On certain integrals involving $\tilde{\omega}_{\mu,v}(x)$

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Summary

In this paper, the author has evaluated $\tilde{\omega}_{\mu,v}(x)$ transforms of a number of functions making use of the self-reciprocal property of some functions.

1. Introduction: The function $\tilde{\omega}_{\mu,v}(x)$ has been defined as

$$\tilde{\omega}_{\mu,v}(x) = \sqrt{x} \int_0^\infty J_\mu(xt)\, J_v\left(\frac{1}{t}\right) \frac{dt}{t}, \quad R(\mu,v) \geqq -\tfrac{1}{2}. \tag{1.1}$$

The integral on the right was first evaluated by Rao [4] and that it plays the role of a transform was conjectured by Watson [6]. Later, Bhatnagar [1] proved in detail that it plays the role of a transform. He also indicated the following properties of the function:

$$\begin{aligned}
&\text{(i)} \quad \tilde{\omega}_{\mu,v}(x) = \tilde{\omega}_{v,\mu}(x),\\
&\text{(ii)} \quad \tilde{\omega}_{v-1,v}(x) = J_{2v-1}(2\sqrt{x}),\\
&\text{(iii)} \quad \tilde{\omega}_{\mu,v}(x) = O(x^{v+\frac{1}{2}},\, x^{\mu+\frac{1}{2}}), \text{ when } x \text{ is small},\\
&\qquad\qquad\quad\;\; = O(x^{-\frac{1}{4}}), \text{ when } x \text{ is large}.
\end{aligned}$$

The object of the present paper is to obtain certain integrals involving the function $\tilde{\omega}_{\mu,v}(x)$. The paper is in continuation of my paper [5].

2. We have [3, pp. 368, (34)]

$$\text{(1)} \qquad \int_0^\infty x \sin\left(\frac{1}{2x}\right) K_0(ax)\, dx = \frac{\pi}{2a} J_1(\sqrt{a})\, K_1(\sqrt{a}).$$

On multiplying both sides by $a^{v+\frac{1}{2}}\tilde{\omega}_{v,v}(ab)$ and integrating with respect to a between the limits $(0,\infty)$, we get

$$\int_0^\infty a^{v-\frac{1}{2}} J_1(\sqrt{a})\, K_1(\sqrt{a})\, \tilde{\omega}_{v,v}(ab)\, da$$

$$= \frac{2}{\pi}\int_0^\infty a^{v+\frac{1}{2}}\tilde{\omega}_{v,v}(ab)\, da \int_0^\infty x \sin\left(\frac{1}{2x}\right) K_0(ax)\, dx$$

$$= \frac{2}{\pi}\int_0^\infty x^{-v-\frac{1}{2}} \sin\left(\frac{1}{2x}\right) dx \int_0^\infty a^{v+\frac{1}{2}} K_0(a)\, \tilde{\omega}_{v,v}\left(\frac{ab}{x}\right) da.$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(v) > -\frac{1}{2}$].

$$= \frac{2}{\pi} b^{v+\frac{1}{2}} \int_0^\infty x^{2v-1} K_0(bx) \sin\left(\tfrac{1}{2}x\right) dx, \text{ since } a^{v+\frac{1}{2}} K_0(a) \text{ is } R_{v,v}. \text{ *)}$$

Evaluating the integral on the right hand side, we get

$$\int_0^\infty a^{v-\frac{1}{2}} J_1(\sqrt{a})\, K_1(\sqrt{a})\, \tilde{\omega}_{v,v}(ab)\, da$$

$$= \frac{2^{2v-1}\{\Gamma(\nu+\frac{1}{2})\}^2}{\pi b^{v+\frac{1}{2}}}\, {}_2F_1(v+\tfrac{1}{2}, v+\tfrac{1}{2}; \tfrac{3}{2}; -\tfrac{1}{4}b^{-2}),\quad R(v) > -\tfrac{1}{2}. \qquad (2.1)$$

Also from (1) we get

$$\int_0^\infty a\, J_1(\sqrt{a})\, K_1(\sqrt{a})\, \tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(ab)\, da$$

$$= \frac{4}{\pi}\int_0^\infty x^2 K_0(x)\, dx \int_0^\infty \sin(a)\, \tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(2abx)\, da,$$

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*) If $f(x) = \int_0^\infty \tilde{\omega}_{\mu,v}(xy) f(y)\, dy$, then $f(x)$ is called $R_{\mu,v}$.

the change in the order of integrations is justified.

$$= \frac{4}{\pi}\int_0^\infty x^2 K_0(x) \sin(2bx)\, dx, \text{ since } \sin(a) \text{ is } R_{\frac{1}{2},\frac{1}{2}}.$$

Hence we obtain

$$\int_0^\infty a\, J_1(\sqrt{a})\, K_1(\sqrt{a})\, \tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(ab)\, da = \frac{32}{\pi} b\, {}_2F_1(2,2;\tfrac{3}{2};-4b^2). \qquad (2.2)$$

Again, on using the result [3, pp. 369, (35)]

$$\int_0^\infty x \cos\left(\frac{1}{2x}\right) K_0(ax)\, dx = -\frac{\pi}{2a} Y_1(\sqrt{a})\, K_1(\sqrt{a}),$$

we get by a method similar to the above

$$\int_0^\infty a^{v-\frac{1}{2}}\, Y_1(\sqrt{a})\, K_1(\sqrt{a})\, \tilde{\omega}_{v,v}(ab)\, da = -\frac{2^{2v-1}\{\Gamma(v)\}^2}{\pi b^{v-\frac{1}{2}}}\, {}_2F_1(v,v;\tfrac{1}{2};-\tfrac{1}{4}b^{-2}),$$

$$R(v) > 0; \qquad (2.3)$$

and

$$\int_0^\infty a\, Y_1(\sqrt{a})\, K_1(\sqrt{a})\, \tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)\, da = \frac{2(8b^2-1)}{(1+4b^2)^{5/2}}. \qquad (2.4)$$

3. We have [3, pp. 344, (34)]

$$\int_1^\infty (x^2-1)^{-\frac{1}{2}} J_v(ax)\, dx = -\frac{\pi}{2} J_{\frac{1}{2}v}(\tfrac{1}{2}a)\, Y_{\frac{1}{2}v}(\tfrac{1}{2}a), \quad R(v) > -1.$$

On multiplying both sides by $a^\lambda\, \tilde{\omega}_{v-\lambda+\frac{1}{2},\, v+\lambda-\frac{1}{2}}(ay)$ and integrating with respect to a between the limits $(0,\infty)$, we get

$$\int_0^\infty a^\lambda J_{\frac{1}{2}v}(\tfrac{1}{2}a)\, Y_{\frac{1}{2}v}(\tfrac{1}{2}a)\, \tilde{\omega}_{v-\lambda+\frac{1}{2},\, v+\lambda-\frac{1}{2}}(ay)\, da$$

$$= -\frac{2}{\pi}\int_1^\infty (x^2-1)^{-\frac{1}{2}} dx \int_0^\infty a^\lambda J_v(ax)\, \tilde{\omega}_{v-\lambda+\frac{1}{2},\, v+\lambda-\frac{1}{2}}(ay)\, da.$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(\lambda+v) \geqq 0$, $R(v-\lambda) \geqq -1$, $R(\lambda) < -\frac{1}{4}$]

$$= -\frac{2}{\pi} y^{\lambda} \int_0^1 x^{2\lambda}(1-x^2)^{-\frac{1}{2}} J_v(xy)\, dx, \text{ since } a^{\lambda} J_v(a) \text{ is } R_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}, \ [1]$$

$$= -\frac{y^{v+\lambda}\,\Gamma(\lambda+\frac{1}{2}v+\frac{1}{2})}{\sqrt{\pi}\; 2^v\, \Gamma(\lambda+\frac{1}{2}v+1)\,\Gamma(v+1)}\, {}_1F_2(\lambda+\tfrac{1}{2}v+\tfrac{1}{2};\, \lambda+\tfrac{1}{2}v+1, v+1;\, -y^2);$$

$$R(2\lambda+v) > -1.$$

{The integral on the right has been evaluated with the help of [3, pp. 26, (34)]}.

Hence we obtain

$$\int_0^\infty a^{\lambda} J_{\frac{1}{2}v}(\tfrac{1}{2}a)\, Y_{\frac{1}{2}v}(\tfrac{1}{2}a)\, \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay)\, da$$

$$= -\frac{y^{v+\lambda}\,\Gamma(\lambda+\frac{1}{2}v+\frac{1}{2})}{\sqrt{\pi}\; 2^v\, \Gamma(\lambda+\frac{1}{2}v+1)\,\Gamma(v+1)}\, {}_1F_2(\lambda+\tfrac{1}{2}v+\tfrac{1}{2};\, \lambda+\tfrac{1}{2}v+1, v+1;\, -\tfrac{1}{4}y^2);$$

$$R(\lambda+v) \geqq 0,\ R(v-\lambda) \geqq -1,\ R(2\lambda+v) > -1,\ R(\lambda) < \tfrac{3}{4}. \qquad (3.1)$$

4. We have [3, pp. 385, (12)]

$$\int_0^\infty \cos\left(\frac{2a}{x}\right)[I_0(x)-L_0(x)]\frac{dx}{x} = 2J_0(2\sqrt{a})\, K_0(2\sqrt{a}).$$

Multiplying both sides by $\tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)$ and integrating with respect to a between the limits $(0,\infty)$, we get

$$2\int_0^\infty J_0(2\sqrt{a})\, K_0(2\sqrt{a})\, \tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)\, da$$

$$= \int_0^\infty \tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)\, da \int_0^\infty \cos\left(\frac{2a}{x}\right)[I_0(x)-L_0(x)]\frac{dx}{x}$$

$$= \int_0^\infty [I_0(x)-L_0(x)]\frac{dx}{x} \int_0^\infty \tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)\cos\left(\frac{2a}{x}\right) da,$$

on changing the order of integrations, which is justified.

$$= \frac{1}{2} \int_0^\infty [I_0(x) - L_0(x)] \cos\left(\tfrac{1}{2} bx\right) dx, \text{ since } \cos(x) \text{ is } R_{-\frac{1}{2},-\frac{1}{2}}, \ [1]$$

$$= \frac{1}{\pi} \int_0^\infty \sin(u)\, K_0\left(\tfrac{1}{2} bu\right) du,$$

$$= \frac{1}{\pi}\left(1 + \frac{b^2}{4}\right)^{-\frac{1}{2}} \log\left(\frac{2 + \sqrt{b^2+4}}{b}\right), \text{ on using [2, pp, 105, (46)].}$$

Thus we obtain

$$\int_0^\infty J_0(2\sqrt{a})\, K_0(2\sqrt{a})\, \tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)\, da = \frac{1}{\pi}(b^2+4)^{-\frac{1}{2}} \log\left[\frac{2 + (b^2+4)^{\frac{1}{2}}}{b}\right]. \quad (4.1)$$

By a similar method we can prove that

$$\int_0^\infty J_0(2\sqrt{a})\, K_0(2\sqrt{a})\, \tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(ab)\, da = \frac{1}{\pi}(b^2+4)^{-\frac{1}{2}} \log\left[\frac{b + (b^2+4)^{\frac{1}{2}}}{2}\right]. \quad (4.2)$$

5. We know that [3, pp. 347, (52)]

$$\int_0^\infty \cos\left(\frac{a}{2x}\right)[\sin(x)\, Y_0(x) - \cos(x)\, J_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a})\, Y_0(\sqrt{a}), \ a > 0.$$

Multiplying both sides by $\tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$I = \int_0^\infty J_0(\sqrt{a})\, Y_0(\sqrt{a})\, \tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)\, da$$

$$= \frac{1}{\pi} \int_0^\infty [\sin(x)\, Y_0(x) - \cos(x)\, J_0(x)] \frac{dx}{x} \int_0^\infty \cos\left(\frac{a}{2x}\right) \tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)\, da.$$

(The change in the order of integrations on the right hand side is justified).

Hence we obtain

$$I = \frac{2}{\pi}\int_0^\infty [\sin(x)\,Y_0(x) - \cos(x)\,J_0(x)]\cos(2bx)\,dx, \text{ since } \cos(x) \text{ is } R_{-\frac{1}{2},-\frac{1}{2}}. \, [1]$$

Evaluating the right hand side with the help of [2, pp.103, (31), and pp.46, (19)] we have

$$\int_0^\infty J_0(\sqrt{a})\,Y_0(\sqrt{a})\,\tilde{\omega}_{-\frac{1}{2},-\frac{1}{2}}(ab)\,da = -\frac{1}{2\pi}(b-b^2)^{-\frac{1}{2}}$$

$$+\frac{1}{\pi^2}\left[(b+b^2)^{-\frac{1}{2}}\log\left\{(1+2b)-2\sqrt{b+b^2}\right\}+(b-b^2)^{-\frac{1}{2}}\sin^{-1}(1-2b)\right],$$

$$0 < b < 1. \qquad (5.1)$$

$$= \frac{1}{\pi^2}\left[(b+b^2)^{-\frac{1}{2}}\log\left\{(2b+1)-2\sqrt{b+b^2}\right\}\right]$$

$$-\frac{1}{\pi^2}\left[(b^2-b)^{-\frac{1}{2}}\log\left\{(2b-1)-2\sqrt{b^2-b}\right\}\right], \quad b > 1. \qquad (5.2)$$

6. On using the result [3, pp.346, (51)]

$$\int_0^\infty [\sin(x)\,J_0(x)+\cos(x)\,Y_0(x)]\sin\left(\frac{a}{2x}\right)\frac{dx}{x} = \pi\,J_0(\sqrt{a})\,Y_0(\sqrt{a}),$$

we get

$$\int_0^\infty J_0(\sqrt{a})\,Y_0(\sqrt{a})\,\tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(ab)\,da$$

$$= \frac{1}{\pi}\int_0^\infty [\sin(x)\,J_0(x)+\cos(x)\,Y_0(x)]\frac{dx}{x}\int_0^\infty \sin\left(\frac{a}{2x}\right)\tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(ab)\,da.$$

[The change in the order of integrations on the right hand side is justified.]

$$= \frac{2}{\pi}\int_0^\infty [\sin(x)\,J_0(x)+\cos(x)\,Y_0(x)]\sin(2bx)\,dx,$$

since $\sin(x)$ is $R_{\frac{1}{2},\frac{1}{2}}$. [1]

On evaluating the integrals on the right hand side, we get

$$\int_0^\infty J_0(\sqrt{a})\, Y_0(\sqrt{a})\, \tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(ab)\, da$$

$$= \frac{1}{2\pi}(b-b^2)^{-\frac{1}{2}} + \frac{1}{\pi^2}(b+b^2)^{-\frac{1}{2}} \log\left[(2b+1)-2\sqrt{b+b^2}\right]$$

$$-\frac{1}{\pi^2}(b-b^2)^{-\frac{1}{2}} \sin^{-1}(1-2b), \qquad 0<b<\tfrac{1}{2}. \qquad (6.1)$$

$$= \frac{1}{2\pi}(b-b^2)^{-\frac{1}{2}} + \frac{1}{\pi^2}(b+b^2)^{-\frac{1}{2}} \log\left[(2b+1)-2(b+b^2)^{\frac{1}{2}}\right]$$

$$+\frac{1}{\pi^2}(b-b^2)^{-\frac{1}{2}} \sin^{-1}(2b-1), \qquad \tfrac{1}{2}<b<1. \qquad (6.2)$$

$$= \frac{1}{\pi^2}(b+b^2)^{-\frac{1}{2}} \log\left[(2b+1)-2(b+b^2)^{\frac{1}{2}}\right]$$

$$+\frac{1}{\pi^2}(b^2-b)^{-\frac{1}{2}} \log\left[(2b-1)-2(b^2-b)^{\frac{1}{2}}\right], \qquad b>1. \qquad (6.3)$$

7. We have the result [3, pp. 333, (7)]

$$\int_0^1 x^{-v}(1-x^2)^{-v-\frac{1}{2}} J_v(ax)\, dx = \sqrt{\pi}\, 2^{-v-1} a^v\, \Gamma(\tfrac{1}{2}-v)\, J_v(\tfrac{1}{2}a)\, J_{-v}(\tfrac{1}{2}a), \qquad R(v)<\tfrac{1}{2}.$$

Therefore we get

$$\int_0^\infty a^{v+\lambda} J_v(\tfrac{1}{2}a)\, J_{-v}(\tfrac{1}{2}a)\, \tilde{\omega}_{v-\lambda+\frac{1}{2},\, v+\lambda-\frac{1}{2}}(ay)\, da$$

$$= \frac{2^{v+1}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-v)} \int_0^\infty a^\lambda\, \tilde{\omega}_{v-\lambda+\frac{1}{2},\, v+\lambda-\frac{1}{2}}(ay)\, da \int_0^1 (1-x^2)^{-v-\frac{1}{2}} J_v(ax) \frac{dx}{x^v}$$

$$= \frac{2^{v+1}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-v)} \int_0^1 x^{-v-\lambda-1}(1-x^2)^{-v-\frac{1}{2}}\, dx \int_0^\infty a^\lambda J_v(a)\, \tilde{\omega}_{v-\lambda+\frac{1}{2},\, v+\lambda-\frac{1}{2}}\left(\frac{ay}{x}\right) da.$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(v)>-\frac{1}{2}$, $R(\lambda)<-\frac{1}{4}$, $R(v+\lambda)>-\frac{1}{2}$, $R(v)<\frac{1}{2}$.]

$$= \frac{2^{v+1} y^\lambda}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-v)} \int_0^1 x^{-v-2\lambda-1}(1-x^2)^{-v-\frac{1}{2}} J_v\left(\frac{y}{x}\right) dx, \quad \text{since } a^\lambda J_v(a) \text{ is } R_{v-\lambda+\frac{1}{2},\, v+\lambda-\frac{1}{2}}.$$

Evaluating the integral on the right hand side with the help of [3, pp. 205, (36)], we get

$$\int_0^\infty a^{v+\lambda} J_v(\tfrac{1}{2}a)\, J_{-v}(\tfrac{1}{2}a)\, \tilde{\omega}_{v-\lambda+\frac{1}{2},\, v+\lambda-\frac{1}{2}}(ay)\, da$$

$$= \frac{2^{4v+2\lambda-1}}{\sqrt{\pi}\, y^{3v+\lambda-1}} G_{13}^{20}\left(\frac{y^2}{4}\,\middle|\, \begin{matrix} 0 \\ v-\frac{1}{2},\, 2v+\lambda-\frac{1}{2},\, v+\lambda-\frac{1}{2} \end{matrix}\right),$$

$$R(2\lambda+v) < \tfrac{3}{2},\quad R(v+\tfrac{1}{2}) > 0,\quad \tfrac{1}{4} > R(v+\lambda) \geqq 0,\quad R(v) < \tfrac{1}{2}. \qquad (7.1)$$

If we take $\lambda = -2v$, we obtain

$$\int_0^\infty a^{-v} J_v(\tfrac{1}{2}a)\, J_{-v}(\tfrac{1}{2}a)\, \tilde{\omega}_{3v+\frac{1}{2},\, -v-\frac{1}{2}}(ay)\, da = -y^{-v} J_v(\tfrac{1}{2}y)\, Y_v(\tfrac{1}{2}y). \qquad (7.2)$$

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Zusammenfassung

In der vorliegenden Arbeit berechnet der Autor $\tilde{\omega}_{\mu,v}(x)$-Transformationen verschiedener Funktionen. Dabei benützt er wesentlich die Tatsache, dass sich gewisse Funktionen unter dieser Transformation selbst reproduzieren.

Résumé

Dans le présent travail, l'auteur soumet diverses fonctions à la transformation $\tilde{\omega}_{\mu,v}(x)$. Il y utilise principalement le fait que certaines fonctions se reproduisent par cette transformation.

Riassunto

Nel presente lavoro l'autore calcola $\tilde{\omega}_{\mu,v}(x)$-trasformazioni di differenti funzioni. Egli impiega in modo essenziale il fatto che certe funzioni sottomesse à questa trasformazione si riproducono da se stesse.