

# A theorem in operational calculus

Autor(en): **Dahiya, R.S.**

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## A Theorem in Operational Calculus

By *R. S. Dahiya, Pilani (Rajasthan), India*

### Summary

The author proves a theorem in the field of Laplace transformations. Two examples show the possible applications.

1. Introduction. The integral equation

$$\Phi(p) = p \int_0^{\infty} e^{-pt} f(t) dt, \quad R(p) > 0. \quad (1.1)$$

represents the classical Laplace transform and the functions  $\Phi(p)$  and  $f(t)$  related by (1.1), are said to be operationally related to each other.  $\Phi(p)$  is called the image and  $f(t)$  the original. Symbolically we can write

$$\Phi(p) \doteq f(t) \quad \text{or} \quad f(t) \doteq \Phi(p).$$

The transforms, we have dealt with, are either the Hankel transform or another transform in which the kernel is designated by  $\tilde{\omega}_{\mu,\nu}(x)$ , where

$$\tilde{\omega}_{\mu,\nu}(x) = \sqrt{x} \int_0^{\infty} J_{\nu}\left(\frac{x}{t}\right) J_{\mu}(t) \frac{dt}{t}. \quad (1.2)$$

The integral on the right was first evaluated by C.V.H. Rao [1] and that it plays the role of a transform was conjectured by Watson [2]. Later on Bhatnagar has proved in detail that it plays the role of a transform. In this paper, we make use of Goldstein's theorem [3] as follows:

Let (i)  $\Phi_1(p) \doteq f_1(t)$ ,

(ii)  $\Phi_2(p) \doteq f_2(t)$ ,

then after applying Goldstein's theorem, we get

$$\int_0^\infty f_1(t) \Phi_2(t) \frac{dt}{t} = \int_0^\infty \Phi_1(t) f_2(t) \frac{dt}{t},$$

where  $\Phi(p)$  is the image and  $f(t)$  is the original.

2. Theorem. Let

(i)  $\psi(p) \doteq f(x)$ ,

(ii)  $t^{\mu-\lambda-\frac{1}{2}} f(t)$  be  $R_{\mu,\nu}$ ,

then

$$x^{2\mu-\lambda} f(x) \doteq \frac{2^{2\mu-\lambda+1} \Gamma\left(\mu + \frac{3}{2}\right) \Gamma\left(\frac{\mu + \nu + 3}{2}\right)}{\Gamma\left(\frac{\lambda + 3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma\left(\frac{\nu - \mu - 1}{2}\right)} \cdot p^2 \int_0^\infty t^{\lambda+1} \psi(t) {}_3F_2\left(\begin{matrix} \mu + \frac{3}{2}, \frac{\mu + \nu + 3}{2}, \frac{\nu - \mu + 3}{2}; \\ \frac{\lambda + 3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right) dt, \quad (2.1)$$

provided  $f(x)$  is absolutely integrable in  $(0, \infty)$ ,  $t^{\mu-\lambda-\frac{1}{2}} f(t)$  and  $x^{2\mu-\lambda} f(x)$  are integrable in  $(0, \infty)$ , and  $R(\mu) \geq -\frac{1}{2}$ ,  $R(\nu) \geq -\frac{1}{2}$ ,  $R(\lambda + 2) \leq 0$ .

Proof: Let

$$p^m \tilde{\omega}_{\mu,\nu}(p) \doteq F(x)$$

then

$$x^\lambda F\left(\frac{1}{x}\right) \doteq p^{\frac{1-\lambda}{2}} \int_0^\infty J_{\lambda+1}(2\sqrt{xp}) x^{\frac{\lambda-1}{2}+m} \tilde{\omega}_{\mu,\nu}(x) dx, \quad (2.2)$$

$R(\lambda + 2) > 0$ ,  $R(\lambda + m + \mu) \geq -\frac{3}{2}$ ,  $R(\lambda + m + \nu) \geq -\frac{3}{2}$ ,  $R(\lambda + 2m) < 0$

$$\text{or, } x^\lambda F\left(\frac{1}{x}\right) \doteq p^{\frac{1-\lambda}{2}} \int_0^\infty J_{\lambda+1}(2\sqrt{pt}) t^{\frac{\lambda-1}{2}+m} \tilde{\omega}_{\mu,\nu}(t) dt, \quad (2.3)$$

$$\text{or, } x^\lambda F\left(\frac{1}{x}\right) \doteq \frac{p^{1-m-\lambda}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\lambda+m+s)}{\Gamma(-m-s+2)} 2^{1-2s} \frac{\Gamma\left(\frac{\mu-s}{2} + \frac{3}{4}\right) \Gamma\left(\frac{\nu-s}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\mu+s}{2} + \frac{1}{4}\right) \Gamma\left(\frac{\nu+s}{2} + \frac{1}{4}\right)} p^{-s} ds, \quad (2.4)$$

$$\text{or, } x^\lambda F\left(\frac{1}{x}\right) \doteq \frac{p^{1-\lambda-m}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\lambda+2m-1} \frac{\Gamma\left(\frac{\lambda+m+s}{2}\right) \Gamma\left(\frac{\lambda+m+s+1}{2}\right) \Gamma\left(\frac{\mu-s}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{-m-s+2}{2}\right) \Gamma\left(\frac{-m-s+3}{2}\right) \Gamma\left(\frac{\mu+s}{2} + \frac{1}{4}\right)} \cdot \frac{\Gamma\left(\frac{\nu-s}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\nu+s}{2} + \frac{1}{4}\right)} p^{-s} ds. \quad (2.5)$$

By putting  $m = \mu - \lambda + \frac{1}{2}$  and then evaluating the integral, we get

$$x^\lambda F\left(\frac{1}{x}\right) \doteq 2^{2\mu-\lambda+1} p^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!} \frac{\Gamma\left(\mu+n+\frac{3}{2}\right) \Gamma\left(\frac{\mu+\nu+3}{2}+n\right)}{\Gamma\left(\frac{\lambda}{2}+n+\frac{3}{2}\right) \Gamma\left(\frac{\lambda}{2}+n+2\right) \Gamma\left(\frac{\nu-\mu}{2}-n-\frac{1}{2}\right)} p^{2n} \quad (2.6)$$

$$\text{or, } x^\lambda F\left(\frac{1}{x}\right) \doteq 2^{2\mu-\lambda+1} p^2 \frac{\Gamma\left(\mu+\frac{3}{2}\right) \Gamma\left(\frac{\mu+\nu+3}{2}\right)}{\Gamma\left(\frac{\lambda+3}{2}\right) \Gamma\left(\frac{\lambda}{2}+2\right) \Gamma\left(\frac{\nu-\mu-1}{2}\right)} \cdot {}_3F_2\left(\begin{matrix} \mu+\frac{3}{2}, \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2}+2; \end{matrix} p^2\right). \quad (2.7)$$

On writing  $\frac{x}{t}$  for  $x$  we have

$$\left(\frac{x}{t}\right)^\lambda F\left(\frac{t}{x}\right) \doteq 2^{2\mu-\lambda+1} \frac{\Gamma\left(\mu + \frac{3}{2}\right) \Gamma\left(\frac{\mu + \nu + 3}{2}\right) p^2 t^2}{\Gamma\left(\frac{\lambda + 3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma\left(\frac{\nu - \mu - 1}{2}\right)} \cdot {}_3F_2\left(\begin{matrix} \mu + \frac{3}{2}, \frac{\mu + \nu + 3}{2}, \frac{\mu - \nu + 3}{2}; \\ \frac{\lambda + 3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right). \quad (\text{A})$$

Let (i)  $p^{\mu-\lambda+\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(p) \doteq F(x), \quad (\text{B})$

(ii)  $\psi(p) \doteq f(x), \quad (\text{C})$

Also from (B),  $(ap)^{\mu-\lambda+\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(ap) \doteq F\left(\frac{x}{a}\right). \quad (\text{D})$

Applying Goldstein's theorem to (C) and (D), we get

$$\int_0^\infty f(t) (at)^{\mu-\lambda+\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(at) \frac{dt}{t} = \int_0^\infty \psi(t) F\left(\frac{t}{a}\right) \frac{dt}{t}. \quad (2.8)$$

On writing  $x$  for  $a$  and multiplying by  $x^\lambda$  to both sides, it follows

$$x^\lambda \int_0^\infty f(t) (xt)^{\mu-\lambda+\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(xt) \frac{dt}{t} = \int_0^\infty \psi(t) F\left(\frac{t}{x}\right) \left(\frac{x}{t}\right)^\lambda t^{\lambda-1} dt. \quad (2.9)$$

Interpreting with the help of (A), we get

$$x^{\mu+\frac{1}{2}} \int_0^\infty f(t) t^{\mu-\lambda-\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(xt) dt \doteq \frac{2^{2\mu-\lambda+1} \Gamma\left(\mu + \frac{3}{2}\right) \Gamma\left(\frac{\mu + \nu + 3}{2}\right)}{\Gamma\left(\frac{\lambda + 3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma\left(\frac{\nu - \mu - 1}{2}\right)} \cdot p^2 \int_0^\infty t^{\lambda+1} \psi(t) {}_3F_2\left(\begin{matrix} \mu + \frac{3}{2}, \frac{\mu + \nu + 3}{2}, \frac{\mu - \nu + 3}{2}; \\ \frac{\lambda + 3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right) dt. \quad (2.10)$$

If  $t^{\mu-\lambda-\frac{1}{2}} f(t)$  is  $R_{\mu,\nu}$ , we obtain the required result.

3. Example 1. Let  $t^{\frac{\mu-\nu+1}{2}} J_{\frac{\mu+\nu}{2}}(t)$  be  $R_{\mu,\nu}$ ,

then  $f(t) = t^{\lambda - \frac{\nu+\mu}{2} + 1} J_{\frac{\mu+\nu}{2}}(t)$

$$\doteq \frac{\Gamma(\lambda+2) p^{-\lambda-1}}{2^{\frac{\mu+\nu}{2}} \Gamma\left(\frac{\mu+\nu+2}{2}\right)} {}_2F_1\left(\begin{matrix} \frac{\lambda}{2} + 1, \frac{\lambda+3}{2}; \\ \frac{\mu+\nu+2}{2}; \end{matrix} -\frac{1}{p^2}\right) \equiv \psi(p)$$

and  $x^{2\mu-\lambda} f(x) = x^{\frac{3\mu-\nu}{2} + 1} J_{\frac{\mu+\nu}{2}}(x)$

$$\doteq \frac{\Gamma(2\mu+2) p^{-2\mu-1}}{2^{\frac{\mu+\nu}{2}} \Gamma\left(\frac{\mu+\nu+2}{2}\right)} {}_2F_1\left(\begin{matrix} \mu+1, \mu + \frac{3}{2}; \\ \frac{\mu+\nu+2}{2}; \end{matrix} -\frac{1}{p^2}\right). \quad (3.1)$$

Hence from (2.1) we get

$$\int_0^\infty {}_2F_1\left(\begin{matrix} \frac{\lambda}{2} + 1, \frac{\lambda+3}{2}; \\ \frac{\mu+\nu+2}{2}; \end{matrix} -\frac{1}{t^2}\right) {}_3F_2\left(\begin{matrix} \mu + \frac{3}{2}, \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right) dt$$

$$\equiv \frac{\Gamma\left(\frac{\lambda+3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma\left(\frac{\nu-\mu-1}{2}\right) \Gamma(2\mu+2) p^{-2\mu-3}}{2^{2\mu-\lambda+1} \Gamma\left(\mu + \frac{3}{2}\right) \Gamma\left(\frac{\mu+\nu+3}{2}\right) \Gamma(\lambda+2)} {}_2F_1\left(\begin{matrix} \mu+1, \mu + \frac{3}{2}; \\ \frac{\mu+\nu+2}{2}; \end{matrix} -\frac{1}{p^2}\right).$$

In particular, if  $\mu = \nu$ , then (3.2)

$$\int_0^\infty {}_2F_1\left(\begin{matrix} \frac{\lambda}{2} + 1, \frac{\lambda+3}{2}; \\ \nu+1; \end{matrix} -\frac{1}{t^2}\right) {}_3F_2\left(\begin{matrix} \nu + \frac{3}{2}, \nu + \frac{3}{2}, \frac{3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right) dt$$

$$\equiv \frac{\sqrt{\pi} \Gamma\left(\frac{\lambda+3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma(2\nu+2) p^{-2\nu-3}}{2^{2\nu-\lambda} \Gamma\left(\nu + \frac{3}{2}\right) \Gamma\left(\frac{2\nu+3}{2}\right) \Gamma(\lambda+2)} {}_1F_0\left(\nu + \frac{3}{2}; -\frac{1}{p^2}\right).$$

4. Example 2. Let

$$t^{n+\mu+\frac{1}{2}} K_n(t) \quad \text{be} \quad R_{\mu,\nu}, \quad \text{where} \quad \nu = \mu + 2n$$

then  $f(t) = t^{n+\lambda+1} K_n(t)$

$$\doteq \frac{\sin [(n + \lambda + 1) \pi] \Gamma(\lambda + 2)}{\sin [(2n + \lambda + 1) \pi] (p^2 - 1)^{\frac{n+\lambda+2}{2}}} Q_{n+\lambda+1}^n \left( \frac{p}{\sqrt{p^2 - 1}} \right) \equiv \psi(p)$$

and

$$x^{2\mu-\lambda} f(x) = x^{2\mu+n+1} K_n(x)$$

$$\doteq \frac{\sin [(2\mu + n + 1) \pi] \Gamma(2\mu + 2)}{\sin [(2\mu + 2n + 1) \pi] (p^2 - 1)^{\frac{2\mu+n+2}{2}}} Q_{2\mu+n+1}^n \left( \frac{p}{\sqrt{p^2 - 1}} \right).$$

Hence from (2.1) we get

$$\begin{aligned} & \int_0^\infty \frac{t^{\lambda+1}}{(t^2-1)^{\frac{2\mu+n+2}{2}}} Q_{2\mu+n+1}^n \left( \frac{t}{\sqrt{t^2-1}} \right) {}_3F_2 \left( \begin{matrix} \mu + \frac{3}{2}, \frac{2\mu+2n+3}{2}, \frac{3-2n}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2} + 2; \end{matrix} ; t^2 p^2 \right) dt \\ &= \frac{\Gamma(2\mu+2) \Gamma\left(\frac{\lambda+3}{2}\right) \Gamma\left(\frac{\lambda}{2}+2\right) \Gamma\left(\frac{2n-1}{2}\right) \sin [(2\mu+n+1)\pi] \sin [(2n+\lambda+1)\pi]}{2^{2\mu-\lambda+1} \Gamma\left(\mu+\frac{3}{2}\right) \Gamma\left(\frac{2\mu+2n+3}{2}\right) \Gamma(\lambda+2) \sin [(2\mu+n+1)\pi] \sin [(n+\lambda+1)\pi]} \\ & \cdot \frac{1}{p^2 (p^2-1)^{\frac{2\mu+n+2}{2}}} Q_{2\mu+n+1}^n \left( \frac{p}{\sqrt{p^2-1}} \right), \quad R(2\mu+2n-\lambda) > 0. \quad (4.1) \end{aligned}$$

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### References

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### Zusammenfassung

Der Autor beweist einen Satz aus dem Gebiet der Laplace-Transformationen und illustriert ihn an zwei Beispielen.

### Résumé

Dans la présente note, l'auteur établit un théorème appartenant à la transformation de Laplace et en donne deux applications.

### Riassunto

L'autore prova un teorema riguardante la trasformazione di Laplace e ne dà due esempi.