Two economic indicators measuring modification of expenditure in dynamic forecast of pension-schemes

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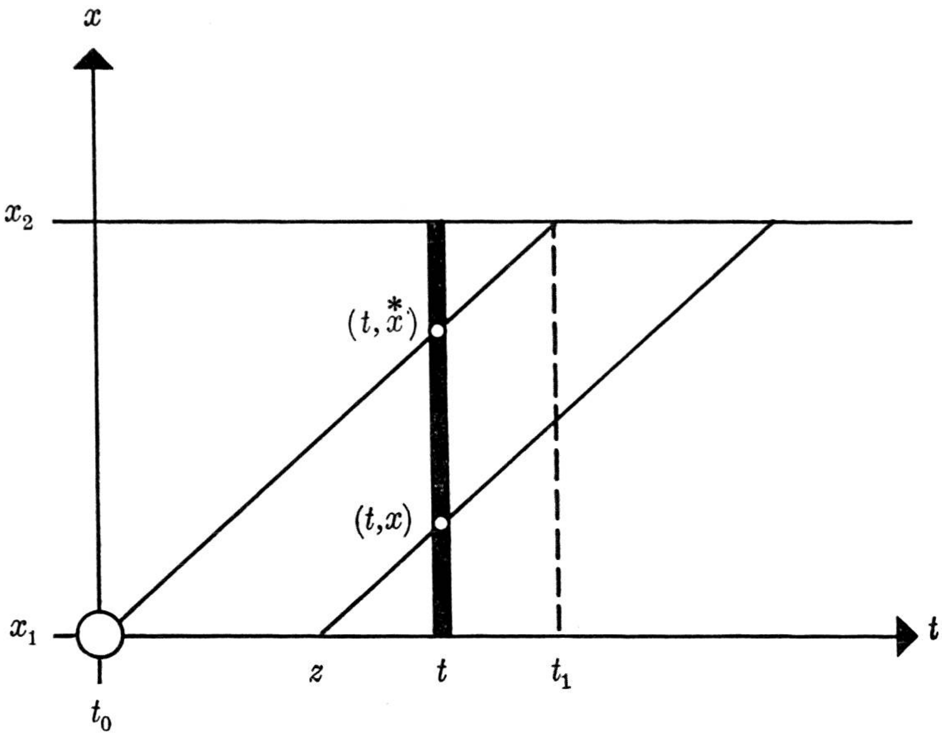
Summary

Three models of evolution are considered: a static, a full-dynamic and a semi-dynamic. The static evolution is only influenced by demographic factors. The full-dynamic model is characterized by the assumption of a long-range growth of the general level of wages and of an automatic adjustment of all pensions according to the same level. In semi-dynamic forecast only the new pensions follow the movement of wages, whereas the pensions in course of payment are not adjusted in the same way. First, two direct problems are dealt with, i.e. deriving the full-dynamic model by means of a first indicator from the static one and similarly the semi-dynamic from the full dynamic forecast of expenditure in using a second indicator. This latter indicator raises an inverse problem leading to a second kind integral equation of Volterra.

I. Models of projection

1. The Lexis diagram for old-age pensioners

1.1. The various connections between the *age* x of a person and the *epoch of observation* t appear most clearly in the following—somewhat modified—Lexis diagram where we consider only old-age pensioners within the limits of age $x_1 \leqq x \leqq x_2$, the lower limit x_1 being a fixed pensionable age and the upper limit x_2 representing ω in usual actuarial notation. Let us further define the variable *epoch z of attaining the pensionable age x_1*. If $t = t_0$ is chosen as the epoch of setting-up the compulsory pension-scheme, then the values $z < t_0$ are associated to the initial population of pensioners having not paid any contributions and values $z > t_0$ have to be linked with the contributing members of the fund.



1.2. The *diagonal straight lines* appearing in our Lexis diagram are of special importance. An old-age pensioner whose pension is allotted at the epoch z will follow such a line issued from the point (z, x_1) up to the end of his life. Indeed, the differential equation of a diagonal line is simply $dx = dt$ and the equation of the "life-line" may thus be written as:

$$x - x_1 = t - z \, , \tag{1}$$

x_1 being a constant and z a parameter.

2. Static and dynamic evolutions

2.1. Let us first recall the principal ideas of the *mathematical theory of evolutions.* If $E(t)$ represents a quantity changing at each moment t and $\varepsilon(t)$ the intensity of relative variation (instantaneous rate of variation), we then have the following relations between E and ε:

$$\varepsilon(t) = \frac{1}{E(t)} \frac{dE(t)}{dt} = \frac{d \ln E(t)}{dt} \tag{2}$$

$$E(t) = E(t_0) \exp \int_{t_0}^{t} \varepsilon(\tau) \, d\tau \, . \tag{2'}$$

It is further obvious that a compound evolution (n evolution-causes $i = 1, 2, \ldots n$):

$$E(t) = C \prod E_i(t) \tag{3}$$

involves:

$$\varepsilon(t) = \sum \varepsilon_i(t) \tag{3'}$$

and vice-versa.

2.2. As a first application of this theory we consider the *evolution of the sum* $\tilde{\Theta}(t)$ *of wages* as a product of the two evolutions $L(t)$ of the population of wage-earners and $U(t)$ of annual average wages. Denoting by $\nu(t)$ and $\eta(t)$ the corresponding intensities, we thus can write:

$$\tilde{\Theta}(t) = \Theta(t_0) \exp \int_{t_0}^{t} (\nu + \eta)\, d\tau\,. \tag{4}$$

This is the formula for a *dynamic wage-evolution* which is characterized by $\eta > 0$. On the other hand, when $\eta = 0$ we define the *static model of evolution* $\Theta(t)$ depending only upon demographic factors. The dynamic model $\tilde{\Theta}$ can be deducted from the static evolution Θ as shown by the formula

$$\tilde{\Theta}(t) = \Theta(t) \exp \int_{t_0}^{t} \eta\, d\tau = u(t)\, \Theta(t)\,, \tag{4'}$$

the symbol of the *first indicator* $u(t)$ being used for the index-number $U(t) : U(t_0)$ showing the evolution of the general level of wages.

2.3. By a similar method we will get the evolution of the expenditure for pensions for which several *economic intensities* have to be regarded. In the general case of a so-called semi-dynamic evolution we assume that the new pensions follow the evolution of wages (intensity η), whereas the pensions in course of payment are adjusted in a different way given by the intensity $\varkappa(t)$ which coincides often with the intensity $\pi(t)$ of evolution of prices. It is essential to distinguish carefully between all these intensities usually linked by the following inequalities:

$$0 \leqq \varkappa \leqq \eta \tag{5}$$

$$\varkappa \gtrless \pi\,. \tag{5'}$$

Generally $\pi < \eta$ although it is possible that $\pi > \eta$, but then we must assume special economic conditions.

Before writing down the appropriate formula for *semi-dynamic expenditure-evolution* $\tilde{\tilde{\Lambda}}(t)$ we have to explain some further symbols:

R_1 average amount of annual pensions allotted to new pensioners according to the same static pension formula

R_2 average amount of annual pensions allotted to the initial population of pensioners

$L(t, x)$ density of pensioners aged of x at the moment t

$\overset{*}{x}(t) = x_1 + (t - t_0)$ age separating the group of initial pensioners from that of the new pensioners.

We are now in a position to write:

$$\tilde{\tilde{\Lambda}}(t) = R_1 \int_{x_1}^{\overset{*}{x}} L(t,x) \exp\left[\int_{t_0}^{z} \eta\, d\tau + \int_{z}^{t} \varkappa\, dt\right] dx + R_2 \exp \int_{t_0}^{t} \varkappa\, d\tau \int_{\overset{*}{x}}^{x_2} L(t,x)\, dx\,, \tag{6}$$

where $z = t - (x - x_1)$ according to (1). This semi-dynamic model includes as special cases:

- the *static model* Λ with $\varkappa = \eta = 0$, depending only upon demographic evolution;
- the *full-dynamic model* $\tilde{\Lambda}$ with $\varkappa = \eta > 0$ leading to a sensible simplification in our formula.

We can now proceed to face *two problems*. First, we have to derive $\tilde{\Lambda}$ from Λ and secondly, we propose to express $\tilde{\tilde{\Lambda}}$ by $\tilde{\Lambda}$. Indeed, we have just seen from (6) that the evaluation of Λ and $\tilde{\Lambda}$ is much easier than that of $\tilde{\tilde{\Lambda}}$.

2.4. The solution of the *first problem* raises no special difficulties, as the *connection between the static* Λ *and the full-dynamic* $\tilde{\Lambda}$ appears to be, according to (6):

$$\tilde{\Lambda}(t) = \Lambda(t) \exp \int_{t_0}^{t} \eta\, d\tau = u(t)\, \Lambda(t)\,. \tag{7}$$

Thus, as shown by (4′) and (7), the same rule is linking $\tilde{\Lambda}$ to Λ as $\tilde{\Theta}$ to Θ (first indicator-rule).

2.5. It is hardly necessary to say that the *solution of our second problem* will involve more difficulties than the first one. However, the introduction of a modifying-function of expenditure will be a powerful tool and enables us to find the appropriate relation. The following notations may help us in this respect:

- First, we obtain for the standardized frequency of the *age structure* $\lambda_1(x)$ of pensioners issued from $z > t_0$, in putting:

$$L_1(t) = \int_{x_1}^{\overset{*}{x}} L(t,x)\,dx \quad \text{and} \quad L_2(t) = \int_{\overset{*}{x}}^{x_2} L(t,x)\,dx\,,$$

$$\lambda_1(x;t) = \frac{L(t,x)}{L_1(t)}\,, \quad \text{hence} \int_{x_1}^{\overset{*}{x}} \lambda_1\,dx = 1\,. \tag{8}$$

- The further symbol defines our *second indicator* $m(t)$ *as a modifying-function of full-dynamic expenditure:*

$$m(t) = \exp \int_{t_0}^{t} (\varkappa - \eta)\,d\tau\,, \tag{9}$$

for which it is quite clear that, according to the inequality (5), $m \leqq 1$.

Our problem now is to express formula (6) in terms of λ_1 and m. Thus the *relation between the full-dynamic* $\tilde{A}$ *and the semi-dynamic* $\tilde{\tilde{A}}$ reads:

$$\tilde{\tilde{A}}(t) = u(t)\,m(t)\left\{R_1 L_1(t)\int_{x_1}^{\overset{*}{x}} \lambda_1(x;t)\,m^{-1}(z)\,dx + R_2 L_2(t)\right\}. \tag{10}$$

Now, the integral $(x_1, \overset{*}{x})$ in this formula defines obviously a correct weighted average of the function $m^{-1}(z)$, where $z = t - (x - x_1)$. In case of monotonic functions we can associate to this average the value of m^{-1} taken at en *average* $\bar{z}$ *of the epoch* z, so that the integral $(x_1, \overset{*}{x}) = m^{-1}(\bar{z})$. In splitting up the expenditure $\tilde{A}$ in two components $\tilde{A}_1$ and $\tilde{A}_2$ corresponding to the populations L_1 and L_2 of pensioners, we obtain finally:

$$\tilde{\tilde{A}}(t) = m(t/\bar{z})\,\tilde{A}_1(t) + m(t)\,\tilde{A}_2(t) = \bar{m}(t)\,\tilde{A}(t) \tag{10'}$$

the symbol $m(t/\bar{z})$ being equivalent to the division $m(t) : m(\bar{z})$ and where $\bar{m}$ is defined as en average indicator in t.

Thus, having calculated $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ according to (7), there is no difficulty to derive the corresponding $\bar{\bar{\Lambda}}_1$ and $\bar{\bar{\Lambda}}_2$. To take a concrete example, let us consider the *case of relative stability* where $\varkappa$ and η are constant evolution-intensities. In a mature scheme (see Lexis diagram $t \geqq t_1$) the population $L_2(t)$ has disappeared. In substituting further in (9) $\exp \varkappa = 1 + h$ and $\exp \eta = 1 + j$ we obtain as an indicator:

$$m(t/\bar{z}) = \left(\frac{1+h}{1+j}\right)^{t-\bar{z}} = \left(\frac{1+h}{1+j}\right)^{\bar{x}-x_1}, \tag{11}$$

where $\bar{x}$ is the average age corresponding to $\bar{z}$. The difference $\bar{x} - x_1$ is generally situated in the neighbourhood of 10 years; $j - h = 1\%$ implies then $m \approx 0{,}90$ and for $j - h = 2\%$, m will be reduced to 0,83 and for $j - h = 3\%$ even to 0,76. Thus, a *substantial economy* of 10 to 24% may be obtained if the adjustment of pensions in course of payment is less intensitive than the one of the new pensions. From the social point of view such a reduction is permissible if h is the annual rate of price-evolution and j that of the general increase of wages.

II. Financial equilibrium

3. Collective mechanism of financing

3.1. There are two fundamental equations dominating the problems of financial equilibrium. The first one refers to the evolution of the mathematical reserve $V(t)$ and leads to the *equation of the technical balance*. It can be written in a static, semi- or full-dynamic form. Let us consider the retrospective method giving — first in *static evolution* — the usual technical balance for an open fund:

$$V(t) = \int_{t_0}^{t} \exp \int_{\tau}^{t} \delta \, d\iota \cdot \Theta(\tau) \left[\beta(\tau) - \alpha(t)\right] d\tau , \tag{12}$$

where we have still to define some more symbols:

δ instantaneous rate of interest,
β rate of contribution of the given financial system,
$\alpha = \Lambda : \Theta$ rate of expenditure or assessment rate.

This equation can be written too:

- for *full-dynamic* evolution with $\tilde{V}$, $\tilde{\Theta}$, $\tilde{\beta}$, $\tilde{\alpha}$
- for *semi-dynamic* evolution with $\tilde{\tilde{V}}$, $\tilde{\Theta}$, $\tilde{\tilde{\beta}}$ and $\tilde{\tilde{\alpha}}$.

In dividing (12) by $\Theta(t)$ we obtain the *wage-related rate* $\gamma(t)$ *of mathematical reserve:*

$$\gamma(t) = \int_{t_0}^{t} \exp \int_{\tau}^{t} \delta \, d\iota \cdot \Theta(\tau/t) \, [\beta(\tau) - \alpha(\tau)] \, d\tau \, , \tag{12'}$$

and in a similar way $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$. In these two cases $\tilde{\Theta}$, $\tilde{\alpha}$ and $\tilde{\tilde{\alpha}}$ may be expressed by (4′), (7) and (10′), whereas $\tilde{\beta}$ and $\tilde{\tilde{\beta}}$ have to be evaluated according to the specific financial system, as we have indicated it in a previous paper[1]). Thus it appears, that it would be rather difficult to express, in a simple way, $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$ as functions of γ.

3.2. The second fundamental equation of financial equilibrium gives a complete interpretation of the evolution of income and expenditure of a pension-scheme at any moment t, i. e. the *budgetary equation.* In writing the derivative of (12′) by t, we obtain a most important differential equation; for the derivation of (12′) it is however essential to express $\tilde{\Theta}$ according to (4). The result for semi-dynamic evolution will then be, all terms depending on t:

$$\tilde{\tilde{\beta}} = \tilde{\tilde{\alpha}} + (\nu + \eta - \delta) \, \tilde{\tilde{\gamma}} + \tilde{\tilde{\gamma}}' \, . \tag{13}$$

It is not the purpose of this paper to discuss in detail our fundamental equation (13). We may simply point out that it enables one to compare the contribution rate of any financial system with the assessment-rate α. The importance of the sign of the factor $(\nu + \eta - \delta)$ is evident, especially when $\gamma' = 0$. Here, we must first of all emphazise the *influence of the modifying-function m on the contribution rate β of any financial system*[2]).

[1]) E. Kaiser, «Functional equations of social mathematics» (International Review on Actuarial and Statistical Problems of Social Security, Geneva, 1962).

[2]) The other problem, up to which extent a given financial system allows adjusting of current pensions without changing the static rate β is dealt with in our paper submitted to the 18th International Congress of Actuaries (Munich, 1968).

A direct influence is seen in considering that $\tilde{\tilde{\alpha}} = \bar{m}\alpha$, this according to (10′), (4′) and (7). But there is an indirect influence by means of the rate of $\tilde{\tilde{\gamma}}$, the influence of which is more difficult to evaluate. This may be a matter of further investigation.

4. *The integral equation of pension adjustment*

4.1. We have been connected, so far, with a direct problem, i.e. to determine the influence of our modifying-functions on evolution of income and expenditure of a pension-scheme. We may now attack an *inverse problem*. Generally, the static evolution of pension-charges is known by the classical long-range estimates and thus the static assessment rate is given too. Now, this static α indicates usually a very expensive evolution; indeed the values of α may grow up to 30% of wages and even more. Generally, only demographic reasons are responsible for such an important growth.

An appropriate policy of pension-adjustment may compensate to a certain extent these rather heavy charges for economy. Thus, we consider a given evolution $\tilde{\tilde{\alpha}} < \alpha$ and attempt to determine the *intensity $\varkappa(t)$ of adjustment of current pensions as an unknown function of an integral equation,* with the understanding that the adjustment of new pensions will follow the intensity $\eta(t)$ of wage-evolution. Notice, that it would not be possible to adjust new pensions only according to $\varkappa < \eta$ as the ratio «Pension: Wages» would tend to zero if t tends to infinity. This is obviously not the case with $m(t/\bar{z})$ as, according to formula (9), we would have:

$$m\,(t/\bar{z}) = \exp \int_{\bar{z}}^{t} (\varkappa - \eta)\, d\tau, \text{ where } t - \bar{z} = \bar{x} - x_1, \text{ i.e. a limited interval.}$$

4.2. In order to find a solution to our problem let us return to formula (10′). In dividing both sides by $\tilde{\Theta}(t)$ we obtain the *semi-dynamic assessment-rate* $\tilde{\tilde{\alpha}}$ as follows, but notice first that $\tilde{\alpha} = \alpha$, as a result of the division (7): (4′).

$$\tilde{\tilde{\alpha}}\,(t) = m\,(t/\bar{z})\,\alpha_1(t) + m\,(t)\,\alpha_2(t)\,. \tag{14}$$

Here, $\tilde{\tilde{\alpha}}$ is a given function and m an unknown one. Hence $\bar{z}$ is unknown too; therefore we must express $m^{-1}(\bar{z})$ by its definition as a weighted average of the function $m^{-1}(z)$, what leads to:

$$\tilde{\tilde{\alpha}}(t)\, m^{-1}(t) = \alpha_1(t) \int_{x_1}^{\overset{*}{x}} \lambda_1(x; t)\, m^{-1}(z)\, dx + \alpha_2(t)\,. \tag{14'}$$

Obviously, we are faced with an integral equation with the *unknown function* $m^{-1}(t)$ for which we will put:

$$\varphi(t) = m^{-1}(t) = \exp - \int_{t_0}^{t} (\varkappa - \eta)\, d\tau\,. \tag{15}$$

If φ is determined, then the *unknown intensity of adjustment* $\varkappa(t)$ is determined too, as we must assume that $\eta(t)$ is a given function describing the evolution of the general level of wages.

In order to find a classical form, our integral equation (14′) needs some transformations. First, we may substitute in (14′) the variable x by z according to (1), i.e. $x = -z + t + x_1$. Then, we can group the *given functions* as follows:

$$K(t-z, t) = \frac{\alpha_1(t) \cdot \lambda_1(t - z + x_1; t)}{\tilde{\tilde{\alpha}}(t)}\,, \tag{16}$$

$$F(t) = \frac{\alpha_2(t)}{\tilde{\tilde{\alpha}}(t)}\,, \tag{16'}$$

where $\tilde{\tilde{\alpha}} < \alpha = \alpha_1 + \alpha_2$.

Now, we obtain the classical form of an *integral equation of Volterra of second kind,* as we can write:

$$\varphi(t) = \int_{t_0}^{t} K(t-z, t)\, \varphi(z)\, dz + F(t)\,. \tag{17}$$

As a conclusion, we may add some comments as to the *possibilities of solving* this integral equation.

- The *mathematical theory of integral equations* indicates various ways for the solution of our equation in this general form. To a great deal, the solution depends upon the *analytical form of the nucleous K,* which appears in (17) already in a particular form.

- If $\lambda_1(x;\, t)$ *is independent from* t, then K degenerates to a product $f(t) \cdot g(t-z)$ and the solution may be obtained by techniques analogous to those applied for solving the *integral equation of renewal* of population set up by our senior Swiss collegue Christian Moser (method of iteration and Laplace-transformations[3])).

- If $t > t_1$ (see Lexis-diagram) we better may consider the original form (14′) of our equation; in this case $\alpha_2 = 0$, $\alpha_1 = \alpha$ and the upper limit $\overset{*}{x} = x_2$. A *first approach to a solution* may be obtained in putting $\overline{m} = \tilde{\bar{\alpha}} : \alpha = \exp\,(\bar{x} - x_1)\,(\varkappa - \eta)$ and in assuming that $\bar{x} - x_1 = 10$. Take, for instance $\overline{m} = 0{,}8$, then the solution would be $\varkappa = 0{,}1\; ln\; 0{,}8 + \eta$; for $\eta = 0{,}05$ we find $\varkappa = 0{,}028$.

Having thus explained, in a very summary way, the nature of our integral equation, we need hardly say that *further theoretical investigations* will be necessary in order to show the range of practical applications in the field of pension-schemes and of social security.

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[3]) A very complete theory of these methods is given in the text-book of Walter Saxer «Versicherungsmathematik II» (Springer 1958).

Zusammenfassung

Es werden drei Entwicklungsmodelle betrachtet: ein statisches, ein volldynamisches und ein semidynamisches. Die statische Entwicklung wird lediglich durch demographische Faktoren beeinflusst. Das volldynamische Modell ist dadurch gekennzeichnet, dass das allgemeine Lohnniveau langfristig zunimmt und dass sämtliche Renten automatisch dieser Lohnentwicklung angepasst werden. Im semidynamischen Modell folgen lediglich die Neurenten der allgemeinen Lohnbewegung, wogegen die laufenden Renten in unterschiedlicher Weise angepasst werden. Es werden zunächst zwei direkte Probleme betrachtet: die Ableitung des volldynamischen Modells aus dem statischen, dies mit Hilfe eines ersten Indikators, und in analoger Weise die Berechnung der semidynamischen Entwicklung der Ausgaben anhand der volldynamischen, gestützt auf einen zweiten Indikator (Reduktionsfaktor). Der letztgenannte Indikator führt zu einem Umkehrproblem, welches anhand einer Integralgleichung von Volterra der zweiten Art gelöst werden kann.

Résumé

On considère trois modèles d'évolution: statique, dynamique et semi-dynamique. L'évolution dite statique n'est influencée que par des facteurs démographiques. Le modèle dynamique est caractérisé par un accroissement à long terme du niveau général des salaires et par un ajustement automatique de toutes les pensions selon la même intensité d'accroissement. Dans les estimations semi-dynamiques ce sont uniquement les pensions nouvelles qui suivent le mouvement des salaires alors que les pensions en cours sont adaptées d'une manière différente. Deux problèmes directs sont envisagés, c'est-à-dire le calcul du modèle dynamique à partir du modèle statique, moyennant un premier indicateur, et la détermination de l'évolution semi-dynamique des dépenses en partant du modèle dynamique, ceci à l'aide d'un deuxième indicateur (facteur de réduction). Ce dernier soulève un problème inverse pouvant être résolu par une équation intégrale de Volterra de seconde espèce.