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Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries**

Band (Jahr): **63 (1963)**

PDF erstellt am: **17.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-966925

The Variation Problem in Case of the Linear Decreasing Rate of Interest

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Summary

In this paper, the author gives approximation formulas for premiums and mathematical reserves of endowment assurances with linear decreasing rate of interest, in connection with the well-known variation problem. Also he shows the relation between the mathematical reserve of endowment assurance with decreasing rate of interest and that with increasing additive extra mortality.

Zusammenfassung

In Anlehnung an frühere Arbeiten befasst sich der Autor mit der Herleitung von approximativen Formeln für Prämie und Deckungskapital der gemischten Versicherung im Falle eines linear sinkenden Zinsfusses. Er weist auch auf den Zusammenhang zwischen den Reserven einer solchen Versicherung hin, wenn einerseits von einem sinkenden Zinsfuss, anderseits von einer steigenden additiven Übersterblichkeit ausgegangen wird.

Résumé

En s'appuyant sur des travaux antérieurs, l'auteur établit des formules approximatives en vue du calcul de la prime et de la réserve mathématique d'une assurance mixte fondée sur un taux d'intérêt linéairement décroissant. Certaines analogies sont également mises en évidence, pour cette forme d'assurance, entre les réserves mathématiques résultant, d'une part, de l'application d'un taux d'intérêt décroissant et, d'autre part, de la prise en considération d'une surmortalité croissante.

Riassunto

Fondandosi su lavori precedenti, l'autore si preoccupa di dedurre formule approssimative per il calcolo del premio e della riserva matematica dell'assicurazione mista, nel caso di un tasso d'interesse decrescente in modo lineare. Egli accenna anche ai rapporti tra le riserve matematiche di una tale assicurazione, supposto, da una parte, un tasso d'interesse decrescente e, dall'altra, una maggior mortalità crescente.

§ 1. In my earlier paper[1]), I showed the following relations:

$$a)\qquad -\frac{d}{d\delta}P_{\overline{xn}|} = \frac{\sum\limits_{0}^{n-1}({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}{N_x - N_{x+n}}. \tag{1}$$

$$b)\qquad \frac{d}{d\delta}\,{}_tV_{\overline{xn}|} = \frac{d}{d\delta}P_{\overline{xn}|}\frac{D_x}{D_{x+t}}\ddot{a}_{\overline{xt}|} + \frac{\sum\limits_{0}^{t-1}({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}{D_{x+t}}$$

$$= \frac{d}{d\delta}P_{\overline{xn}|}\frac{D_x}{D_{x+t}}\ddot{a}_{\overline{xt}|}\left(1-\frac{\ddot{a}_{\overline{xn}|}}{\ddot{a}_{\overline{xt}|}}\frac{\sum\limits_{0}^{t-1}({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}{\sum\limits_{0}^{n-1}({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}\right)$$

$$\overset{2)}{\sim} \frac{d}{d\delta}P_{\overline{xn}|}\frac{D_x}{D_{x+t}}\ddot{a}_{\overline{xt}|}\left(1-\frac{1+\dfrac{I\,a_{\overline{xt-1}|}}{\ddot{a}_{\overline{xt}|}}}{1+\dfrac{I\,a_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}}}\right)$$

$$\sim \frac{d}{d\delta}P_{\overline{xn}|}\frac{D_x}{D_{x+t}}\ddot{a}_{\overline{xt}|}\left\{1-\frac{t}{n}-\frac{t(i+q_{\overline{xn}|})}{6}\left(1-\frac{t}{n}\right)\right.$$

$$\left.-\frac{1}{n}\left(1-\frac{t}{n}\right)-\frac{t}{12n}\left(1-\frac{t^2}{n^2}\right)(n\,q_{\overline{xn}|})^2-\frac{t}{24n}\left(1-\frac{t^3}{n^3}\right)(n\,q_{\overline{xn}|})^3\right\} \tag{2}$$

$$\sim \frac{d}{d\delta}P_{\overline{xn}|}\frac{D_x}{D_{x+t}}\ddot{a}_{\overline{xt}|}\left(1-\frac{t}{n}\right) \tag{3}$$

(for the case where[3]) n and x are not large).

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[1]) Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker, Bd. 59 and Bd. 60.

[2]) $-\dfrac{d}{d\delta}\,\ddot{a}_{\overline{xn}|} = I\,a_{\overline{xn-1}|}$

$$\frac{I\,a_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}} \sim \frac{n-1}{2}\left\{1-\frac{(n+1)\,(i+q_{\overline{xn}|})}{6}-\frac{(n\,q_{\overline{xn}|})^2}{12}\left(1+\frac{n\,q_{\overline{xn}|}}{2}\right)\right\},$$

where $\dfrac{\sum\limits_{0}^{n-1} l_{x+t}}{l_x} = n\left(1-\dfrac{n-1}{2}\,q_{\overline{xn}|}\right).$

[3]) We assume $10 \lesssim n \lesssim 30$ in this paper. Moreover in this case we take $n(i+q_{\overline{xn}|}) \lesssim 1.5$ and $n\,q_{\overline{xn}|} \lesssim 0.5$.

Or
$$\sim \frac{d}{d\delta} P_{\overline{xn}|}\, t\left(1 - \frac{t}{n}\right) \tag{3'}$$
(for the case where[1]) t is small, or x is large).

c)
$$-\frac{\sum_{0}^{n-1}\left(\frac{d}{d\delta}\, {}_tV_{\overline{xn}|} + \frac{d}{d\delta} P_{\overline{xn}|}\right) D_{x+t}}{N_x - N_{x+n}} = \frac{1}{2}\left(\frac{d^2}{d\delta^2} P_{\overline{xn}|} - \frac{d}{d\delta} P_{\overline{xn}|}\right). \tag{4}$$

Using (2) we have for the case where n and x are not large
$$\frac{1}{2}\frac{d^2}{d\delta^2} P_{\overline{xn}|} - \frac{1}{2}\frac{d}{d\delta} P_{\overline{xn}|} \sim -\frac{d}{d\delta} P_{\overline{xn}|} \left\{\frac{n+1}{2} - \left(1 - \frac{1}{2n}\right)\frac{I\, a_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}} \right.$$
$$\left. - \frac{1}{6} + \frac{I^2 a_{\overline{xn-1}|}}{2n\, \ddot{a}_{\overline{xn}|}} - \frac{n^2 (i + q_{\overline{xn}|})}{72} - \frac{2n}{15}\frac{(n\, q_{\overline{xn}|})^2}{12}\left(1 + \frac{n}{2} q_{\overline{xn}|}\right)\right\}, \tag{5}$$
where $\frac{d^2}{d\delta^2}\ddot{a}_{\overline{xn}|} = I^2 a_{\overline{xn-1}|}$,

or using (3′) we get for the case where x is large
$$\frac{1}{2}\frac{d^2}{d\delta^2} P_{\overline{xn}|} - \frac{1}{2}\frac{d}{d\delta} P_{\overline{xn}|} \sim -\frac{d}{d\delta} P_{\overline{xn}|}\left\{1 + \frac{I\, a_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}} - \frac{I^2 a_{\overline{xn-1}|}}{n\, \ddot{a}_{\overline{xn}|}}\right\}. \tag{6}$$

d) In case of linear increasing extra mortality such as

$q'_{x+t} = q_{x+t} + c\frac{t}{n}$ (c is a constant) we have[2])
$$-\lim_{c\to 0}\frac{{}_tV_{\overline{xn}|}^{\,q_{x+t}+c\frac{t}{n}} - {}_tV_{\overline{xn}|}}{c} \sim 0.73\left(0.63 - \frac{t}{n}\right)\lim_{c\to 0}\frac{{}_tV_{\overline{xn}|}^{\,q_{x+t}+c} - {}_tV_{\overline{xn}|}}{c}$$
$$\sim 0.73\left(0.63 - \frac{t}{n}\right)\frac{d}{d\delta}\, {}_tV_{\overline{xn}|}. \tag{7}$$

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[1]) We assume in this case $n(i + q_{\overline{xn}|}) \gtrsim 1.5$ and $n q_{\overline{xn}|} \gtrsim 0.5$.

[2]) The better approximation is as follows:
$$-\lim_{c\to 0}\frac{{}_tV_{\overline{xn}|}^{\,q_{x+t}+c\frac{t}{n}} - {}_tV_{\overline{xn}|}}{c} \sim 0.46\left(1 - \frac{8}{5}\frac{t}{n} + \frac{3\left(1 - \frac{t}{2n}\right)}{n}\right)\frac{d}{d\delta}\, {}_tV_{\overline{xn}|} \quad \text{for small } n\text{, say } n \sim 10.$$
$$\sim 0.46\left(1 - \frac{8}{5}\frac{t}{n}\right)\frac{d}{d\delta}\, {}_tV_{\overline{xn}|} \quad \text{for not small } n.$$

§ 2. Now we assume the rates of interest such as

$$\delta+\delta',\quad \delta+\delta'\frac{n-1}{n},\quad \ldots,\quad \delta+\delta'\frac{n-t}{n},\quad \ldots,\quad \delta+\frac{\delta'}{n},\quad \delta.$$

The following relation is fundamental.

$$\lim_{\delta'\to 0}\frac{P_{\overline{xn}|}^{\delta+\delta'\frac{n-t}{n}}-P_{\overline{xn}|}}{\delta'}=-\frac{\sum_0^{n-1}(n-t)\,({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}{n\,(N_x-N_{x+n})}$$

$$=\frac{d}{d\delta}P_{\overline{xn}|}+\frac{\sum_0^{n-1}t\,({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}{n\,(N_x-N_{x+n})}.$$

To evaluate $\dfrac{\sum_0^{n-1}t\,({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}{n\,(N_x-N_{x+n})}$, differentiate the equation (1) with respect to δ, and we have

$$\frac{d^2}{d\delta^2}P_{\overline{xn}|}+\frac{d}{d\delta}P_{\overline{xn}|}\frac{\frac{d}{d\delta}\ddot{a}_{\overline{xn}|}}{\ddot{a}_{\overline{xn}|}}=-\frac{\sum_0^{n-1}\left(\frac{d}{d\delta}\,{}_tV_{\overline{xn}|}+\frac{d}{d\delta}P_{\overline{xn}|}\right)D_{x+t}}{N_x-N_{x+n}}$$

$$+\frac{\sum_0^{n-1}t\,({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}{N_x-N_{x+n}}.$$

Carrying (4) to the right hand side of this equation, we have

$$\frac{\sum_0^{n-1}t\,({}_tV_{\overline{xn}|}+P_{\overline{xn}|})\,D_{x+t}}{n\,(N_x-N_{x+n})}=\frac{1}{2n}\frac{d^2}{d\delta^2}P_{\overline{xn}|}+\frac{1}{2n}\frac{d}{d\delta}P_{\overline{xn}|}-\frac{1}{n}\frac{Ia_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}}\frac{d}{d\delta}P_{\overline{xn}|}.$$

Therefore we have

$$\lim_{\delta'\to 0}\frac{P_{\overline{xn}|}^{\delta+\delta'\frac{n-t}{n}}-P_{\overline{xn}|}}{\delta'}=\frac{d}{d\delta}P_{\overline{xn}|}\left(1+\frac{1}{2n}-\frac{Ia_{\overline{xn-1}|}}{n\ddot{a}_{\overline{xn}|}}\right)+\frac{1}{2n}\frac{d^2}{d\delta^2}P_{\overline{xn}|}\tag{8}$$

$$=D\frac{d}{d\delta}P_{\overline{xn}|},\tag{8'}$$

where $$D=1+\frac{1}{2n}-\frac{Ia_{\overline{xn-1}|}}{n\ddot{a}_{\overline{xn}|}}+\frac{1}{2n}\frac{\frac{d^2}{d\delta^2}P_{\overline{xn}|}}{\frac{d}{d\delta}P_{\overline{xn}|}}.$$

Consequently
$$\frac{P_{\overline{xn}|}^{\delta+\delta'\frac{t}{n}} - P_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}} = (1-D)\frac{d}{d\delta}P_{\overline{xn}|}. \tag{9}$$

The approximation to D is obtained as follows:

(i) For the case where x and n are not large. From (8), using the approximation (5), we have

$$D \sim \frac{1}{2} + \frac{2}{3n} - \frac{I\,a_{\overline{xn-1}|}}{2n^2\,\ddot{a}_{\overline{xn}|}} - \frac{I^2\,a_{\overline{xn-1}|}}{2n^2\,\ddot{a}_{\overline{xn}|}} + \frac{n\,(i+q_{\overline{xn}|})}{72} + \frac{(n\,q_{\overline{xn}|})^2}{90}\left(1+\frac{n}{2}q_{\overline{xn}|}\right).$$

On the other hand, we may write

$$\frac{I\,a_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}} \sim \frac{n-1}{2}\left\{1 - \frac{(n+1)\,(i+q_{\overline{xn}|})}{6} - \frac{(n\,q_{\overline{xn}|})^2}{12}\left(1+\frac{n\,q_{\overline{xn}|}}{2}\right)\right\},$$

$$\frac{I^2\,a_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}} \sim \frac{(n-1)\,(2n-1)}{6}\left\{1 - \frac{(n-1)\,(i+q_{\overline{xn}|})}{4} - \frac{(n\,q_{\overline{xn}|})^2}{8}\left(1+\frac{n\,q_{\overline{xn}|}}{2}\right)\right\}.$$

Using these approximations, we may have

$$D \sim \frac{1}{3} + \frac{2}{3n} + \frac{(n-1)\,(i+q_{\overline{xn}|})}{18} + \frac{(n\,q_{\overline{xn}|})^2}{31}\left(1+\frac{n}{2}q_{\overline{xn}|}\right). \tag{10}$$

From (10) we have, for the case where $ni \lesssim 1$ and $n q_{\overline{xn}|} \lesssim \frac{1}{3}$, $D \sim 0.4$.

Examples:

$n = 30 \qquad x = 30$

C.S.O. (41) Table, 2½%		J^PM(3) Table, 3½%	
	$q_{30\,\overline{30}|} = 0.0063$		$q_{30\,\overline{30}|} = 0.010$
the true value of	$D = 0.401$		$D = 0.417$
by (10)	$= 0.407$	by (10)	$= 0.431$

(ii) For the case where x is large. Carrying (6) into the formula (8), we obtain

$$D \sim 1 - 2\frac{I\,a_{\overline{xn-1}|}}{n\,\ddot{a}_{\overline{xn}|}} + \frac{I^2\,a_{\overline{xn-1}|}}{n^2\,\ddot{a}_{\overline{xn}|}}.$$

Hence we may have

$$D \sim \frac{1}{3} + \frac{1}{2n} + \frac{(n-1)\,(i+q_{\overline{xn}|})}{12} + \frac{(n\,q_{\overline{xn}|})^2}{24}\left(1+\frac{n}{2}q_{\overline{xn}|}\right). \tag{11}$$

Examples:

	$n = 30 \quad x = 50$ C.S.O. Table, $2\frac{1}{2}\%$	$J^{PM(3)}$, $3\frac{1}{2}\%$	$n = 20 \quad x = 50$ $J^{PM(3)}$, $3\frac{1}{2}\%$
	$q_{50\,\overline{30}|} = 0.0215$	$q_{50\,\overline{30}|} = 0.029$	$q_{50\,\overline{20}|} = 0.026$
the true value of D	$= 0.491$	0.557	0.456
by (11)	$= 0.482$	0.549	0.469
by (10)	$= 0.448$	0.494	0.440

§ 3. As to the premium reserve we have

$$\frac{{}_tV_{x\overline{n}|}^{\delta+\delta'\frac{t}{n}} - {}_tV_{x\overline{n}|}}{\delta'}\Bigg|_{\delta'\to 0} = \frac{P_{x\overline{n}|}^{\delta+\delta'\frac{t}{n}} - P_{x\overline{n}|}}{\delta'}\Bigg|_{\delta'\to 0} \frac{D_x}{D_{x+t}}\,\ddot{a}_{x\overline{t}|}\left(1 - \frac{\ddot{a}_{x\overline{n}|}}{\ddot{a}_{x\overline{t}|}}\,\frac{\sum_0^{t-1} t\left({}_tV_{x\overline{n}|} + P_{x\overline{n}|}\right)D_{x+t}}{\sum_0^{n-1} t\left({}_tV_{x\overline{n}|} + P_{x\overline{n}|}\right)D_{x+t}}\right)$$

$$\sim \frac{P_{x\overline{n}|}^{\delta+\delta'\frac{t}{n}} - P_{x\overline{n}|}}{\delta'}\Bigg|_{\delta'\to 0} \frac{D_x}{D_{x+t}}\,\ddot{a}_{x\overline{t}|}\left(1 - \frac{\dfrac{I^2 a_{x\overline{t-1}|}}{\ddot{a}_{x\overline{t}|}}}{\dfrac{I^2 a_{x\overline{n-1}|}}{\ddot{a}_{x\overline{n}|}}}\right),$$

using for $\dfrac{I^2 a_{x\overline{n-1}|}}{\ddot{a}_{x\overline{n}|}}$ the same approximation as on p. 37 as well as (3) and (9), we may have[1])

$$\sim (1-D)\left(1+\frac{t}{n}\right)\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}, \tag{12}$$

when n and x are not large.

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[1]) The better approximation is as follows:

$$\frac{{}_tV_{x\overline{n}|}^{\delta+\delta'\frac{t}{n}} - {}_tV_{x\overline{n}|}}{\delta'}\Bigg|_{\delta'\to 0} \sim (1-D)\left(1+\frac{t}{n}\right)\left\{1+\frac{1}{n}-\frac{t(i+q_{x\overline{n}|})}{6}\left(1-\frac{3}{2\left(1+\frac{n}{t}\right)}\right)\right\}\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}$$

$$\sim (1-D)\left(1+\frac{t}{n}\right)\left(1+\frac{1}{n}\right)\frac{d}{d\delta}\,{}_tV_{x\overline{n}|} \quad \text{for small } n, \text{ say } n \sim 10$$

$$\sim (1-D)\left(1+\frac{t}{n}\right)\frac{d}{d\delta}\,{}_tV_{x\overline{n}|} \quad \text{for not small } n.$$

Consequently, from the fundamental relation of

$$\frac{{}_tV_{\overline{xn}|}^{\delta+\delta'\frac{n-t}{n}}-{}_tV_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}}+\frac{{}_tV_{\overline{xn}|}^{\delta+\delta'\frac{t}{n}}-{}_tV_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}}=\frac{d}{d\delta}\,{}_tV_{\overline{xn}|}$$

we obtain[1]), for the case where n and x are not large,

$$\frac{{}_tV_{\overline{xn}|}^{\delta+\delta'\frac{n-t}{n}}-{}_tV_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}}\sim D\left(1-\frac{1-D}{D}\,\frac{t}{n}\right)\frac{d}{d\delta}\,{}_tV_{\overline{xn}|}. \tag{13}$$

Putting $D\sim 0.4$, we have[1]) for the case where $ni\lesssim 1$ and $nq_{\overline{xn}|}\lesssim\frac{1}{3}$

$$\frac{{}_tV_{\overline{xn}|}^{\delta+\delta'\frac{n-t}{n}}-{}_tV_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}}\sim 0.4\left(1-\frac{3}{2}\,\frac{t}{n}\right)\frac{d}{d\delta}\,{}_tV_{\overline{xn}|}. \tag{13'}$$

It is worth while to notice that from (7) and (13) we have, when

$$ni\lesssim 1 \qquad \text{and} \qquad nq_{\overline{xn}|}\lesssim\tfrac{1}{3},$$

$$-\frac{{}_tV_{\overline{xn}|}^{q_{x+t}+c\frac{t}{n}}-{}_tV_{\overline{xn}|}}{\underset{c\to 0}{c}}\sim\frac{{}_tV_{\overline{xn}|}^{\delta+\delta'\frac{n-t}{n}}-{}_tV_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}}. \tag{14}$$

This interesting relation is also obtained as follows: Clearly we have

$$\frac{P_{\overline{xn}|}^{q_{x+t}+c\frac{t}{n}}-P_{\overline{xn}|}}{\underset{c\to 0}{c}}-\frac{P_{\overline{xn}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)}-P_{\overline{xn}|}}{\underset{\beta\to 0}{\beta}}=v\,\frac{Ia_{\overline{xn-1}|}}{n\ddot{a}_{\overline{xn}|}}-\frac{P_{\overline{xn}|}^{\delta+\delta'\frac{t}{n}}-P_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}},$$

$$\frac{P_{\overline{xn}|}^{q_{x+t}+c\frac{n-t}{n}}-P_{\overline{xn}|}}{\underset{c\to 0}{c}}-\frac{P_{\overline{xn}|}^{q_{x+t}\left(1+\frac{n-t}{n}\beta\right)}-P_{\overline{xn}|}}{\underset{\beta\to 0}{\beta}}=v\left(1-\frac{Ia_{\overline{xn-1}|}}{n\ddot{a}_{\overline{xn}|}}\right)-\frac{P_{\overline{xn}|}^{\delta+\delta'\frac{n-t}{n}}-P_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}}.$$

Therefore

$$\frac{{}_tV_{\overline{xn}|}^{q_{x+t}+c\frac{t}{n}}-{}_tV_{\overline{xn}|}}{\underset{c\to 0}{c}}-\frac{{}_tV_{\overline{xn}|}^{\delta+\delta'\frac{n-t}{n}}-{}_tV_{\overline{xn}|}}{\underset{\delta'\to 0}{\delta'}}$$

$$=\frac{d}{d\delta}\,{}_tV_{\overline{xn}|}+\frac{v}{n}\,\frac{D_x}{D_{x+t}}\,\ddot{a}_{\overline{xt}|}\left(\frac{Ia_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}}-\frac{Ia_{\overline{xt-1}|}}{\ddot{a}_{\overline{xt}|}}\right)+\frac{{}_tV_{\overline{xn}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)}-{}_tV_{\overline{xn}|}}{\underset{\beta\to 0}{\beta}}.$$

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[1]) When $n\sim 10$, the formulas (13) and (13′) may be slightly modified.

And when $ni \lesssim 1$ and $nq_{\overline{xn}|} \lesssim \frac{1}{3}$ we see easily that

$$-\frac{d}{d\delta}\,{}_tV_{\overline{xn}|} = \frac{\ddot{a}_{x+t\,\overline{n-t}|}}{\ddot{a}_{\overline{xn}|}}\left(\frac{Ia_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}} - \frac{Ia_{x+t\,\overline{n-t}|}}{\ddot{a}_{x+t\,\overline{n-t}|}}\right)$$

$$\sim \frac{v}{n}\,\frac{D_x}{D_{x+t}}\,\ddot{a}_{\overline{xt}|}\left(\frac{Ia_{\overline{xn-1}|}}{\ddot{a}_{\overline{xn}|}} - \frac{Ia_{\overline{xt-1}|}}{\ddot{a}_{\overline{xt}|}}\right),$$

and $\lim\limits_{\beta\to 0}\dfrac{{}_tV_{\overline{xn}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)} - {}_tV_{\overline{xn}|}}{\beta}$ is small. Therefore we get the above relation.

§ 4. The formula (8) is suitable for the case where δ' is small. When δ' is not small the following modified formula is better.

$$\frac{P_{\overline{xn}|}^{i+i'\frac{n-t}{n}} - P_{\overline{xn}|}}{i'} = vD\frac{d}{d\delta}P_{\overline{xn}|} + \frac{v\sum\limits_{0}^{n-1}\left(\left|\Delta\,{}_tV_{\overline{xn}|}\right| + \left|\Delta P_{\overline{xn}|}\right|\right)D_{x+t}}{N_x - N_{x+n}}$$

$$-\frac{v\sum\limits_{0}^{n-1}t\left(\left|\Delta\,{}_tV_{\overline{xn}|}\right| + \left|\Delta P_{\overline{xn}|}\right|\right)D_{x+t}}{n\,(N_x - N_{x+n})},$$

where $\Delta P_{\overline{xn}|} = P_{\overline{xn}|}^{i+i'\frac{n-t}{n}} - P_{\overline{xn}|}$, $\quad \Delta\,{}_tV_{\overline{xn}|} = {}_tV_{\overline{xn}|}^{i+i'\frac{n-t}{n}} - {}_tV_{\overline{xn}|}$.

Using the following approximations

$$\left|\Delta P_{\overline{xn}|}\right| \sim i'vD\left|\frac{d}{d\delta}P_{\overline{xn}|}\right|,$$

$$\left|\Delta\,{}_tV_{\overline{xn}|}\right| \sim i'vD\left|\frac{d}{d\delta}\,{}_tV_{\overline{xn}|}\right|\left(1 - \frac{1-D}{D}\,\frac{t}{n}\right),$$

we have

$$\frac{\sum\limits_{0}^{n-1}\left(\left|\Delta\,{}_tV_{\overline{xn}|}\right| + \left|\Delta P_{\overline{xn}|}\right|\right)D_{x+t}}{N_x - N_{x+n}} \sim i'vD\left|\frac{d}{d\delta}P_{\overline{xn}|}\right|\left\{1 + \frac{n}{12}\left(3 - \frac{1}{D}\right)\right\} \tag{15}$$

and

$$\frac{\sum\limits_{0}^{n-1}t\left(\left|\Delta\,{}_tV_{\overline{xn}|}\right| + \left|\Delta P_{\overline{xn}|}\right|\right)D_{x+t}}{n\,(N_x - N_{x+n})} \sim i'vD\left|\frac{d}{d\delta}P_{\overline{xn}|}\right|\left\{\frac{1}{2} + \frac{n}{5}\left(\frac{2}{3} - \frac{1}{4D}\right)\right\}. \tag{16}$$

Therefore we may write when x and n are not large

$$P_{\overline{xn}|}^{i+i'\frac{n-t}{n}} - P_{\overline{xn}|} \sim i'vD\frac{d}{d\delta}P_{\overline{xn}|}\left\{1 - i'v\left[\frac{1}{2} + \frac{n}{60}\left(7 - \frac{2}{D}\right)\right]\right\}. \tag{17}$$

When $t \equiv 0$, we get $D = 1$, (16) $= 0$, and we may write from (17)

$$P_{\overline{xn}|}^{i+i'} - P_{\overline{xn}|} \sim i'v\frac{d}{d\delta}P_{\overline{xn}|}\left\{1 - i'v\left(1 + \frac{n}{6}\right)\right\}$$ (for the case where x and n are not large).