline{n}|}-P_{\overline{n}|})\frac{\ddot{a}_{\overline{n}|}z_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}-\beta z_{x:\overline{n}|}(\ddot{a}_{\overline{n}|}-\ddot{a}_{x:\overline{n}|})}.$$

From the definition of $z_{x:\overline{n}|}$, applying the "equated time of payments" method to $\ddot{a}_{\overline{n}|}$, $\ddot{a}_{x:\overline{n}|}$ and $\ddot{a}'_{x:\overline{n}|}$, we have, when β is small,

$$z_{x:\overline{n}|}\sim\frac{1}{\beta}\left(\frac{q'_{x:\overline{n}|}}{q_{x:\overline{n}|}}-1\right)\sim\frac{1}{\beta}\frac{\ddot{e}_{x:\overline{n}|}-\ddot{e}'_{x:\overline{n}|}}{n-\ddot{e}_{x:\overline{n}|}},$$

hence

$$\Delta P_{x:\overline{n}|}\sim\beta(P_{x:\overline{n}|}-P_{\overline{n}|})\frac{\ddot{a}_{\overline{n}|}\dfrac{1}{\beta}\left(\dfrac{q'_{x:\overline{n}|}}{q_{x:\overline{n}|}}-1\right)}{\ddot{a}_{x:\overline{n}|}-\left(\dfrac{q'_{x:\overline{n}|}}{q_{x:\overline{n}|}}-1\right)(\ddot{a}_{\overline{n}|}-\ddot{a}_{x:\overline{n}|})}, \tag{27}$$

$$\sim\beta(P_{x:\overline{n}|}-P_{\overline{n}|}). \tag{28}$$

Formula (28) is a well-known interpolation formula, but it should be noticed that when β is small

$$\ddot{a}_{\overline{n}|}\left(\frac{q'_{x:\overline{n}|}}{q_{x:\overline{n}|}}-1\right)\sim\beta\left\{\ddot{a}_{x:\overline{n}|}-\left(\frac{q'_{x:\overline{n}|}}{q_{x:\overline{n}|}}-1\right)(\ddot{a}_{\overline{n}|}-\ddot{a}_{x:\overline{n}|})\right\}.$$

For, if not so, we can not obtain (28). This notice is useful in case of Premium Reserve.

$$q'_x=q_x+c. \tag{ii}$$

From (25) we have

$$\Delta P_{x:\overline{n}|}=\frac{\sum_0^{n-1}D_{x+t}\,v\,\Delta q_{x+t}(1-{}_{t+1}V'_{x:\overline{n}|})}{N_x-N_{x+n}}.$$

Putting $\Delta q_{x+t}=c$,

$$\Delta P_{x:\overline{n}|}\sim cv\frac{\sum_0^{n-1}D_{x+t}(1-{}_{t+1}V_{x:\overline{n}|})}{N_x-N_{x+n}}.$$

(assuming that c is small).

a) Putting ${}_{t+1}V_{x:\overline{n}|}=\dfrac{t+1}{n}$ in this approximation, we have, when x and n are not large

$$\Delta P_{x:\overline{n}|}\sim cv\left(1-\frac{S_x-S_{x+n}-nN_{x+n}}{n(N_x-N_{x+n})}\right). \tag{29}$$

b) Another approximation is obtained as follows. Clearly

$$\sum_0^{n-1} v D_{x+t}\,{}_{t+1}V_{x:\overline{n}|} = \sum_0^{n-1} D_{x+t}\,(P_{x:\overline{n}|} + {}_tV_{x:\overline{n}|}) - \sum_0^{n-1} C_{x+t}\,(1 - {}_{t+1}V_{x:\overline{n}|})\,.$$

Therefore

$$\Delta P_{x:\overline{n}|} = cv\,\frac{\sum\limits_0^{n-1} D_{x+t}\,(1-{}_{t+1}V_{x:\overline{n}|})}{N_x - N_{x+n}} + cv\,\frac{\sum\limits_0^{n-1} D_{x+t}\,({}_{t+1}V_{x:\overline{n}|} - {}_{t+1}V'_{x:\overline{n}|})}{N_x - N_{x+n}}$$

$$\sim c\left(v + \frac{\sum\limits_0^{n-1} C_{x+t}\,(1-{}_{t+1}V_{x:\overline{n}|})}{N_x - N_{x+n}} - \frac{\sum\limits_0^{n-1} D_{x+t}\,({}_tV_{x:\overline{n}|} + P_{x:\overline{n}|})}{N_x - N_{x+n}}\right),$$

(assuming that c is small).

Using (17), (25) and (26) we have

$$\Delta P_{x:\overline{n}|} = c\left(v + \frac{d}{d\delta}P_{x:\overline{n}|} + \lim_{\beta\to 0}\frac{P'_{x:\overline{n}|} - P_{x:\overline{n}|}}{\beta}\right) + cv\sum_0^{n-1} D_{x+t}\,({}_{t+1}V_{x:\overline{n}|} - {}_{t+1}V'_{x:\overline{n}|})$$

$$\sim c\left(v + \frac{d}{d\delta}P_{x:\overline{n}|} + (P_{x:\overline{n}|} - P_{\overline{n}|})\right), \tag{30}$$

where the mortality rate of $P'_{x:\overline{n}|}$ is $q'_x = (1+\beta)\,q_x$ and we assume that c is small (see Appendix).

Example: J^{PM} Table, $i = 4\%$, $c = 0{,}0073$.

$$\Delta P_{30:\overline{30}|} = 0{,}00504\,,$$
by formula (29) $= 0{,}00485$,
by formula (30) $= 0{,}00499$.

6. Mortality rate amd Premium Reserve.

When $q'_x = q_x + \Delta q_x$ we have easily (see Appendix)

$$\Delta P_{x:\overline{n}|} = \frac{\sum\limits_0^{n-1}(1-{}_{t+1}V'_{x:\overline{n}|})\,\Delta q_{x+t}\,D_{x+t}\,v}{N_x - N_{x+n}} = \frac{\sum\limits_0^{n-1}(1-{}_{t+1}V_{x:\overline{n}|})\,\Delta q_{x+t}\,D'_{x+t}\,v}{N'_x - N'_{x+n}},$$

$$\Delta_t V_{x:\overline{n}|} = \frac{\sum\limits_0^{t-1}(\Delta P_{x:\overline{n}|})\,D_{x+t} - \sum\limits_0^{t-1}(1-{}_{t+1}V'_{x:\overline{n}|})\,\Delta q_{x+t}\,D_{x+t}\,v}{D_{x+t}}$$

$$= \frac{\sum\limits_0^{t-1}(\Delta P_{x:\overline{n}|})\,D'_{x+t} - \sum\limits_0^{t-1}(1-{}_{t+1}V_{x:\overline{n}|})\,\Delta q_{x+t}\,D'_{x+t}\,v}{D'_{x+t}}$$

where D', N' and V' are calculated on the basis of $q'_x = q_x + \Delta q_x$.

Therefore we may write

$$P_{x:\overline{n}|}-P_{\overline{n}|}=\frac{\sum_{0}^{n-1}(1-{}_{t+1}V_{\overline{n}|})\,C_{x+t}}{N_x-N_{x+n}},$$

and hence when $q_x'=(1+\beta)\,q_x$ and β is small (see Appendix).

$$\Delta P_{x:\overline{n}|}=\beta\,(P_{x:\overline{n}|}-P_{\overline{n}|})+\frac{\beta}{N_x-N_{x+n}}\left(\sum_{0}^{n-1}C_{x+t}\,({}_{t+1}V_{\overline{n}|}-{}_{t+1}V'_{x:\overline{n}|})\right)$$

$$\sim\beta\,(P_{x:\overline{n}|}-P_{\overline{n}|})+\frac{\beta}{N_x-N_{x+n}}\left(\sum_{0}^{n-1}C_{x+t}\,({}_{t+1}V_{\overline{n}|}-{}_{t+1}V_{x:\overline{n}|})\right).\quad(31)$$

(i) $$q_x'=(1+\beta)\,q_x.$$

In this case, we have

$${}_tV_{x:\overline{n}|}-{}_tV_{\overline{n}|}=\frac{1}{D_{x+t}}\left\{\sum_{0}^{t-1}(P_{x:\overline{n}|}-P_{\overline{n}|})\,D_{x+t}-\sum_{0}^{t-1}C_{x+t}\,(1-{}_{x+t}V_{x:\overline{n}|})\right\},$$

$${}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}=\frac{1}{D_{x+t}}\left\{\sum_{0}^{t-1}(P'_{x:\overline{n}|}-P_{x:\overline{n}|})\,D_{x+t}-\beta\sum_{0}^{t-1}C_{x+t}\,(1-{}_{t+1}V'_{x:\overline{n}|})\right\};$$

hence easily

$${}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}=\{(P'_{x:\overline{n}|}-P_{x:\overline{n}|})-\beta\,(P_{x:\overline{n}|}-P_{\overline{n}|})\}\,\ddot{a}_{x:\overline{t}|}\,\frac{D_x}{D_{x+t}}-$$

$$-\frac{\beta}{D_{x+t}}\left\{\sum_{0}^{t-1}C_{x+t}\,({}_{t+1}V_{\overline{n}|}-{}_{t+1}V'_{x:\overline{n}|})\right\}+\beta\,({}_tV_{x:\overline{n}|}-{}_tV_{\overline{n}|}),$$

therefore

$$\Delta\,{}_tV_{x:\overline{n}|}\sim\beta\,({}_tV_{x:\overline{n}|}-{}_tV_{\overline{n}|})+\quad(32)$$

$$+\beta\left\{\frac{\sum_{0}^{n-1}C_{x+t}\,({}_{t+1}V_{\overline{n}|}-{}_{t+1}V'_{x:\overline{n}|})}{N_x-N_{x+n}}\;\frac{\sum_{0}^{t-1}D_{x+t}}{D_{x+t}}-\frac{\sum_{0}^{t-1}C_{x+t}\,({}_{t+1}V_{\overline{n}|}-{}_{t+1}V'_{x:\overline{n}|})}{D_{x+t}}\right\}.$$

From (32) we see that the simple approximation $\Delta\,{}_tV_{x:\overline{n}|}\sim\beta\,({}_tV_{x:\overline{n}|}-{}_tV_{\overline{n}|}$ is not always good. Really, in case of ${}_tV_{x:\overline{n}|}\sim{}_tV_{\overline{n}|}$ it may result in failure.

(ii) $$q_x'=q_x+c.$$

In this case, we have

$$\Delta\,{}_tV_{x:\overline{n}|}=\frac{1}{D_{x+t}}\left\{\sum_{0}^{t-1}\Delta P_{x:\overline{n}|}\,D_{x+t}-c\sum_{0}^{t-1}(1-{}_{t+1}V_{x:\overline{n}|}-\Delta\,{}_{t+1}V_{x:\overline{n}|})\,D_{x+t}\,v\right\}=\quad(32)$$

$$=\Delta P_{x:\overline{n}|}\,\ddot{a}_{x:\overline{n}|}\,\frac{D_x}{D_{x+t}}-\frac{c\,v\sum_{0}^{t-1}(1-{}_{t+1}V_{x:\overline{n}|})\,D_{x+t}}{D_{x+t}}-c\,v\,\frac{\sum_{0}^{t-1}D_{x+t}\,({}_{t+1}V_{x:\overline{n}|}-{}_{t+1}V'_{x:\overline{n}|})}{D_{x+t}}.$$

Now, carrying the following relations a) and b) to the right hand side of this equation,

$$\text{a)}\quad \frac{v\sum_0^{t-1} D_{x+t}\,{}_{t+1}V_{x:\overline{n}|}}{D_{x+t}} = \frac{\sum_0^{t-1} D_{x+t}\,({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})}{D_{x+t}} - \frac{\sum_0^{t-1} C_{x+t}\,(1-{}_{t+1}V_{x:\overline{n}|}}{D_{x+t}} =$$

$$= \frac{d}{d\delta}\,{}_tV_{x:\overline{n}|} - \left(\frac{d}{d\delta}P_{x:\overline{n}|}\right)\ddot{a}_{x:\overline{t}|}\frac{D_x}{D_{x+t}} + \lim_{\beta\to 0}\frac{{}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}}{\beta} - \lim_{\beta\to 0}\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{\beta}\,\ddot{a}_{x:\overline{t}|}\frac{D_x}{D_{x+t}},$$

(where ${}_tV'_{x:\overline{n}|}$ and $P'_{x:\overline{n}|}$ represent Premium Reserve and Premium respectively at mortality rate $q'_x = (1+\beta)\,q_x$),

$$\text{b)}\quad \varDelta P_{x:\overline{n}|} = c\left(v + \frac{d}{d\delta}P_{x:\overline{n}|} + \lim_{\beta\to 0}\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{\beta}\right) + cv\,\frac{\sum_0^{n-1} D_{x+t}\,({}_{t+1}V_{x:\overline{n}|}-{}_{t+1}V'_{x:\overline{n}|}}{N_x - N_{x+n}},$$

(where $q'_x = q_x + c$, but $P'_{x:\overline{n}|}$ calculated at mortality rate $q'_x = (1+\beta)\,q_x$; see formula (30)), we obtain, when c is small,

$$\varDelta\,{}_tV_{x:\overline{n}|} \sim c\left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|} + \lim_{\beta\to 0}\frac{{}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}}{\beta}\right). \tag{33}$$

We see easily when $t < n$ in (33)

$$\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|} + \lim_{\beta\to 0}\frac{{}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}}{\beta} < 0$$

from follwowing four relations

$$\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|} = -\frac{\sum_0^{n-1}({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})\,D_{x+t}}{N_x - N_{x+n}}\,\ddot{a}_{x:\overline{t}|}\,\frac{D_x}{D_{x+t}} + \frac{\sum_0^{t-1}({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})\,D_{x+t}}{D_{x+t}},$$

$$\lim_{\beta\to 0}\frac{{}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}}{\beta} = \frac{\sum_0^{n-1}(1-{}_{t+1}V_{x:\overline{n}|})\,C_{x+t}}{N_x - N_{x+n}}\,\ddot{a}_{x:\overline{t}|}\,\frac{D_x}{D_{x+t}} - \frac{\sum_0^{t-1}(1-{}_{t+1}V_{x:\overline{n}|})\,C_{x+t}}{D_{x+t}},$$

$$v\sum_0^{t-1} D_{x+t}\,{}_{t+1}V_{x:\overline{n}|} = \sum_0^{t-1} D_{x+t}\,({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|}) - \sum_0^{t-1}(1-{}_{t+1}V_{x:\overline{n}|})\,C_{x+t}$$

and
$${}_tV_{x:\overline{n}|} < {}_{t+1}V_{x:\overline{n}|}.$$

Approximation formulas in this paragraph are also of use in case of two independent mortality tables, particularly if the value of q'_x of one table (denote q'_x table) is near to the value of q_x of another table (denote q_x table). Here we show some of them.

a) $$\ddot{a}_{x:\overline{n}|} - \ddot{a}'_{x:\overline{n}|} \sim \left(\frac{q'_{x:\overline{n}|}}{q_{x:\overline{n}|}} - 1\right)(\ddot{a}_{\overline{n}|} - \ddot{a}_{x:\overline{n}|}) = \frac{\ddot{e}_{x:\overline{n}|} - \ddot{e}'_{x:\overline{n}|}}{n - \ddot{e}_{x:\overline{n}|}}(\ddot{a}_{\overline{n}|} - \ddot{a}_{x:\overline{n}|}).$$

b) $$A'_{x:\overline{n}|} - A_{x:\overline{n}|} \sim \frac{\ddot{e}_{x:\overline{n}|} - \ddot{e}'_{x:\overline{n}|}}{n - \ddot{e}_{x:\overline{n}|}}(A_{x:\overline{n}|} - v^n).$$

c) $P'_{x:\overline{n}|} - P_{x:\overline{n}|} \sim \beta(P_{x:\overline{n}|} - P_{\overline{n}|})$ where

$$\beta \sim \frac{\ddot{e}_{x:\overline{n}|} - \ddot{e}'_{x:\overline{n}|}}{(n - \ddot{e}_{x:\overline{n}|})\left\{1 - \dfrac{\ddot{e}_{x:\overline{n}|} - \ddot{e}'_{x:\overline{n}|}}{n}\left(1 + \dfrac{4}{3n}(n - \ddot{e}_{x:\overline{n}|})\right)\right\}} \sim \frac{\ddot{e}_{x:\overline{n}|} - \ddot{e}'_{x:\overline{n}|}}{n - \ddot{e}_{x:\overline{n}|}},$$

$$(\text{when } q'_{x:\overline{n}|} : q_{x:\overline{n}|} \sim 1).$$

Here $\ddot{e}_{x:\overline{n}|}$, $\ddot{a}_{x:\overline{n}|}$, $A_{x:\overline{n}|}$ and $P_{x:\overline{n}|}$ are calculated on the basis of the mortality table q_x and $\ddot{e}'_{x:\overline{n}|}$, $\ddot{a}'_{x:\overline{n}|}$, $A'_{x:\overline{n}|}$ and $P'_{x:\overline{n}|}$ are calculated on the basis of the mortality table q'_x.

II. Inequalites

7. Some well-known inequalities.

(i) $$\frac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}} > \frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}},$$

$$(\text{when } \ddot{a}_{x:\overline{n}|} > \ddot{a}_{x+t:\overline{n-t}|},\ t = 1, 2, \ldots, n-1).$$

From the definition of $(IA)_{x:\overline{n}|}$ we have

$$\frac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}} = \frac{D_x}{D_x}\frac{A_{x:\overline{n}|}}{A_{x:\overline{n}|}} + \frac{D_{x+1}}{D_x}\frac{A_{x+1,\overline{n-1}|}}{A_{x:\overline{n}|}} + \ldots > \frac{1}{D_x}(D_x + D_{x+1} + \ldots) = \ddot{a}_{x:\overline{n}|},$$

$$(\text{when } \ddot{a}_{x:\overline{n}|} > \ddot{a}_{x+t:\overline{n-t}|}).$$

On the other hand

$$\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{D_{x+1}}{D_x}\frac{\ddot{a}_{x+1:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{D_{x+2}}{D_x}\frac{\ddot{a}_{x+2:\overline{n-2}|}}{\ddot{a}_{x:\overline{n}|}} + \ldots < \ddot{a}_{x:\overline{n}|},$$

$$(\text{when } \ddot{a}_{x:\overline{n}|} > \ddot{a}_{x+t:\overline{n-t}|}).$$

Therefore we have

$$\frac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}} > \frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}},$$

$$(\text{when } \ddot{a}_{x:\overline{n}|} > \ddot{a}_{x+t:\overline{n-t}|},\ t = 1, 2, \ldots, n-1).$$

From this inequality we obtain, when

$$\frac{d}{d\delta} P_{x:\overline{n}|} = -\frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \left(\frac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}} - \frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} \right) < 0 .$$

(ii) $$\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} > \frac{(Ia)_{x+t:\overline{n-t-1}|}}{\ddot{a}_{x+t:\overline{n-t}|}}, \quad (t = 1, 2, \ldots, n-1).$$

Clearly

$$\ddot{a}_{x:\overline{n}|} = 1 + p_x v + p_x p_{x+1} v^2 + \ldots + p_x p_{x+1} \cdots p_{x+t-1} v^t \ddot{a}_{x+t:\overline{n-t}|},$$

$$(Ia)_{x:\overline{n-1}|} = a_{x:\overline{n-1}|} + p_x v\, a_{x+1:\overline{n-2}|} + \ldots + p_x p_{x+1} \cdots p_{x+t-1} v^t (Ia)_{x+t:\overline{n-t-1}|},$$

hence $$\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} - \frac{(Ia)_{x+t:\overline{n-t-1}|}}{\ddot{a}_{x+t:\overline{n-1}|}} = \frac{1}{\ddot{a}_{x:\overline{n}|}\ddot{a}_{x+t:\overline{n-t}|}} \cdot$$

$$\cdot \left\{ a_{x:\overline{n-1}|}\ddot{a}_{x+t:\overline{n-t}|} + p_x v\, a_{x+1:\overline{n-2}|}\ddot{a}_{x+t:\overline{n-t}|} + \ldots - (Ia)_{x+t:\overline{n-t-1}|} - p_x v (Ia)_{x+t:\overline{n-t-1}|} \cdots \right\}.$$

Therefore, when $\ddot{a}_{x+s:\overline{n-s}|} > \ddot{a}_{x+t:\overline{n-t}|} > \ddot{a}_{x+t+\nu:\overline{n-t-\nu}|}$ for $0 \leqq s < t$ and $1 \leqq \nu \leqq n-t-1$ we have

$$\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} > \frac{(Ia)_{x+t:\overline{n-t-1}|}}{\ddot{a}_{x+t:\overline{n-t}|}}.$$

Hence we obtain under the same conditions

$$\frac{d}{d\delta}\, {}_tV_{x:\overline{n}|} = -\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \left(\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} - \frac{(Ia)_{x+t:\overline{n-t-1}|}}{\ddot{a}_{x+t:\overline{n-t}|}} \right) < 0 .$$

8. Steffensen's inequality.

Let $f(\nu)$ be a constantly decreasing function and let $\varphi(\nu)$ and $\psi(\nu)$ satisfy the following conditions

$$\frac{\sum_{\alpha}^{\gamma} \varphi(\nu)}{\sum_{\alpha}^{\beta} \varphi(\nu)} \geqq \frac{\sum_{\alpha}^{\gamma} \psi(\nu)}{\sum_{\alpha}^{\beta} \psi(\nu)}, \tag{34}$$

for all γ, where $\alpha \leqq \gamma \leqq \beta$ and all of them are integers. Then with the exception when (34) always hold equality, we have

$$\frac{\sum_{\alpha}^{\beta} \varphi(\nu) f(\nu)}{\sum_{\alpha}^{\beta} \varphi(\nu)} > \frac{\sum_{\alpha}^{\beta} \psi(\nu) f(\nu)}{\sum_{\alpha}^{\beta} \psi(\nu)} \tag{35}$$

This inequality is useful in our study.

(i) $$\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} < \frac{n-1}{2}.$$

In (34) if we put $\alpha = 0$, $\beta = n-1$, $\varphi(\nu) = 1$, $\psi(\nu) = \nu$, then for $\nu < n-1$ we have

$$\frac{\sum_{0}^{\nu} \varphi(\nu)}{\sum_{0}^{n-1} \varphi(\nu)} = \frac{\nu+1}{n},$$

$$\frac{\sum_{0}^{\nu} \psi(\nu)}{\sum_{0}^{n-1} \psi(\nu)} = \frac{\nu+1}{n}\,\frac{\nu}{n-1} < \frac{\nu+1}{n}.$$

Therefore from (35), putting $f(\nu) = \dfrac{D_{x+\nu}}{D_x}$ we have

$$\frac{\ddot{a}_{x:\overline{n}|}}{n} > \frac{(Ia)_{x:\overline{n-1}|}}{\frac{n}{2}(n-1)},$$

in consequence $$\frac{n-1}{2} > \frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}. \tag{36}$$

In the same way we have $\dfrac{n-1}{2} > \dfrac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}}$.

(ii) $$\frac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}} > \frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}.$$

When $m < n$ easily we have $\dfrac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} > \dfrac{\ddot{a}_{\overline{m}|}}{\ddot{a}_{x:\overline{m}|}}$.

Therefore, in (35) putting $f(\nu) = -\nu$ we obtain

$$-\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} > -\frac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}},$$

in consequence $$\frac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}} > \frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}.$$

(iii) $$\frac{\ddot{a}_{\overline{n}|}}{\ddot{a}'_{\overline{n}|}} > \frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}}, \qquad (i' > i).$$

The following inequality is clear,

$$\frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} > \frac{\ddot{a}_{\overline{m}|}}{\ddot{a}_{x:\overline{m}|}}, \qquad (n > m).$$

Therefore we have when $i' > i$

$$\frac{\ddot{a}_{\overline{n}|}}{\ddot{a}'_{\overline{n}|}} > \frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}}, \quad \left(\text{putting } f(\nu) = \left(\frac{v'}{v}\right)^{\nu} \text{ in } (35)\right).$$

Now clearly when $i' > i$ for any positive number t, $(t < n-1)$

$$\frac{\ddot{a}_{\overline{n}|}}{\ddot{a}'_{\overline{n}|}} > \frac{\ddot{a}_{\overline{n-t}|}}{\ddot{a}'_{\overline{n-t}|}} > 1, \tag{37}$$

and

$$\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}} > 1, \qquad (n \geqq 2).$$

Hence we may choose k', k such as

$$\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}} = \frac{\ddot{a}_{\overline{n-k'}|}}{\ddot{a}'_{\overline{n-k'}|}}, \quad \left(\ddot{a}_{\overline{n-k}|} = \frac{1-v^{n-k}}{1-v}\right),$$

$$\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{e}_{x:\overline{n}|}} = \frac{\ddot{a}_{\overline{n-k}|}}{\ddot{a}'_{\overline{n-k}|}},$$

and

$$k' \sim \frac{n-\ddot{e}_{x:\overline{n}|}}{3}\left(1+\frac{n}{6}(i+i')\right).$$

When k is given at interest rate i, we have the approximation formula to $\ddot{a}'_{x:\overline{n}|}$ at new interest i' as

$$\ddot{a}'_{x:\overline{n}|} \sim \ddot{e}_{x:\overline{n}|}\frac{\ddot{a}'_{\overline{n-k'}|}}{n-k'}, \quad \text{where } k' = k + \frac{n-\ddot{e}_{x:\overline{n}|}}{18}(i'-i).$$

We notice from

$$\frac{\ddot{a}_{\overline{n}|}}{\ddot{a}'_{\overline{n}|}} > \frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}}, \qquad (i' > i),$$

putting $i' = i$, $i = 0$, we have

$$\ddot{a}_{x:\overline{n}|} > \frac{\ddot{e}_{x:\overline{n}|}}{n}\ddot{a}_{\overline{n}|}.$$

(iv)

$$v^{\frac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}}} > A_{x:\overline{n}|} > v^{\ddot{e}_{x:\overline{n}|}}.$$

Easily we may choose s, $(0 < s)$, such as

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_{\overline{n-s}|}.$$

When $i' > i$ using relations

$$\frac{D_{x+t}}{D_x}\,\ddot{a}_{x+t:\overline{n-t}|} \leqq v^t\,\ddot{a}_{\overline{n-s-t}|}, \quad (t = 0, 1, \ldots, n-1).$$

we have $$\ddot{a}'_{x:\overline{n}|} < \ddot{a}'_{\overline{n-s}|}, \qquad (i' > i),$$

in consequence $$\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}} > \frac{\ddot{a}_{\overline{n-s}|}}{\ddot{a}'_{\overline{n-s}|}}$$

and $$A'_{x:\overline{n}|} > v'^{\,n-s}$$

hence clearly we see $k < s$ and

$$\frac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}} < n - s.$$

Therefore we have $$v^{\frac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}}} > A_{x:\overline{n}|} > v^{\mathring{e}_{x:\overline{n}|}}, \qquad (\mathring{e}_{x:\overline{n}|} > n - s).$$

The value of s is obtained from

$$A_{x:\overline{n}|} = v^{n-s}.$$

(v) $(1+i')P_{x:\overline{n}|(i',q')} - (1+i)P_{x:\overline{n}|(i,q')} > (1+i')P_{x:\overline{n}|(i',q)} - (1+i)P_{x:\overline{n}|(i,q)}$,

where $i' > i$ and $q'_x = (1+\beta)\,q_x > q_x$.

Under the conditions that $q_{x+t}\,(1 - {}_{t+1}V_{x:\overline{n}|})$ decrease constantly and ${}_tV_{x:\overline{n}|(i')} < {}_tV_{x:\overline{n}|(i)}$ for $i' > i$, we have the above inequality using Steffensen's inequality. Here we show it:

$$P_{x:\overline{n}|(i',q')} - P_{x:\overline{n}|(i',q)} = \frac{\beta \sum_0^{n-1} D_{x+t\,(i',q')}\, v'\, q_{x+t}\,(1 - {}_{t+1}V_{x:\overline{n}|(i',q')})}{\ddot{a}_{x:\overline{n}|(i',q')}\, D_{x\,(i',q')}} >$$

$$> \frac{\beta \sum_0^{n-1} D_{x+t\,(i',q')}\, v'\, q_{x+t}\,(1 - {}_{t+1}V_{x:\overline{n}|(i,q)})}{\ddot{a}_{x:\overline{n}|(i',q')}\, D_{x\,(i',q')}}$$

(see Appendix).

Now clearly $$\frac{\ddot{a}_{x:\overline{t}|(i',q')}}{\ddot{a}_{x:\overline{n}|(i',q')}} \geqq \frac{\ddot{a}_{x:\overline{t}|(i,q')}}{\ddot{a}_{x:\overline{n}|(i,q')}}, \quad (i' > i,\ t = 0, 1, \ldots, n-1),$$

where (i,q) represents the calculation basis.

Hence using (35), we have

$$\frac{\beta \sum_{0}^{n-1} D_{x+t\,(i',q')}\, v'\, q_{x+t}\,(1-{}_{t+1}V_{x:\overline{n}|(i,q)})}{\ddot{a}_{x:\overline{n}|(i',q')}\, D_{x\,(i',q')}} >$$

$$> \frac{v'\,\beta \sum_{0}^{n-1} D_{x+t\,(i,q')}\, v\, q_{x+t}\,(1-{}_{t+1}V_{x:\overline{n}|(i,q)})}{v\,\ddot{a}_{x:\overline{n}|(i,q')}\, D_{x\,(i,q')}} = \frac{v'}{v}\,(P_{x:\overline{n}|(i,q')} - P_{x:\overline{n}|(i,q)})\,.$$

Therefore we have our inequality. Moreover, we may write usually roughly speaking for about $n < \frac{2}{i}$)

$$P_{x:\overline{n}|(i,q')} - P_{x:\overline{n}|(i',q')} < P_{x:\overline{n}|(i,q)} - P_{x:\overline{n}|(i',q)}\,.$$

9. Jensen's inequality for the convex function.

Let $\varphi(t)$ be a continuous function such as $\frac{d^2}{dt^2}\,\varphi(t)$ exists and >0.

Then

$$\varphi\left(\frac{\sum_{1}^{n} t\,p_t}{\sum_{1}^{n} p_t}\right) < \frac{\sum_{1}^{n} p_t\,\varphi(t)}{\sum_{1}^{n} p_t}\,, \qquad (38)$$

where $p_1, p_2, \ldots, p_n > 0$.

Applying (38) we have easily following well-known inequalities:

a) As $\ddot{a}_{x:\overline{n}|}$ is convex in respect to i, when $j' > i'$ and $j' - j = i' - i$,

$$\ddot{a}_{x:\overline{n}|(j')} - \ddot{a}_{x:\overline{n}|(j)} > \ddot{a}_{x:\overline{n}|(i')} - \ddot{a}_{x:\overline{n}|(i)}\,,$$

or

$$\ddot{a}_{x:\overline{n}|(j)} - \ddot{a}_{x:\overline{n}|(j')} < \ddot{a}_{x:\overline{n}|(i)} - \ddot{a}_{x:\overline{n}|(i')}\,.$$

b)

$$\frac{\sum S_r\, v^{n_r}}{\sum S_r} > v^{\frac{\Sigma S_r n_r}{\Sigma S_r}}\,, \qquad \text{therefore } n < \frac{\sum S_r\, n_r}{\sum S_r}\,,$$

where n is the equated time of payments.

c) As A_x is convex in respect to i, clearly $A_x > v^{\mathring{e}_x}$, and hence easily $\ddot{a}_x < \ddot{a}_{\overline{\mathring{e}_x}|}$ we obtain.

d) $P_{x:\overline{n}|}$ is convex in respect to i when

$$\ddot{a}_{x+t:\overline{n-t}|} > \ddot{a}_{x+t+1:\overline{n-t-1}|}\,, \qquad (t = 0, 1, 2, \ldots, n-2)\,.$$

We noticed in paragraph 3

$$\frac{\sum_0^{n-1}\left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n|}}+\frac{d}{d\delta}P_{x:\overline{n|}}\right)D_{x+t}}{N_x-N_{x+n}}=\frac{1}{2}\frac{d}{d\delta}P_{x:\overline{n|}}-\frac{1}{2}\frac{d^2}{d\delta^2}P_{x:\overline{n|}}.$$

Therefore, when $\frac{d}{d\delta}\,{}_tV_{x:\overline{n|}}<0,\quad (t=1,2,\ldots,n-1),$

easily we have $\frac{d^2}{d\delta^2}P_{x:\overline{n|}}>0.$

Hence we obtain $P_{x:\overline{n|}(j)}-P_{x:\overline{n|}(j')}<P_{x:\overline{n|}(i)}-P_{x:\overline{n|}(i')}$,
when $j'>i'$ and $j'-j=i'-i$.

e) ${}_tV_{x:\overline{n|}}$ is convex in respect to i, when $0<t<T$
$\left(\text{ordinarily } T \text{ is larger than } \frac{n-1}{2}\left(1+\frac{n}{8}i\right)\right)$,
and after then in $[T<t<n]$ becomes concave in respect to i. The reason is as follows:

Now the following relation is clear.

$$\frac{d}{d\delta}\,{}_tV_{x:\overline{n|}}=\frac{d}{d\delta}P_{x:\overline{n|}}\frac{\sum_0^{t-1}D_{x+t}}{D_{x+t}}+\frac{\sum_0^{t-1}({}_tV_{x:\overline{n|}}+P_{x:\overline{n|}})\,D_{x+t}}{D_{x+t}}.$$

Using this relation and its differential, we have

$$\frac{1}{2}\left(\frac{d^2}{d\delta^2}\,{}_tV_{x:\overline{n|}}-\frac{d}{d\delta}\,{}_tV_{x:\overline{n|}}\right)=$$

$$=\frac{1}{2}\left(\frac{d^2}{d\delta^2}P_{x:\overline{n|}}-\frac{d}{d\delta}P_{x:\overline{n|}}\right)\frac{\sum_0^{t-1}D_{x+t}}{D_{x+t}}+\frac{\sum_0^{t-1}\left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n|}}+\frac{d}{d\delta}P_{x:\overline{n|}}\right)D_{x+t}}{D_{x+t}}.$$

Hence

$$\frac{1}{2}\frac{d^2}{d\delta^2}\,{}_tV_{x:\overline{n|}}=\frac{1}{2}\left(\frac{d^2}{d\delta^2}P_{x:\overline{n|}}-\frac{d}{d\delta}P_{x:\overline{n|}}+\frac{d}{d\delta}P_{x:\overline{n|}}\right)\frac{\sum_0^{t-1}D_{x+t}}{D_{x+t}}-$$

$$-\frac{1}{D_{x+t}}\left\{-\sum_0^{t-1}\left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n|}}+\frac{d}{d\delta}P_{x:\overline{n|}}+\frac{1}{2}({}_tV_{x:\overline{n|}}+P_{x:\overline{n|}})\right)\right\}.$$

On the other hand clearly

$$\frac{1}{2}\left(\frac{d^2}{d\delta^2}P_{x:\overline{n}|}-\frac{d}{d\delta}P_{x:\overline{n}|}\right)=-\frac{\sum_0^{n-1}\left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|}+\frac{d}{d\delta}P_{x:\overline{n}|}\right)D_{x+t}}{N_x-N_{x+n}},$$

and

$$\frac{d}{d\delta}P_{x:\overline{n}|}=-\frac{\sum_0^{n-1}({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})\,D_{x+t}}{N_x-N_{x+n}}.$$

Moreover the positive value

$$-\left(\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|}+\frac{d}{d\delta}P_{x:\overline{n}|}+\tfrac{1}{2}({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})\right)=$$

$$=-\left(\frac{D_{x+t+1}}{D_{x+t}}\frac{d}{d\delta}\,{}_{t+1}V_{x:\overline{n}|}-\tfrac{1}{2}({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})\right)$$

is about $\frac{v}{2}$ at $t=0$, and then constantly increases till the maximum at $t\sim\frac{n-1}{2}\left(1+\frac{n}{8}i\right)$, after then decreases constantly till about $\frac{v}{2}$ at $t=n-1$, $\left(\text{assuming that } \frac{d}{d\delta}\,{}_tV_{x:\overline{n}|}<0\right)$.

Therefore from the theory of the mean value when $t<T$ $\left(T \text{ is larger than } \frac{n-1}{2}\left(1+\frac{n}{8}i\right)\right)$ we have

$$\frac{d^2}{d\delta^2}\,{}_tV_{x:\overline{n}|}>0$$

and when $T<t<n$ in consequence

$$\frac{d^2}{d\delta^2}\,{}_tV_{x:\overline{n}|}<0,\quad \left(\text{we assume } \frac{d}{d\delta}\,{}_tV_{x:\overline{n}|}<0\right).$$

Really, it is easy to verify $\frac{d^2}{d\delta^2}\,{}_{n-1}V_{x:\overline{n}|}<0$.

The estimation of the value T is difficult, and experiences tell us in case of ordinary interest rate $T\sim 0{,}8$ to $0{,}9$.

(i) $$\frac{\ddot{a}'_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}>\left(\frac{v'}{v}\right)^{\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}},\quad (n\geqq 2).$$

In (38), putting $\varphi(t) = \left(\frac{v'}{v}\right)^t$

we obtain directly this inequality

or
$$\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}} < \left(\frac{v}{v'}\right)^{\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}} . \tag{39}$$

Therefore from (36) when $i' > i$

$$\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}} < \left(\frac{v}{v'}\right)^{\frac{n-1}{2}}, \qquad (n \geqq 2) . \tag{40}$$

In the same way, when $i' > i$ we have

$$\frac{A_{x:\overline{n}|}}{A'_{x:\overline{n}|}} < \left(\frac{v}{v'}\right)^{\frac{(IA)_{x:\overline{n}|}}{A_{x:\overline{n}|}}} < \left(\frac{v}{v'}\right)^{n-s} < \left(\frac{v}{v'}\right)^{\mathring{e}_{x:\overline{n}|}} .$$

(ii)
$$P'_{x:\overline{n+m}|} - P_{x:\overline{n+m}|} > P'_{x:\overline{n}|} - P_{x:\overline{n}|}, \qquad (i' > i),$$

where $P'_{x:\overline{n}|}$ represents Net Level Premium at interest rate i', and m is a positive integer.

As $i' > i$ we may write the obove inequality in the following way:

$$P_{x:\overline{n+m}|} - P'_{x:\overline{n+m}|} < P_{x:\overline{n}|} - P'_{x:\overline{n}|} .$$

In case $m = 1$ using (40) easily we have

$$P_{x:\overline{n+1}|} - P'_{x:\overline{n+1}|} < P_{x:\overline{n}|} - P'_{x:\overline{n}|} .$$

Therefore generally we may have the inequality (ii).

(iii)
$$\frac{1 - {}_tV'_{x:\overline{n}|}}{1 - {}_tV_{x:\overline{n}|}} > \frac{1 - {}_tV'_{x:\overline{n+m}|}}{1 - {}_tV_{x:\overline{n+m}|}} > \frac{1 - {}_tV'_x}{1 - {}_tV_x}, \quad (i' > i) \tag{41}$$

where ${}_tV'_{x:\overline{n}|}$ represents Premium Reserve at interest rate i', and m is a positive integer.

Using $\ddot{a}_{x+1:\overline{n-1}|} = \frac{1}{p_x v}(\ddot{a}_{x:\overline{n}|} - 1)$ and (40) we have when $i' > i$

$$\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}} \frac{\ddot{a}'_{x+1:\overline{n-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}} > \frac{\ddot{a}_{x:\overline{n+1}|}}{\ddot{a}'_{x:\overline{n+1}|}} \frac{\ddot{a}'_{x+1:\overline{n}|}}{\ddot{a}_{x+1:\overline{n}|}},$$

in consequence
$$\frac{1 - {}_1V'_{x:\overline{n}|}}{1 - {}_1V_{x:\overline{n}|}} > \frac{1 - {}_1V'_{x:\overline{n+1}|}}{1 - {}_1V_{x:\overline{n+1}|}} .$$

Therefore $$\frac{1-{}_tV'_{x:\overline{n}|}}{1-{}_tV_{x:\overline{n}|}} > \frac{1-{}_tV'_{x:\overline{n+1}|}}{1-{}_tV_{x:\overline{n+1}|}},$$

and generally we have when $i' > i$ the inequality (iii).

From (41) we obtain the following inequalities when $i' > i$:

$$\begin{gathered}
\frac{\ddot{a}_{x:\overline{t+1}|}}{\ddot{a}'_{x:\overline{t+1}|}} > \frac{\ddot{a}_{x:\overline{m+t+1}|}}{\ddot{a}'_{x:\overline{m+t+1}|}} \frac{\ddot{a}'_{x+t:\overline{m+1}|}}{\ddot{a}_{x+t:\overline{m+1}|}}, \\
\frac{\ddot{a}_{x:\overline{t+1}|}}{\ddot{a}'_{x:\overline{t+1}|}} \frac{\ddot{a}'_{x+t:\overline{t+1}|}}{\ddot{a}_{x+t:\overline{t+1}|}} > \frac{\ddot{a}_{x:\overline{2t+1}|}}{\ddot{a}'_{x:\overline{2t+1}|}}, \\
\frac{\ddot{a}_{x:\overline{t+1}|}}{\ddot{a}'_{x:\overline{t+1}|}} \frac{\ddot{a}_{x+t:\overline{t}|}}{\ddot{a}'_{x+t:\overline{t}|}} > \frac{\ddot{a}_{x:\overline{2t}|}}{\ddot{a}'_{x:\overline{2t}|}}.
\end{gathered} \tag{42}$$

(iv) $$\frac{1-{}_tV_{x:\overline{n}|\,(q')}}{1-{}_tV_{x:\overline{n}|}} > \frac{1-{}_tV_{x:\overline{n+m}|\,(q')}}{1-{}_tV_{x:\overline{n+m}|}} > \frac{1-{}_tV_{x\,(q')}}{1-{}_tV_x}. \tag{43}$$

This inequalities is held, under the conditions

$$1 \leqq \frac{p_x}{p'_x} \leqq \frac{p_{x+1}}{p'_{x+1}} \leqq \cdots \leqq \frac{p_{x+t}}{p'_{x+t}} \leqq \frac{p_{x+t+1}}{p'_{x+t+1}} \leqq \cdots.$$

For the proof of (41), the inequality

$$\left(\frac{v}{v'}\right)^{n-1} > \frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}'_{x:\overline{n}|}} \frac{\ddot{a}_{x+1:\overline{n}|}}{\ddot{a}'_{x+1:\overline{n}|}}, \qquad (i' > i;\ n \geqq 2),$$

plays a leading part, and now the inequality

$$\frac{p_{x+1}\,p_{x+2}\,p_{x+3}\cdots p_{x+n-1}}{p'_{x+1}\,p'_{x+2}\,p'_{x+3}\cdots p'_{x+n-1}} > \frac{\ddot{a}_{x:\overline{n}|}\,\ddot{a}_{x+1:\overline{n}|}}{\ddot{a}_{x:\overline{n}|\,(q')}\,\ddot{a}_{x+1:\overline{n}|\,(q')}}$$

is the basis of the proof.

It is troublesome to prove (43) directly, but the following two inequalities are proved easily, under the given conditions.

$$\frac{p_{x+1}\,p_{x+3}\,p_{x+5}\cdots p_{x+2m-1}}{p'_{x+1}\,p'_{x+3}\,p'_{x+5}\cdots p'_{x+2m-1}} > \frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|\,(q')}}, \tag{44}$$

where when n is even $p_{x+2m-1} = p_{x+n-1}$,

when n is odd $p_{x+2m-1} = p_{x+n-2}$,

$$\frac{p_{x+2}\,p_{x+4}\,p_{x+6}\cdots p_{x+2m}}{p'_{x+2}\,p'_{x+4}\,p'_{x+6}\cdots p'_{x+2m}} > \frac{\ddot{a}_{x+1:\overline{n}|}}{\ddot{a}_{x+1:\overline{n}|(q')}} \tag{45}$$

where when n is even $p_{x+2m} = p_{x+n-2}$,
when n is odd $p_{x+2m} = p_{x+n-1}$.

Therefore we have (41) under the given conditions, and the inequality (43) is proved in the same way as (41).

Here we notice that, when $i' > i$ ordinarily

$$\frac{1-{}_tV'_x}{1-{}_tV_x} > 1 \quad \text{is held, but}$$

$$\frac{1-{}_tV_{x(q')}}{1-{}_tV_x} > 1 \quad \text{is assured not at all under the given conditions.}$$

From (43) we have under the same conditions

$$\frac{\ddot{a}_{x:\overline{t+1}|}}{\ddot{a}_{x:\overline{t+1}|(q')}}\,\frac{\ddot{a}_{x+t:\overline{m+1}|}}{\ddot{a}_{x+t:\overline{m+1}|(q')}} > \frac{\ddot{a}_{x:\overline{m+t+1}|}}{\ddot{a}_{x:\overline{m+t+1}|(q')}},$$

$$\frac{\ddot{a}_{x:\overline{t+1}|}}{\ddot{a}_{x:\overline{t+1}|(q')}}\,\frac{\ddot{a}_{x+t:\overline{t+1}|}}{\ddot{a}_{x+t:\overline{t+1}|(q')}} > \frac{\ddot{a}_{x:\overline{2t+1}|}}{\ddot{a}_{x:\overline{2t+1}|(q')}},$$

$$\frac{\ddot{a}_{x:\overline{t+1}|}}{\ddot{a}_{x:\overline{t+1}|(q')}}\,\frac{\ddot{a}_{x+t:\overline{t}|}}{\ddot{a}_{x+t:\overline{t}|(q')}} > \frac{\ddot{a}_{x:\overline{2t}|}}{\ddot{a}_{x:\overline{2t}|(q')}}.$$

On the other hand clearly

$$\frac{p_{x+1}}{p'_{x+1}} > \frac{\ddot{a}_{x:\overline{3}|}}{\ddot{a}_{x:\overline{3}|(q')}},$$

$$\frac{p_{x+2}}{p'_{x+2}} > \frac{\ddot{a}_{x+1:\overline{3}|}}{\ddot{a}_{x+1:\overline{3}|(q')}},$$

$$\frac{p_{x+3}}{p'_{x+3}} > \frac{\ddot{a}_{x+2:\overline{3}|}}{\ddot{a}_{x+2:\overline{3}|(q')}} \quad \text{and so on.}$$

Therefore, if we assume (43), reversely we have the two inequalities (44) and (45) above mentioned.

(v)
$$P_{x:\overline{n+m}|(q')} - P_{x:\overline{n+m}|} > P_{x:\overline{n}|(q')} - P_{x:\overline{n}|}\cdot$$
$$\cdot\left(1 \leqq \frac{p_x}{p'_x} \leqq \frac{p_{x+1}}{p'_{x+1}} \leqq \cdots \leqq \frac{p_{x+t}}{p'_{x+t}} \leqq \cdots\right).$$

Clearly, under the given conditions

$$\frac{\ddot{a}_{x:\overline{2}|}}{\ddot{a}_{x:\overline{2}|(q')}}\,\frac{\ddot{a}_{x+1:\overline{n}|}}{\ddot{a}_{x+1:\overline{n}|(q')}} > \frac{\ddot{a}_{x:\overline{n+1}|}}{\ddot{a}_{x:\overline{n+1}|(q')}},$$

hence we have $\qquad \dfrac{p_x\, p_{x+1} \cdots p_{x+n-1}}{p'_x\, p'_{x+1} \cdots p'_{x+n-1}} > \dfrac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|(q')}}\,\dfrac{\ddot{a}_{x:\overline{n+1}|}}{\ddot{a}_{x:\overline{n+1}|(q')}}.$

From this inequality easily we obtain

$$P_{x:\overline{n+1}|(q')} - P_{x:\overline{n+1}|} > P_{x:\overline{n}|(q')} - P_{x:\overline{n}|},$$

and generally, under the given conditions

$$P_{x:\overline{n+m}|(q')} - P_{x:\overline{n+m}|} > P_{x:\overline{n}|(q')} - P_{x:\overline{n}|}.$$

(vi) Lastly we notice that the conditions

$$1 \leqq \frac{p_x}{p'_x} \leqq \frac{p_{x+t}}{p'_{x+1}} \leqq \cdots \leqq \frac{p_{x+t}}{p'_{x+t}} \leqq \cdots$$

is reduced, in case of $q'_x = (1+\beta)\, q_x$ or $q'_x = q_x + c$ to the following well-used condition

$$q_x \leqq q_{x+1} \leqq \cdots \leqq q_{x+t} \leqq q_{x+t+1} \leqq \cdots.$$

Appendix

When we assume two calculation bases (i, q_x) and $(i + \Delta i, q_x + \Delta q_x)$, following relations are clear $(t = 0, 1, 2, \ldots, n-1)$:

$$(P_{x:\overline{n}|} + {}_tV_{x:\overline{n}|})(1+i) - q_{x+t} = {}_{t+1}V_{x:\overline{n}|}(1 - q_{x+t}). \tag{A}$$

$$\begin{aligned}(P_{x:\overline{n}|} + \Delta P_{x:\overline{n}|} + {}_tV_{x:\overline{n}|} + \Delta\, {}_tV_{x:\overline{n}|})(1 + i + \Delta i) - q_{x+t} - \Delta q_{x+t} = \\ = ({}_{t+1}V_{x:\overline{n}|} + \Delta\, {}_{x+1}V_{x:\overline{n}|})(1 - q_{x+t} - \Delta q_{x+t}).\end{aligned} \tag{B}$$

Hence (B) — (A)

$$\begin{aligned}&(\Delta\, {}_tV_{x:\overline{n}|} + \Delta P_{x:\overline{n}|})(1+i) - \Delta\, {}_{t+1}V_{x:\overline{n}|}\, p_{x+t} = \\ &= (1 - {}_{t+1}V_{x:\overline{n}|} - \Delta\, {}_{t+1}V_{x:\overline{n}|})\, \Delta q_{x+t} - ({}_tV_{x:\overline{n}|} + \Delta\, {}_tV_{x:\overline{n}|} + P_{x:\overline{n}|} + \Delta P_{x:\overline{n}|})\, \Delta i.\end{aligned}$$

Hence clearly

$$\begin{aligned}&D_{x+t}\, \Delta\, {}_xV_{x:\overline{n}|} - D_{x+t+1}\, \Delta\, {}_{t+1}V_{x:\overline{n}|} + D_{x+t}\, \Delta P_{x:\overline{n}|} = \\ &= (1 - {}_{t+1}V_{x:\overline{n}|} - \Delta\, {}_{t+1}V_{x:\overline{n}|})\, \Delta q_{x+t}\, D_{x+t}\, v - ({}_tV_{x:\overline{n}|} + \Delta\, {}_tV_{x:\overline{n}|} + P_{x:\overline{n}|} + \Delta P_{x:\overline{n}|} D_{x+t}\, v\, \Delta i.\end{aligned} \tag{C}$$

Therefore, carrying $\Delta_0 V_{x:\overline{n}|}=0$ and $\Delta_n V_{x:\overline{n}|}=0$ to (C) we easily obtain

$$\Delta P_{x:\overline{n}|}=\frac{1}{N_x-N_{x+n}}\left\{\sum_0^{n-1}(1-{}_{t+1}V_{x:\overline{n}|}-\Delta\,{}_{t+1}V_{x:\overline{n}|})\,\Delta q_{x+t}\,D_{x+t}\,v-\right.$$
$$\left.-\Delta i\sum_0^{n-1}({}_tV_{x:\overline{n}|}+\Delta\,{}_tV_{x:\overline{n}|}+P_{x:\overline{n}|}+\Delta P_{x:\overline{n}|})\,D_{x+t}\,v\right\},$$

and

$$\Delta\,{}_tV_{x:\overline{n}|}=\frac{1}{D_{x+t}}\left\{\Delta P_{x:\overline{n}|}\sum_0^{t-1}D_{x+t}-\sum_0^{t-1}(1-{}_{t+1}V_{x:\overline{n}|}-\Delta\,{}_{t+1}V_{x:\overline{n}|})\cdot\right.$$
$$\left.\cdot\Delta q_{x+t}\,D_{x+t}\,v-\Delta i\sum_0^{t-1}({}_tV_{x:\overline{n}|}+\Delta\,{}_tV_{x:\overline{n}|}+P_{x:\overline{n}|}+\Delta P_{x:\overline{n}|})\,D_{x+t}\,v\right\}.$$

Therefore in case of the change of interest rate, putting $\Delta q_{x+t}=0$ we have

$$\Delta P_{x:\overline{n}|}=\frac{-\Delta i}{N_x-N_{x+n}}\left\{\sum_0^{n-1}({}_tV_{x:\overline{n}|}+\Delta\,{}_tV_{x:\overline{n}|}+P_{x:\overline{n}|}+\Delta P_{x:\overline{n}|})\,D_{x+t}\,v\right\},\qquad\text{(D)}$$

$$\Delta\,{}_tV_{x:\overline{n}|}=\frac{1}{D_{x+t}}\left\{\Delta P_{x:\overline{n}|}\sum_0^{t-1}D_{x+t}+v\,\Delta i\sum_0^{t-1}({}_tV_{x:\overline{n}|}+\Delta\,{}_tV_{x:\overline{n}|}+P_{x:\overline{n}|}+\Delta P_{x:\overline{n}|})\,D_{x+t}\right\},\qquad\text{(D}'\text{)}$$

and in case of the change of mortality, putting $\Delta i=0$ we have

$$\Delta P_{x:\overline{n}|}=\frac{1}{N_x-N_{x+n}}\left\{\sum_0^{n-1}(1-{}_{t+1}V_{x:\overline{n}|}-\Delta\,{}_{t+1}V_{x:\overline{n}|})\,\Delta q_{x+t}\,D_{x+t}\,v\right\},\qquad\text{(E)}$$

$$\Delta\,{}_tV_{x:\overline{n}|}=\frac{1}{D_{x+t}}\left\{\Delta P_{x:\overline{n}|}\sum_0^{t-1}D_{x+t}-\sum_0^{t-1}(1-{}_{t+1}V_{x:\overline{n}|}-\Delta\,{}_{t+1}V_{x:\overline{n}|})\,\Delta q_{x+t}\,D_{x+t}\,v\right\}.\qquad\text{(E}'\text{)}$$

From (D) and (E) we have

$$\frac{d}{d\delta}P_{x:\overline{n}|}=-\frac{\sum_0^{n-1}D_{x+t}({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})}{N_x-N_{x+n}},$$

$$\lim_{c\to0}\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{c}=\frac{\sum_0^{n-1}(1-{}_{t+1}V_{x:\overline{n}|})\,D_{x+t}\,v}{N_x-N_{x+n}},\qquad(q'_x=q_x+c),$$

and

$$\lim_{\beta\to0}\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{\beta}=\frac{\sum_0^{n-1}C_{x+t}(1-{}_{t+1}V_{x:\overline{n}|})}{N_x-N_{x+n}},\qquad(q'_x=(1+\beta)\,q_x).$$

Hence we obtain

$$\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{c}\Bigg|_{c\to 0}-\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{\beta}\Bigg|_{\beta\to 0}=v+\frac{d}{d\delta}P_{x:\overline{n}|}. \tag{F}$$

When $q'_x=(1+\beta)\,q_x$, clearly

$$P'_{x:\overline{n}|}-P_{x:\overline{n}|}=\beta(P_{x:\overline{n}|}-P_{\overline{n}|})+\beta\frac{\sum_0^{n-1}({}_{t+1}V_{\overline{n}|}-{}_{t+1}V_{x:\overline{n}|})}{N_x-N_{x+n}},$$

therefore

$$\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{\beta}\Bigg|_{\beta\to 0}=(P_{x:\overline{n}|}-P_{\overline{n}|})\frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}\frac{\sum_0^{n-1}C_{x+t}\,\ddot{a}_{x+t+1:\overline{n-t-1}|}}{\sum_0^{n-1}C_{x+t}\,\ddot{a}_{\overline{n-t-1}|}}\sim$$

$$\sim(P_{x:\overline{n}|}-P_{\overline{n}|})\frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}\left(1-\frac{n-2}{3}q_{x:\overline{n}|}\right) \tag{G}$$

assuming

$$\frac{1}{D_x}\sum_0^{n-1}C_{x+t}\,\ddot{a}_{\overline{n-t-1}|}=\ddot{a}_{\overline{n}|}-\ddot{a}_{x:\overline{n}|},$$

$$\frac{1}{D_x}\sum_0^{n-1}C_{x+t}\,\ddot{a}_{x+t+1:\overline{n-t-1}|}\sim\ddot{a}_{x:\overline{n}|}q_{x:\overline{n}|}\left(1+\frac{n-2}{3}q_{x:\overline{n}|}\right)\frac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}.$$

From (D) and (E) we have

$$\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|}=\left(\frac{d}{d\delta}P_{x:\overline{n}|}\right)\frac{\sum_0^{t-1}D_{x+t}}{D_{x+t}}+\frac{\sum_0^{t-1}D_{x+t}\,({}_tV_{x:\overline{n}|}+P_{x:\overline{n}|})}{D_{x+t}},$$

$$\frac{{}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}}{c}\Bigg|_{c\to 0}=\left(\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{c}\Bigg|_{c\to 0}\right)\frac{\sum_0^{t-1}D_{x+t}}{D_{x+t}}-\frac{\sum_0^{t-1}D_{x+t}\,v(1-{}_{t+1}V_{x:\overline{n}|})}{D_{x+t}},$$

$$(q'_x=q_x+c).$$

$$\frac{{}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}}{\beta}\Bigg|_{\beta\to 0}=\left(\frac{P'_{x:\overline{n}|}-P_{x:\overline{n}|}}{\beta}\Bigg|_{\beta\to 0}\right)\frac{\sum_0^{t-1}D_{x+t}}{D_{x+t}}-\frac{\sum_0^{t-1}C_{x+t}\,(1-{}_{t+1}V_{x:\overline{n}|})}{D_{x+t}},$$

$$(q'_x=(1+\beta)\,q)_x$$

Hence we obtain

$$\frac{{}_tV'_{x:\overline{n}|}-{}_tV'_{x:\overline{n}|}}{c}\Bigg|_{c\to 0}-\frac{{}_tV'_{x:\overline{n}|}-{}_tV_{x:\overline{n}|}}{\beta}\Bigg|_{\beta\to 0}=\frac{d}{d\delta}\,{}_tV_{x:\overline{n}|}. \tag{F'}$$

Now we show some formulas derived from the so-called mean value theorem.

When $f(x)$ and $g(x)$ are differentiable in $[a,b]$ we may choose ξ such as

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)} \tag{H}$$

where $a<\xi<b$ and we assume $g'(x)=\frac{d}{dx}g(x)\neq 0$.

In (H) if we put $a=i$, $b=i+\Delta i$, $f(i)=\ddot{a}_{x:\overline{n|}(i)}$ and $g(i)=\ddot{a}_{\overline{n|}(i)}$ we have

$$\Delta\ddot{a}_{x:\overline{n|}}=\Delta\ddot{a}_{\overline{n|}}\frac{\left[-\frac{d}{d\delta}\ddot{a}_{x:\overline{n|}}\right]^{i=\xi}}{\left[-\frac{d}{d\delta}\ddot{a}_{\overline{n|}}\right]^{i=\xi}}\sim\Delta\ddot{a}_{\overline{n|}}\frac{(Ia)_{x:\overline{n-1|}(i)}}{(Ia)_{\overline{n-1|}(i)}}\frac{1-\frac{(I^2a)_{x:\overline{n-1|}(i)}}{(Ia)_{x:\overline{n-1|}(i)}}(\xi-i)}{1-\frac{(I^2a)_{\overline{n-1|}(i)}}{(Ia)_{\overline{n-1|}(i)}}(\xi-i)},$$

where

$$(Ia)_{x:\overline{n-1|}}=-\frac{d}{d\delta}\ddot{a}_{x:\overline{n|}}\sim\frac{n-1}{2}\left(1-\frac{n+1}{6}(i+q_{x:\overline{n|}})\right)\ddot{a}_{x:\overline{n|}},$$

$$(I^2a)_{x:\overline{n-1|}}=\frac{d^2}{d\delta^2}\ddot{a}_{x:\overline{n|}}\sim\frac{(n-1)^2}{3}\left(1-\frac{n+1}{4}(i+q_{x:\overline{n|}})\right)\ddot{a}_{x:\overline{n|}},$$

$$(I^2a)_{\overline{n-1|}}=\frac{d^2}{d\delta^2}\ddot{a}_{\overline{n|}}\sim\frac{(n-1)^2}{3}\left(1-\frac{n+1}{4}i\right)\ddot{a}_{\overline{n|}}.$$

Hence we may write when Δi is small

$$\Delta\ddot{a}_{x:\overline{n|}}\sim\Delta\ddot{a}_{\overline{n|}}\frac{(Ia)_{x:\overline{n-1|}(i)}}{(Ia)_{\overline{n-1|}(i)}}\left(1+h\frac{n^2-1}{18}q_{x:\overline{n|}}\right), \tag{I}$$

where $h=\xi-i\sim\frac{1}{2}\Delta i$.

In the same way we have

$$\Delta\ddot{a}_{x:\overline{n|}}=\Delta\left(nv^{\frac{n-1}{2}}\right)\frac{\left[-\frac{d}{d\delta}\ddot{a}_{x:\overline{n|}}\right]^{i=\xi}}{\left[\frac{n(n-1)}{2}v^{\frac{n-1}{2}}\right]^{i=\xi}}\sim$$

$$\sim\Delta\left(nv^{\frac{n-1}{2}}\right)\frac{(Ia)_{x:\overline{n-1|}(i)}}{\frac{n(n-1)}{2}v_{(i)}^{\frac{n-1}{2}}}\frac{1-\frac{2}{3}(n-1)(\xi-i)\left(1-\frac{n+1}{12}(i+q_{x:\overline{n|}})\right)}{1-\frac{n-1}{2}(\xi-i)}\sim$$

$$\sim\Delta\left(nv^{\frac{n-1}{2}}\right)\frac{(Ia)_{x:\overline{n-1|}(i)}}{\frac{n(n-1)}{2}v_{(i)}^{\frac{n-1}{2}}}\left(1-\frac{n-1}{6}h\left(1-\frac{n+1}{3}(i+q_{x:\overline{n|}})\right)\right), \tag{J}$$

where $h=\xi-i\sim\frac{1}{2}\Delta i$.

Example: C. S. O. Table, $i = 2{,}5\,\%$, $i + \Delta i = 3\,\%$.

$$(Ia)_{30:\overline{29|}} = 238{,}5745, \qquad (Ia)_{\overline{29|}} = 271{,}7488,$$
$$-\Delta \ddot{a}_{\overline{30|}} = 1{,}2651, \qquad -\Delta\,(30 v^{14,5}) = 1{,}4282.$$

Hence by formula (I):

$$-\Delta \ddot{a}_{30:\overline{30|}} \sim 1{,}2651 \frac{238{,}5745}{271{,}7488}\left(1 + \frac{0{,}005}{2}\,\frac{899}{18}\,0{,}0063\right) =$$
$$= 1{,}1107 \cdot 1{,}00079 = 1{,}1116,$$

and by formula (J):

$$-\Delta \ddot{a}_{30:\overline{43|}} \sim 1{,}1204(1 - 0{,}00817) = 1{,}1112,$$

true value of $-\Delta \ddot{a}_{30:\overline{30|}} = 1{,}1121$.

The approximation to $\Delta A_{x:\overline{n|}}$ is obtained as follow

$$\Delta A_{x:\overline{n|}} = \Delta\,(v^n) \frac{\left[-\dfrac{d}{d\delta} A_{x:\overline{n|}}\right]^{i=\xi}}{[n v^n]^{i=\xi}} \sim$$

$$\sim \Delta (v^n) \frac{(IA)_{x:\overline{n|}(i)}}{n v^n_{(i)}} \; \frac{1 - (\xi - i)\dfrac{(I^2 A)_{x:\overline{n|}(i)}}{(IA)_{x:\overline{n|}(i)}}}{1 - (\xi - i)\, n},$$

where $\quad (IA)_{x:\overline{n|}} = -\dfrac{d}{d\delta} A_{x:\overline{n|}} \sim n\left(1 - \dfrac{n-1}{2}\, q_{x:\overline{n|}}\right) A_{x:\overline{n|}},$

$$(I^2 A)_{x:\overline{n|}} = \frac{d^2}{d\delta^2} A_{x:\overline{n|}} \sim n^2\left(1 - \frac{n-1}{2}\, q_{x:\overline{n|}}\right)^2 A_{x:\overline{n|}}.$$

Therefore when i is small, we may write

$$\Delta A_{x:\overline{n|}} \sim \Delta (v^n) \frac{(IA)_{x:\overline{n|}(i)}}{n v^n_{(i)}}\left(1 + h\,\frac{n\,(n-1)}{2}\, q_{x:\overline{n|}}\right), \tag{K}$$

where $\quad h = \xi - i \sim \tfrac{1}{2}\Delta i$.

Example: C. S. O. Table, $i = 2{,}5\,\%$, $i + \Delta i = 3\,\%$.

$(IA)_{30:\overline{30|}} = 13{,}5002$, $\quad 30 v^{30} = 14{,}3023$, $\quad -\Delta (v^{30}) = 0{,}06476$,

hence by formula (K):

$$-\Delta A_{30:\overline{30|}} \sim 0{,}06476 \frac{13{,}5002}{14{,}3023}\left(1 + \frac{0{,}005}{2}\,\frac{870}{2}\,0{,}0063 =\right.$$

$= 0{,}006112 \cdot 1{,}00685 = 0{,}06154$, true value of $-\Delta A_{30:\overline{30|}} = 0{,}06139$.

The approximation to $\Delta P_{x:\overline{n}|}$ is, as already mentioned

$$\Delta P_{x:\overline{n}|} \sim \Delta P_{\overline{n}|} \frac{v - \dfrac{(IA)_{x:\overline{n-1}|(i)}}{(\ddot{a}_{x:\overline{n}|(i)})^2}}{v - \dfrac{(Ia)_{\overline{n-1}|(i)}}{(\ddot{a}_{\overline{n}|(i)})^2}}, \tag{L}$$

or

$$\Delta P_{x:\overline{n}|} \sim \Delta \left(\frac{1}{n} v^{\frac{n+1}{2}}\right) \frac{v - \dfrac{(Ia)_{x:\overline{n-1}|(i)}}{(\ddot{a}_{x:\overline{n}|(i)})^2}}{\dfrac{n+1}{2n} v_{(i)}^{\frac{n+1}{2}}}. \tag{M}$$

Example: C.S.O. Table, $i = 2{,}5\,\%$, $i + \Delta i = 3\,\%$.

$$\Delta P_{\overline{30}|} = -\Delta\left(\frac{v^{30}}{\ddot{a}_{\overline{30}|}}\right) = 0{,}001815,$$

$$\frac{(Ia)_{30:\overline{29}|}}{(\ddot{a}_{30:\overline{30}|})^2} = 0{,}6084, \qquad \frac{(Ia)_{\overline{29}|}}{(\ddot{a}_{\overline{30}|})^2} = 0{,}5905,$$

hence by formula (L)

$$-\Delta P_{30:\overline{30}|} \sim 0{,}001815 \frac{0{,}9756 - 0{,}6084}{0{,}9756 - 0{,}5905} = 0{,}00173$$

and by formula (M) $\quad -\Delta P_{30:\overline{30}|} \sim 0{,}00165 \dfrac{0{,}3672}{0{,}3524} = 0{,}00172,$

true value of $-\Delta P_{30:\overline{30}|} = 0{,}00172.$

The approximation to $\Delta_t V_{x:\overline{n}|}$ is, also as already mentioned

$$\Delta_t V_{x:\overline{n}|} \sim \Delta_t V_{\overline{n}|} \frac{\dfrac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \left(\dfrac{(Ia)_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}} - \dfrac{(Ia)_{x+t:\overline{n-t-1}|}}{\ddot{a}_{x+t:\overline{n-t}|}}\right)}{\dfrac{\ddot{a}_{\overline{n-t}|}}{\ddot{a}_{\overline{n}|}} \left(\dfrac{(Ia)_{\overline{n-1}|}}{\ddot{a}_{\overline{n}|}} - \dfrac{(Ia)_{\overline{n-t-1}|}}{\ddot{a}_{\overline{n-t}|}}\right)}. \tag{N}$$

Example: C.S.O. Table, $i = 2{,}5\,\%$, $i + \Delta i = 3\,\%$.

$$-\Delta_{10} V_{\overline{30}|} = -\Delta\left(1 - \frac{\ddot{a}_{\overline{20}|}}{\ddot{a}_{\overline{30}|}}\right) = 0{,}014225$$

hence $\quad -\Delta_{10} V_{30:\overline{30}|} \sim 0{,}014225 \dfrac{0{,}7434\,(12{,}048 - 8{,}293)}{0{,}7448\,(12{,}667 - 8{,}682)} = 0{,}01338,$

true value of $-\Delta_{10} V_{30:\overline{30}|} = 0{,}01336.$

For whole life, we may write in the same way

$$\Delta A_x \sim \Delta v^{\ddot{e}_x} \frac{(IA)_{x(i)}}{\ddot{e}_x\, v_{(i)}^{\ddot{e}_x}},$$

$$\Delta \ddot{a}_x \sim \Delta \ddot{a}_{\overline{\ddot{e}_x}|} \frac{(Ia)_{x(i)}}{(Ia)_{\overline{\ddot{e}_x - 1}|(i)}},$$

$$\text{where} \quad (IA)_x = \frac{R_x}{D_x} \quad \text{and} \quad (Ia)_x = \frac{S_{x+1}}{D_x}.$$

Example: C.S.O. Table, $i = 2{,}5\,\%$, $i + \Delta i = 3\,\%$.

$$\ddot{e}_{30} = 38{,}2421,$$

$$\Delta A_{30} \sim 0{,}066042 \frac{13{,}6974}{14{,}8752} = 0{,}06081 \text{ (true value} = 0{,}06063\text{)},$$

$$\Delta \ddot{a}_{30} \sim 1{,}80608 \frac{399{,}7716}{392{,}2805} = 1{,}8234 \quad \text{(true value} = 1{,}8264\text{)}.$$

Zusammenfassung

Es werden Näherungsformeln hergeleitet für Rentenwert, Nettoprämie und mathematische Reserve. Diese Formeln können beim Fehlen vollständiger Kommutationstabellen zur Abschätzung genannter Versicherungswerte nützliche Dienste leisten, ebenso zur Abschätzung des Einflusses von Grundlagenänderungen auf vorhandene Versicherungswerte und bei der technischen Behandlung erhöhter Risiken.

Résumé

L'auteur développe des formules d'approximation pour la valeur de la rente, la prime pure et la réserve mathématique. Ces formules peuvent rendre d'utiles services pour l'évaluation de ces éléments actuariels dans le cas où des tables de commutations complètes font défaut, de même que pour évaluer l'influence de changements de bases sur des valeurs actuarielles existantes, ainsi que pour le traitement technique de risques aggravés.

Riassunto

L'autore deduce formule d'approssimazione per rappresentare il valore attuale delle rendite, dei premi puri e delle riserve matematiche. Se non si dispone di valori di commutazione completi, queste formule rendono utili servizi per l'evaluazione delle dette quantità come pure per valutare le differenze resultanti da un cambio delle basi techniche o per il trattamento tecnico dei rischi tarati.