

On certain integrals involving , $v(x)$

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On certain integrals involving $\tilde{\omega}_{\mu,v}(x)$

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Summary

In this paper, the author has evaluated $\tilde{\omega}_{\mu,v}(x)$ transforms of a number of functions making use of the self-reciprocal property of some functions.

1. Introduction: The function $\tilde{\omega}_{\mu,v}(x)$ has been defined as

$$\tilde{\omega}_{\mu,v}(x) = \sqrt{x} \int_0^{\infty} J_{\mu}(xt) J_{\nu}\left(\frac{1}{t}\right) \frac{dt}{t}, \quad R(\mu, \nu) \geq -\frac{1}{2}. \quad (1.1)$$

The integral on the right was first evaluated by Rao [4] and that it plays the role of a transform was conjectured by Watson [6]. Later, Bhatnagar [1] proved in detail that it plays the role of a transform. He also indicated the following properties of the function:

- (i) $\tilde{\omega}_{\mu,v}(x) = \tilde{\omega}_{v,\mu}(x)$,
- (ii) $\tilde{\omega}_{v-1,v}(x) = J_{2v-1}(2\sqrt{x})$,
- (iii) $\tilde{\omega}_{\mu,v}(x) = O(x^{v+\frac{1}{2}}, x^{\mu+\frac{1}{2}})$, when x is small,
 $= O(x^{-\frac{1}{4}})$, when x is large.

The object of the present paper is to obtain certain integrals involving the function $\tilde{\omega}_{\mu,v}(x)$. The paper is in continuation of my paper [5].

2. We have [3, pp. 368, (34)]

$$(1) \quad \int_0^{\infty} x \sin\left(\frac{1}{2x}\right) K_0(ax) dx = \frac{\pi}{2a} J_1(\sqrt{a}) K_1(\sqrt{a}).$$

On multiplying both sides by $a^{v+\frac{1}{2}}\tilde{\omega}_{v,v}(ab)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$\begin{aligned} & \int_0^{\infty} a^{v-\frac{1}{2}} J_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{v,v}(ab) da \\ &= \frac{2}{\pi} \int_0^{\infty} a^{v+\frac{1}{2}} \tilde{\omega}_{v,v}(ab) da \int_0^{\infty} x \sin\left(\frac{1}{2x}\right) K_0(ax) dx \\ &= \frac{2}{\pi} \int_0^{\infty} x^{-v-\frac{1}{2}} \sin\left(\frac{1}{2x}\right) dx \int_0^{\infty} a^{v+\frac{1}{2}} K_0(a) \tilde{\omega}_{v,v}\left(\frac{ab}{x}\right) da. \end{aligned}$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(v) > -\frac{1}{2}$].

$$= \frac{2}{\pi} b^{v+\frac{1}{2}} \int_0^{\infty} x^{2v-1} K_0(bx) \sin\left(\frac{1}{2}x\right) dx, \text{ since } a^{v+\frac{1}{2}} K_0(a) \text{ is } R_{v,v}. *$$

Evaluating the integral on the right hand side, we get

$$\begin{aligned} & \int_0^{\infty} a^{v-\frac{1}{2}} J_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{v,v}(ab) da \\ &= \frac{2^{2v-1} \{\Gamma(v + \frac{1}{2})\}^2}{\pi b^{v+\frac{1}{2}}} {}_2F_1\left(v + \frac{1}{2}, v + \frac{1}{2}; \frac{3}{2}; -\frac{1}{4}b^{-2}\right), \quad R(v) > -\frac{1}{2}. \quad (2.1) \end{aligned}$$

Also from (1) we get

$$\begin{aligned} & \int_0^{\infty} a J_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(ab) da \\ &= \frac{4}{\pi} \int_0^{\infty} x^2 K_0(x) dx \int_0^{\infty} \sin(a) \tilde{\omega}_{\frac{1}{2},\frac{1}{2}}(2abx) da, \end{aligned}$$

*) If $f(x) = \int_0^{\infty} \tilde{\omega}_{\mu,v}(xy) f(y) dy$, then $f(x)$ is called $R_{\mu,v}$.

the change in the order of integrations is justified.

$$= \frac{4}{\pi} \int_0^{\infty} x^2 K_0(x) \sin(2bx) dx, \text{ since } \sin(a) \text{ is } R_{\frac{1}{2}, \frac{1}{2}}.$$

Hence we obtain

$$\int_0^{\infty} a J_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da = \frac{32}{\pi} b {}_2F_1(2, 2; \frac{3}{2}; -4b^2). \quad (2.2)$$

Again, on using the result [3, pp. 369, (35)]

$$\int_0^{\infty} x \cos\left(\frac{1}{2x}\right) K_0(ax) dx = -\frac{\pi}{2a} Y_1(\sqrt{a}) K_1(\sqrt{a}),$$

we get by a method similar to the above

$$\int_0^{\infty} a^{v-\frac{1}{2}} Y_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{v, v}(ab) da = -\frac{2^{2v-1} \{\Gamma(v)\}^2}{\pi b^{v-\frac{1}{2}}} {}_2F_1(v, v; \frac{1}{2}; -\frac{1}{4} b^{-2}),$$

$R(v) > 0;$ (2.3)

and

$$\int_0^{\infty} a Y_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da = \frac{2(8b^2-1)}{(1+4b^2)^{5/2}}. \quad (2.4)$$

3. We have [3, pp. 344, (34)]

$$\int_1^{\infty} (x^2-1)^{-\frac{1}{2}} J_v(ax) dx = -\frac{\pi}{2} J_{\frac{1}{2}v}(\frac{1}{2}a) Y_{\frac{1}{2}v}(\frac{1}{2}a), \quad R(v) > -1.$$

On multiplying both sides by $a^\lambda \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$\begin{aligned} & \int_0^{\infty} a^\lambda J_{\frac{1}{2}v}(\frac{1}{2}a) Y_{\frac{1}{2}v}(\frac{1}{2}a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \\ &= -\frac{2}{\pi} \int_1^{\infty} (x^2-1)^{-\frac{1}{2}} dx \int_0^{\infty} a^\lambda J_v(ax) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da. \end{aligned}$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(\lambda + v) \geq 0$, $R(v - \lambda) \geq -1$, $R(\lambda) < -\frac{1}{4}$]

$$\begin{aligned}
 &= -\frac{2}{\pi} y^\lambda \int_0^1 x^{2\lambda} (1-x^2)^{-\frac{1}{2}} J_v(xy) dx, \text{ since } a^\lambda J_v(a) \text{ is } R_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}, [1] \\
 &= -\frac{y^{v+\lambda} \Gamma(\lambda + \frac{1}{2}v + \frac{1}{2})}{\sqrt{\pi} 2^v \Gamma(\lambda + \frac{1}{2}v + 1) \Gamma(v+1)} {}_1F_2(\lambda + \frac{1}{2}v + \frac{1}{2}; \lambda + \frac{1}{2}v + 1, v + 1; -y^2); \\
 & \hspace{15em} R(2\lambda + v) > -1.
 \end{aligned}$$

{The integral on the right has been evaluated with the help of [3, pp. 26, (34)]}.

Hence we obtain

$$\begin{aligned}
 &\int_0^\infty a^\lambda J_{\frac{1}{2}v}(\frac{1}{2}a) Y_{\frac{1}{2}v}(\frac{1}{2}a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \\
 &= -\frac{y^{v+\lambda} \Gamma(\lambda + \frac{1}{2}v + \frac{1}{2})}{\sqrt{\pi} 2^v \Gamma(\lambda + \frac{1}{2}v + 1) \Gamma(v+1)} {}_1F_2(\lambda + \frac{1}{2}v + \frac{1}{2}; \lambda + \frac{1}{2}v + 1, v + 1; -\frac{1}{4}y^2); \\
 & \hspace{10em} R(\lambda + v) \geq 0, R(v - \lambda) \geq -1, R(2\lambda + v) > -1, R(\lambda) < \frac{3}{4}. \quad (3.1)
 \end{aligned}$$

4. We have [3, pp. 385, (12)]

$$\int_0^\infty \cos\left(\frac{2a}{x}\right) [I_0(x) - L_0(x)] \frac{dx}{x} = 2J_0(2\sqrt{a}) K_0(2\sqrt{a}).$$

Multiplying both sides by $\tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$\begin{aligned}
 &2 \int_0^\infty J_0(2\sqrt{a}) K_0(2\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da \\
 &= \int_0^\infty \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da \int_0^\infty \cos\left(\frac{2a}{x}\right) [I_0(x) - L_0(x)] \frac{dx}{x} \\
 &= \int_0^\infty [I_0(x) - L_0(x)] \frac{dx}{x} \int_0^\infty \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) \cos\left(\frac{2a}{x}\right) da,
 \end{aligned}$$

on changing the order of integrations, which is justified.

$$\begin{aligned}
 &= \frac{1}{2} \int_0^{\infty} [I_0(x) - L_0(x)] \cos\left(\frac{1}{2}bx\right) dx, \text{ since } \cos(x) \text{ is } R_{-\frac{1}{2}, -\frac{1}{2}}, \quad [1] \\
 &= \frac{1}{\pi} \int_0^{\infty} \sin(u) K_0\left(\frac{1}{2}bu\right) du, \\
 &= \frac{1}{\pi} \left(1 + \frac{b^2}{4}\right)^{-\frac{1}{2}} \log\left(\frac{2 + \sqrt{b^2 + 4}}{b}\right), \text{ on using [2, pp, 105, (46)].}
 \end{aligned}$$

Thus we obtain

$$\int_0^{\infty} J_0(2\sqrt{a}) K_0(2\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da = \frac{1}{\pi} (b^2 + 4)^{-\frac{1}{2}} \log\left[\frac{2 + (b^2 + 4)^{\frac{1}{2}}}{b}\right]. \quad (4.1)$$

By a similar method we can prove that

$$\int_0^{\infty} J_0(2\sqrt{a}) K_0(2\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da = \frac{1}{\pi} (b^2 + 4)^{-\frac{1}{2}} \log\left[\frac{b + (b^2 + 4)^{\frac{1}{2}}}{2}\right]. \quad (4.2)$$

5. We know that [3, pp. 347, (52)]

$$\int_0^{\infty} \cos\left(\frac{a}{2x}\right) [\sin(x) Y_0(x) - \cos(x) J_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a}) Y_0(\sqrt{a}), \quad a > 0.$$

Multiplying both sides by $\tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$\begin{aligned}
 I &= \int_0^{\infty} J_0(\sqrt{a}) Y_0(\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da \\
 &= \frac{1}{\pi} \int_0^{\infty} [\sin(x) Y_0(x) - \cos(x) J_0(x)] \frac{dx}{x} \int_0^{\infty} \cos\left(\frac{a}{2x}\right) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da.
 \end{aligned}$$

(The change in the order of integrations on the right hand side is justified).

Hence we obtain

$$I = \frac{2}{\pi} \int_0^{\infty} [\sin(x) Y_0(x) - \cos(x) J_0(x)] \cos(2bx) dx, \text{ since } \cos(x) \text{ is } R_{-\frac{1}{2}, -\frac{1}{2}}. [1]$$

Evaluating the right hand side with the help of [2, pp.103, (31), and pp.46, (19)] we have

$$\begin{aligned} \int_0^{\infty} J_0(\sqrt{a}) Y_0(\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da &= -\frac{1}{2\pi} (b-b^2)^{-\frac{1}{2}} \\ &+ \frac{1}{\pi^2} [(b+b^2)^{-\frac{1}{2}} \log \{(1+2b) - 2\sqrt{b+b^2}\} + (b-b^2)^{-\frac{1}{2}} \sin^{-1}(1-2b)], \\ &0 < b < 1. \quad (5.1) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\pi^2} [(b+b^2)^{-\frac{1}{2}} \log \{(2b+1) - 2\sqrt{b+b^2}\}] \\ &\quad - \frac{1}{\pi^2} [(b^2-b)^{-\frac{1}{2}} \log \{(2b-1) - 2\sqrt{b^2-b}\}], \quad b > 1. \quad (5.2) \end{aligned}$$

6. On using the result [3, pp.346, (51)]

$$\int_0^{\infty} [\sin(x) J_0(x) + \cos(x) Y_0(x)] \sin\left(\frac{a}{2x}\right) \frac{dx}{x} = \pi J_0(\sqrt{a}) Y_0(\sqrt{a}),$$

we get

$$\begin{aligned} &\int_0^{\infty} J_0(\sqrt{a}) Y_0(\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da \\ &= \frac{1}{\pi} \int_0^{\infty} [\sin(x) J_0(x) + \cos(x) Y_0(x)] \frac{dx}{x} \int_0^{\infty} \sin\left(\frac{a}{2x}\right) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da. \end{aligned}$$

[The change in the order of integrations on the right hand side is justified.]

$$\begin{aligned} &= \frac{2}{\pi} \int_0^{\infty} [\sin(x) J_0(x) + \cos(x) Y_0(x)] \sin(2bx) dx, \\ &\hspace{15em} \text{since } \sin(x) \text{ is } R_{\frac{1}{2}, \frac{1}{2}}. [1] \end{aligned}$$

On evaluating the integrals on the right hand side, we get

$$\begin{aligned} & \int_0^{\infty} J_0(\sqrt{a}) Y_0(\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da \\ &= \frac{1}{2\pi} (b-b^2)^{-\frac{1}{2}} + \frac{1}{\pi^2} (b+b^2)^{-\frac{1}{2}} \log [(2b+1) - 2\sqrt{b+b^2}] \\ & \quad - \frac{1}{\pi^2} (b-b^2)^{-\frac{1}{2}} \sin^{-1}(1-2b), \quad 0 < b < \frac{1}{2}. \quad (6.1) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\pi} (b-b^2)^{-\frac{1}{2}} + \frac{1}{\pi^2} (b+b^2)^{-\frac{1}{2}} \log [(2b+1) - 2(b+b^2)^{\frac{1}{2}}] \\ & \quad + \frac{1}{\pi^2} (b-b^2)^{-\frac{1}{2}} \sin^{-1}(2b-1), \quad \frac{1}{2} < b < 1. \quad (6.2) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\pi^2} (b+b^2)^{-\frac{1}{2}} \log [(2b+1) - 2(b+b^2)^{\frac{1}{2}}] \\ & \quad + \frac{1}{\pi^2} (b^2-b)^{-\frac{1}{2}} \log [(2b-1) - 2(b^2-b)^{\frac{1}{2}}], \quad b > 1. \quad (6.3) \end{aligned}$$

7. We have the result [3, pp. 333, (7)]

$$\int_0^1 x^{-v}(1-x^2)^{-v-\frac{1}{2}} J_v(ax) dx = \sqrt{\pi} 2^{-v-1} a^v \Gamma(\frac{1}{2}-v) J_v(\frac{1}{2}a) J_{-v}(\frac{1}{2}a),$$

$R(v) < \frac{1}{2}.$

Therefore we get

$$\begin{aligned} & \int_0^{\infty} a^{v+\lambda} J_v(\frac{1}{2}a) J_{-v}(\frac{1}{2}a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \\ &= \frac{2^{v+1}}{\sqrt{\pi} \Gamma(\frac{1}{2}-v)} \int_0^{\infty} a^\lambda \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \int_0^1 (1-x^2)^{-v-\frac{1}{2}} J_v(ax) \frac{dx}{x^v} \\ &= \frac{2^{v+1}}{\sqrt{\pi} \Gamma(\frac{1}{2}-v)} \int_0^1 x^{-v-\lambda-1} (1-x^2)^{-v-\frac{1}{2}} dx \int_0^{\infty} a^\lambda J_v(a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}\left(\frac{ay}{x}\right) da. \end{aligned}$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(v) > -\frac{1}{2}$, $R(\lambda) < -\frac{1}{4}$, $R(v+\lambda) > -\frac{1}{2}$, $R(v) < \frac{1}{2}$.]

$$= \frac{2^{v+1} y^\lambda}{\sqrt{\pi} \Gamma(\frac{1}{2}-v)} \int_0^1 x^{-v-2\lambda-1} (1-x^2)^{-v-\frac{1}{2}} J_v\left(\frac{y}{x}\right) dx, \quad \text{since } a^\lambda J_v(a) \text{ is}$$

$R_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}.$

Evaluating the integral on the right hand side with the help of [3, pp.205, (36)], we get

$$\int_0^{\infty} a^{v+\lambda} J_v\left(\frac{1}{2}a\right) J_{-v}\left(\frac{1}{2}a\right) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da$$

$$= \frac{2^{4v+2\lambda-1}}{\sqrt{\pi} y^{3v+\lambda-1}} G_{13}^{20} \left(\frac{y^2}{4} \middle| \begin{matrix} 0 \\ v-\frac{1}{2}, 2v+\lambda-\frac{1}{2}, v+\lambda-\frac{1}{2} \end{matrix} \right),$$

$$R(2\lambda+v) < \frac{3}{2}, \quad R(v+\frac{1}{2}) > 0, \quad \frac{1}{4} > R(v+\lambda) \geq 0, \quad R(v) < \frac{1}{2}. \quad (7.1)$$

If we take $\lambda = -2v$, we obtain

$$\int_0^{\infty} a^{-v} J_v\left(\frac{1}{2}a\right) J_{-v}\left(\frac{1}{2}a\right) \tilde{\omega}_{3v+\frac{1}{2}, -v-\frac{1}{2}}(ay) da = -y^{-v} J_v\left(\frac{1}{2}y\right) Y_v\left(\frac{1}{2}y\right). \quad (7.2)$$

In the end, I wish to express my sincere thanks to Dr. S. C. Mitra, Research Professor in Mathematics, (B. I. T. S.), Pilani, for the help and guidance in the preparation of this paper.

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Zusammenfassung

In der vorliegenden Arbeit berechnet der Autor $\tilde{\omega}_{\mu,v}(x)$ -Transformationen verschiedener Funktionen. Dabei benützt er wesentlich die Tatsache, dass sich gewisse Funktionen unter dieser Transformation selbst reproduzieren.

Résumé

Dans le présent travail, l'auteur soumet diverses fonctions à la transformation $\tilde{\omega}_{\mu,v}(x)$. Il y utilise principalement le fait que certaines fonctions se reproduisent par cette transformation.

Riassunto

Nel presente lavoro l'autore calcola $\tilde{\omega}_{\mu,v}(x)$ -trasformazioni di differenti funzioni. Egli impiega in modo essenziale il fatto che certe funzioni sottomesse à questa trasformazione si riproducono da se stesse.