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Some Notes on the Qualifying Period in Disability Insurance

II. Problems of Maximum Likelihood Estimation

By Jan M. Hoem

1. Introduction

§ 1.1. We have studied parts of the theory of disability insurance in two previous papers (Hoem, 1968, 1969a) and now turn to some of the estimation problems which arise in connection with a qualifying period. Introducing a slight change from our previous presentation we shall make a *four*-state time-continuous Markov chain our object of study. The four states will be called " active" (or state a), " disabled" (or state i), "dead while active" (or state d), and "dead while disabled" (or state D). They may be represented as in figure 1.1, where arrows indicate possible transitions.

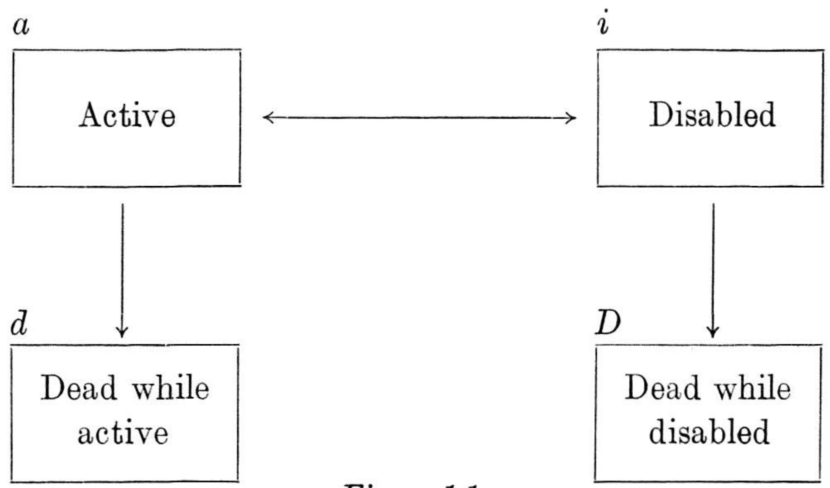

Figure 1.1.

§ 1.2. The forces of transition between these states have a prominent position in the theory, and this paper is entirely devoted to problems connected with finding their maximum likelihood estimators. These forces are generally assumed to be continuous functions of the exact age of the insured. In the ages relevant to insurance the variation in the function value of such a force during a single age year is probably very small. If we are only interested in a single age year, we may therefore use constant forces of transition as a reasonable approximation to the more general model.

In fact, one common way of estimating the values of a continuous force function for a set of ages (say the force of mortality for the central ages) is to estimate one function value for each age year and then possibly fit some smooth analytical function to the set of estimates. (Cf. the Gompertz-Makeham technique.)
(Cf. the Gompertz-Makeham technique)

In the first instance, therefore, interest centers on the estimation procedure for one age year where one temporarily behaves as if the force function has a single value, i.e. is a constant. This is the position which we will adopt.

Our model will thus contain four basic parameters:

μ^a is the force of mortality for active persons.
This is the force of transition from state a to d.

μ^i is the force of mortality for the disabled.
This is the force of transition from state i to D.

ν is the force of disablement.
This is the force of transition from state a to i.

ϱ is the force of recovery.
This is the force of transition from state i to a.

We also define

$\alpha = \mu^a + \nu$, which is the total force of decrement from state a,
$\beta = \mu^i + \varrho$, which is the total force of decrement from state i,

and

$$\lambda = \alpha - \beta,$$

and assume that $\alpha > 0$, $\beta > 0$, $\lambda \neq 0$.

§ 1.3. Methods for estimating these parameters in a situation with complete information of the sample paths of the disability process has been studied by Sverdrup (1965). Situations with incomplete information have been studied by authors like Fix and Neyman (1951) and Høyland (1967).

§ 1.4. As noted in a previous paper (Hoem, 1968 b) an insurer offering disability annuities with a qualifying period of length $\varkappa$ (insurance form A of the paper mentioned) does not really observe sample paths of the actual disability process. Instead he must be content with observing paths of what we have called the *registered process*. This secondary process also has four states, which we call " non-recipient (of disability benefit) " (or state n), " recipient " (or state r), " dead while a non-recipient " (or state b), and " dead while a recipient " (or state B). (For further description of the registered process, the reader is referred to Hoem (1969 a).)

We shall consider maximum likelihood estimation of μ^a, μ^i, ν, and ϱ from the information available to such an insurer.

§ 1.5. Before going into mathematical technicalities we shall give some consideration to the situation in which the data arise.

Assume then that the insurer keeps an individual record for each person insured from his entry into the portfolio and until his policy expires. When the insurer sets out to estimate the forces of transition for a specific age year, say year y, he will collect all records for persons who were insured with him in that age year. We shall split these into three groups.

(i) Part of the records relate to persons who took out their policies at ages below y and who did not increase their sums assured during age year y. We shall call this the unaltered old stock for year y.

(ii) Another part of the records relate to old stock where the sums assured were increased during the age year. We shall call this the altered old stock for age year y.

(iii) The rest of the records relate to persons who took out their policies during age year y. This will be called the new stock for the age year.

Adopting a purist position, we shall not in the present paper use the information contained in the records of the unaltered old stock.

Similarly for each policy of the altered old stock we shall discard information relating to the time before the (first) increase of the sum assured during age year y. This plainly leads to a waste of information which needs some explanation.

Part of the old stock were registered as recipients of disability benefit when they entered age year y. The rest were then registered as non-recipients. Of the latter the major part would probably really be active, but some of the insured would be disabled but still within the qualifying period. Since a large part of these would recover without reaching the end of the qualifying period and before making possible increases in the sums assured, their disability at the beginning of age year y would never be recorded by the insurer.

To utilize information about members of the old stock registered as non-recipients at the beginning of the age year would raise problems which we are not prepared to tackle in the present context.

We *might* have utilized information of members of the old stock registered as recipients at the beginning of the age year beside the information which we shall actually use. This would not have caused great complication. To simplify matters we shallleave it out nonetheless.

This leaves us with the information of the new stock, and with such information of the altered old stock as relates to the time after the (first) increase in the sum assured. The reason why we have so restricted ourselves is the fact that a raise in the sum assured will only be granted to an insured person who is actually active (and not only a non-recipient), and that similarly a prospective customer will only be permitted to take out disability insurance provided he is active. Thus *each of the sample paths corresponding to the records which we have retained will start in state a of the actual disability process.*

For the new stock there may be increases in the sums assured, and for the altered old stock there may be further increases after the first one. At each increase the company will know that the policy belongs to state a of the actual disability process. This may be regarded as the end of the period of observation of one sample path and the start of a new and independent path.

When the basic observational period is one age year, the period of observation for each sample path will then be at most one year long. The length of this period will vary from one sample path to another.

§ 1.6. We shall conclude this chapter by introducing transition probabilities of the actual and the registered disability process. Let $S(t)$ be the state of the actual process at time t, and let $S_\varkappa(t)$ similarly be the state of the registered process. We introduce

$$P_{jk}(t) = P\{S(\tau+t)=k|S(\tau)=j\},$$

$$Q_j(t) = P\{S(\tau+\xi)=j \quad \text{for} \quad 0<\xi\leqq t|S(\tau)=j\}, \quad \text{and}$$

$$R_j(t) = P\{S_\varkappa(\tau+\xi)=n \quad \text{for} \quad 0<\xi\leqq t \text{ and } S(\tau+t)=j|S(\tau)=a\}$$

for $j = a, i;$ $k = a, i, d, D;$ $\tau \geqq 0$, and $t > 0$. Here

$$Q_a(t) = e^{-\alpha t} \quad \text{and} \quad Q_i(t) = e^{-\beta t}, \tag{1.1}$$

$$P_{aa}(t) = \{(r_1+\alpha)e^{r_2 t} - (r_2+\alpha)e^{r_1 t}\}/(r_1-r_2), \quad \text{and} \tag{1.2}$$

$$P_{ai}(t) = \nu\{e^{r_1 t} - e^{r_2 t}\}/(r_1-r_2), \tag{1.3}$$

with

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \tfrac{1}{2}(-\alpha-\beta\pm\sqrt{\lambda^2+4\varrho\nu}). \tag{1.4}$$

(Sverdrup, 1965.) Quite similar formulae hold for $P_{ia}(t)$ and $P_{ii}(t)$.

2. The special case $\varrho = 0$

§ 2.1. We shall prove unable to find explicit expressions for the maximum likelihood estimators of the four forces of transition in the case where they are all positive. Before proceeding to this more difficult situation, however, we shall consider the special case where recovery is impossible ($\varrho = 0$). In fact we shall find a complete solution only in a subcase even to this simpler situation. We hope nevertheless that the results which we do find throw some light on the questions involved.

In this chapter, then, $\varrho = 0$.

§ 2.2 Assume that a given number N of independent sample paths of the registered process have been observed, and that the period of observation (including possibly time from death to end of age year) of sample path no. j has length Z_j. $Z_1, Z_2, \ldots, Z_N$ will be taken to constitute a set of independent, identically distributed random variables (Sverdrup, 1965) with a distribution function $G(z)$, where $G(0) = 0$, $G(1) = 1$.

Let $S(t, j)$ and $S_\varkappa(t, j)$ be the states in the actual and the registered process, respectively, observed at time t of sample path no. j. Let $N_{jk} = 1$ if $S_\varkappa(Z_j, j) = k$, $N_{jk} = 0$ otherwise, for $k = n, r, b, B$. (Time is now reckoned from the beginning of the observational period for each sample path. Each path thus has a "clock" of its own.) Obviously for each j exactly one of N_{jn}, N_{jr}, N_{jb}, and N_{jB} equals 1 and the rest equal 0.

To establish the likelihood of sample path no. j we consider four cases:

(i) $P\{N_{jn} = 1 | Z_j = z\} = R_a(z) + R_i(z)$.

(ii) If $N_{jr} = 1$, a disablement must have taken place at some moment $U_j - \varkappa$ and it must have been registered at time U_j. With the usual notation we get

$$P\{N_{jr} = 1 \quad \text{and} \quad u < U_j < u + du | Z_j = z\} = Q_a(u-\varkappa)\nu\, Q_i(z-u+\varkappa)\, du\,.$$

(iii) If $N_{jb} = 1$, the insured must have died while a non-recipient at some time W_j. We get

$$P\{N_{jb} = 1 \text{ and } w < W_j < w + dw | Z_j = z\} = R_a(w)\mu^a dw + R_i(w)\mu^i dw\,.$$

(iv) If $N_{jB} = 1$, a disablement must have taken place at some moment $U_j - \varkappa$, it must have been registered at time U_j, and then the insured must have died at some time W_j. We get

$$P\{N_{jB} = 1,\ u < U_j < u + du, \text{ and } w < W_j < w + dw | Z_j = z\}$$
$$= Q_a(u-\varkappa)\nu\, Q_i(w - u + \varkappa)\mu^i\, du\, dw\,.$$

In case (iii) $0 < w < z$, and in cases (ii) and (iv) $\varkappa < u < w < z$. When $\varrho = 0$, we get $R_a(t) = e^{-\alpha t}$, and

$$R_i(t) = \int_{\max(0,\, t-\varkappa)}^{t} Q_a(\tau)\nu Q_i(t-\tau)\, d\tau = \nu e^{-\alpha t}\{e^{\lambda \min(t,\varkappa)} - 1\} / \lambda\,.$$

Utilizing these results, we may write the likelihood for sample path no. j in the form

$$\lambda^{-(n_{jn}+n_{jb})}\nu^{n_{jr}+n_{jB}}(\mu^i)^{n_{jB}}(\nu e^{\lambda \min(z_j,\varkappa)}+\mu^a-\mu^i)^{n_{jn}}\cdot$$

$$\cdot[\lambda\mu^a+\nu\mu^i(e^{\lambda\min(w_j,\varkappa)}-1)]^{n_{jb}}\exp\{-\alpha z_j n_{jn}-\mu^i z_j n_{jr}-\lambda(u_j-\varkappa)(n_{jr}+n_{jB})$$

$$-\alpha w_j n_{jb}-\mu^i w_j n_{jB}\}\ (du_j)^{n_{jr}+n_{jB}}(dw_j)^{n_{jb}+n_{jB}}dG(z_j)\,.$$

The likelihood for all sample paths is therefore

$$\Lambda(\mu^a,\mu^i,\nu)\prod_{j=1}^{N}\{(du_j)^{n_{jr}+n_{jB}}(dw_j)^{n_{jb}+n_{jB}}\}\,dG(z_1)\,dG(z_2)\ldots dG(z_N)$$

where $\quad\Lambda(\mu^a,\mu^i,\nu)=\lambda^{-\Sigma(n_{jn}+n_{jB})}\nu^{\Sigma(n_{jr}+n_{jB})}(\mu^i)^{\Sigma n_{jB}}\cdot$

$$\cdot\prod_{j=1}^{N}\{(\nu e^{\lambda\min(z_j,\varkappa)}+\mu^a-\mu^i)^{n_{jn}}[\lambda\mu^a+\nu\mu^i(e^{\lambda\min(w_j,\varkappa)}-1)]^{n_{jb}}\}\cdot \tag{2.1}$$

$$\cdot\exp\{-\alpha\Sigma z_j n_{jn}-\mu^i\Sigma z_j n_{jr}-\lambda\Sigma(u_j-\varkappa)(n_{jr}+n_{jB})-\alpha\Sigma w_j n_{jb}-\mu^i\Sigma w_j n_{jB}\}.$$

Here all summations should be taken over j from $j=1$ to $j=N$.

If the distribution function G is independent of μ^a, μ^i, and ν, maximum likelihood estimators for these parameters will be those (if any) which maximize $\Lambda(\mu^a, \mu^i, \nu)$.

In case G depends on μ^a, μ^i, and ν, possible maximum likelihood estimators will be influenced by characteristics of G. Then the estimators found by maximizing $\Lambda(\mu^a, \mu^i, \nu)$ need not be m. l. estimators. Even so, they have some interest, and we shall concentrate on this set of estimators.

We introduce

$A=\Sigma N_{jn}$, which is the number of sample paths never registered to leave state n,

$L_{nn}=\Sigma Z_j N_{jn}$, which is the total registered living time for these paths,

$I=\Sigma N_{jr}$, which is the number of insured studied that receive disability benefits by the end of the period of observation,

$L_{nr}=\Sigma U_j N_{jr}$, which is the total registered living time as non-recipients of these persons,

$L_r = \Sigma Z_j N_{jr}$, which is their total registered living time,

$D_n = \Sigma N_{jb}$, which is the number of insured persons who die while non-recipients,

$L_{nb} = \Sigma W_j N_{jb}$, which is their total registered living time,

$D_r = \Sigma N_{jB}$, which is the number of insured persons who die while receiving disability benefits,

$L_{nB} = \Sigma U_j N_{jB}$, which is their total registered living time as non-recipients, and

$L_B = \Sigma W_j N_{jB}$, which is their total registered living time.

We also introduce

$L_n = L_{nn} + L_{nr} + L_{nb} + L_{nB}$, which is the total registered living time as non-recipients,

$L = L_{nn} + L_r + L_{nb} + L_B$, which is the total registered living time,

$D = D_n + D_r$, which is the total number of deaths registered, and

$J = I + D_r$, which is the total number of disablements registered.

Changing to random variables in (2.1) we then get

$$\log \Lambda\,(\mu^a, \mu^i, \nu) = -(A + D_n)\log\lambda + J\log\nu + D_r\log\mu^i - \alpha(L_{nn} + L_{nb})$$
$$- \mu^i(L_r + L_B) - \lambda(L_{nr} + L_{nB} - \varkappa J) + \sum_{j=1}^{N} N_{jn}\log(\nu e^{\lambda\min(Z_j,\varkappa)} + \mu^a - \mu^i)$$
$$+ \sum_{j=1}^{N} N_{jb}\log\{\lambda\mu^a + \nu\mu^i(e^{\lambda\min(W_j,\varkappa)} - 1)\}\,. \tag{2.2}$$

Because of the last two terms in (2.2) it is impossible to find nice analytical formulae for the estimators. In the next paragraph we restrict ourselves to a special case.

§ 2.3. Now assume that

$$\mu^a = \mu^i\,(=\mu)\,. \tag{2.3}$$

We introduce $L_{\varkappa n} = \sum_{j=1}^{N} N_{jn}\min(Z_j, \varkappa)$, $L_{\varkappa b} = \sum_{j=1}^{N} N_{jb}\min(W_j, \varkappa)$,

and

$$M_\varkappa = L_n - L_{\varkappa n} - L_{\varkappa b} - \varkappa J\,. \tag{2.4}$$

Formula (2.2) then reduces to

$$\log \Lambda(\mu,\nu) = J\log\nu - \nu M_\varkappa + D\log\mu - \mu L. \tag{2.5}$$

The maximizing values of ν and μ are

$$\hat{\nu}_\varkappa = J/M_\varkappa \quad \text{and} \quad \hat{\mu} = D/L. \tag{2.6}$$

The formula for $\hat{\mu}$ is as me might have expected. The denominator of $\hat{\nu}_\varkappa$ may need some comment.

In a situation without a qualifying period, $\varkappa = 0$ and $\hat{\nu}$ of (2.6) would be $\hat{\nu}_0 = J/L_n$, where L_n would then be the total actual living time L_a *as active*. In the present context, however, L_n is the total registered living time as *non-recipients*. Some kind of correction must therefore be made to L_n before it can be used as an "estimate" of L_a in the denominator in $\hat{\nu}$. It appears that $M_\varkappa$ actually is such a "corrected estimate" of L_a.

For each of the persons who do receive disability benefits during some period, it is certain that his actual living time as active is $\varkappa$ less than his registered living time U as a non-recipient. This explains the subtraction of $\varkappa J$ in (2.4).

It is also probable that at least some of those who were never registered as recipients, were actually disabled for a period not exceeding $\varkappa$. It would be desirable to adjust L_n for the effect of this, and the subtraction of $L_{\varkappa n} + L_{\varkappa b}$ in (2.4) is such an adjustment. It is surprising, though, that one should have to subtract this much and not some smaller quantity.

§ 2.4. We shall state some properties of the estimators in (2.6) when $\varrho = 0$ and (2.3) holds.

By (2.5) the vector $(J, D, L, M_\varkappa, Z_1, \dots, Z_N)$ will be sufficient. If G is completely specified, the Z_j are superfluous here, and $(J, D, L, M_\varkappa)$ will be minimal sufficient by the properties of the Darmois-Koopman class of probability measures.

As $N \to \infty$, $\hat{\nu}_\varkappa$ and $\hat{\mu}$ will be consistent. Moreover $\sqrt{N}(\hat{\nu}_\varkappa - \nu)$ and $\sqrt{N}(\hat{\mu} - \mu)$ will be asyptotically independent and normally distributed with means 0 and asymptotic variances

$$\text{as. var. } \sqrt{N}(\hat{\nu}_\varkappa - \nu) = \nu/E\bar{M}_\varkappa \quad \text{and}$$

$$\text{as. var. } \sqrt{N}(\hat{\mu} - \mu) = \mu/E\bar{L},$$

where $\bar{M}_\varkappa = M_\varkappa/N$ and $\bar{L}=L/N$ (Hoem, 1969 b, § 5.2). Here

$$E\bar{M}_\varkappa = E\{N_{jn}\max(0,Z_j-\varkappa) + (N_{jr}+N_{jB})(U_j-\varkappa) + N_{jB}\max(0,W_j-\varkappa)\}$$

and $E\bar{L}$ is the expected living time under observation for each person.

If G is completely specified, $\hat{\nu}_\varkappa$ and $\hat{\mu}$ will be optimal Fisher consistent estimators for ν and μ, respectively (Sverdrup, 1965, Appendix B; Hoem, 1969 b).

3. The case where all parameters are positive

§ 3.1. We now turn to the case where μ^a, μ^i, ν, and ϱ are all positive. Some reflection shows that knowledge of $R_a(t)$ is essential to establish the likelihood of any sample path of the registered process. Our first goal therefore is to find a formula for $R_a(t)$, and various expression are established in the paragraphs below. The formulae turn out to be rather involved, and we are unable to find explicit expressions for the four maximum likelihood estimators. By *numerical* methods, however, our result can be used to provide estimates of the forces of transition.

§ 3.2. For $\Delta t > 0$ we have

$$R_a(t+\Delta t) = R_a(t)\,(1-\alpha\Delta t) + R_i(t)\varrho\Delta t + o(\Delta t)\,, \quad \text{and}$$

$$R_i(t+\Delta t) = R_i(t)\,(1-\beta\Delta t) - R_a(t-\varkappa)\nu Q_i(\varkappa)\Delta t + R_a(t)\nu\Delta t + o(\Delta t)\,,$$

with $R_a(0)=1$, $R_i(0)=0$, and conventionally $R_a(t)=0$ for $t<0$.

Rearranging these equations, dividing by Δt and letting $\Delta t \to 0$, we get

$$\begin{aligned} R_a'(t) &= -\alpha R_a(t) + \varrho R_i(t)\,, \quad \text{and} \\ R_i'(t) &= \nu R_a(t) - \beta R_i(t) - \nu e^{-\beta\varkappa} R_a(t-\varkappa)\,, \end{aligned} \tag{3.1}$$

since $Q_i(t) = e^{-\beta t}$. For $t<\varkappa$, $R_a(t-\varkappa) = 0$, and (3.1) reduces to what substantially are the Kolmogorov differential equations for $P_{aa}(t)$ and $P_{ai}(t)$ (Sverdrup, 1965, [4] and [5]). Thus by (1.2) to (1.4),

$$R_a(t) = \{(r_1+\alpha)e^{r_2 t}-(r_2+\alpha)e^{r_1 t}\}/(r_1-r_2) \quad \text{for } 0\leqq t\leqq \varkappa,$$
$$\text{and} \quad R_i(t) = \nu(e^{r_1 t}-e^{r_2 t})/(r_1-r_2) \quad \text{for } 0\leqq t\leqq \varkappa. \tag{3.2}$$

Since R_a and R_i must be continuous, (3.2) holds also for $t=\varkappa$.

For $t>\varkappa$ (3.1) is a set of differential-difference equations of the kind studied by Bellman and Cooke (1963, Chapter Six). We introduce some of their notation. With $\delta=\nu e^{-\beta\varkappa}$, let

$$\mathbf{A}_0 = \begin{pmatrix}1, 0\\0, 1\end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix}\alpha, -\varrho\\-\nu, \beta\end{pmatrix}, \quad \text{and} \quad \mathbf{B}_1 = \begin{pmatrix}0, 0\\\delta, 0\end{pmatrix}.$$

Furthermore, let $\mathbf{R}(t) = \begin{pmatrix}R_a(t)\\R_i(t)\end{pmatrix}$. Then (3.1) may be written in the form

$$\mathbf{A}_0\mathbf{R}'(t)+\mathbf{B}_0\mathbf{R}(t)+\mathbf{B}_1\mathbf{R}(t-\varkappa) = \mathbf{0} \tag{3.3}$$

with $\mathbf{0}=\begin{pmatrix}0\\0\end{pmatrix}$. By Bellman and Cooke's theorem 6.2, (3.3) has a unique solution for $t\geqq 0$ which satisfies (3.2), is continuous for $t\geqq 0$, and has a continuous derivative at least for $t\geqq\varkappa$. One form of the solution is given by their theorem 6.3, and is found as follows: Let

$$\prod(z) = \begin{pmatrix}\Pi_a(z)\\\Pi_i(z)\end{pmatrix} = e^{-\varkappa z}\mathbf{R}(\varkappa)-\mathbf{B}_1 e^{\varkappa z}\int_0^{\varkappa}\mathbf{R}(t)\,e^{-zt}\,dt$$

with complex z. Then

$$\Pi_a(z) = e^{-\varkappa z}R_a(\varkappa) \quad \text{and} \quad \Pi_i(z) = e^{-\varkappa z}R_i(\varkappa)-\delta e^{-\varkappa z}\int_0^{\varkappa}R_a(t)\,e^{-zt}\,dt.$$

Furthermore let

$$\mathbf{H}(z) = \mathbf{A}_0 z+\mathbf{B}_0+\mathbf{B}_1 e^{-\varkappa z} = \begin{pmatrix}z+\alpha & , & -\varrho\\-\nu+\delta e^{-\varkappa z}, & & z+\beta\end{pmatrix}.$$

Then for any sufficiently large real number c,

$$\mathbf{R}(t) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}e^{tz}\,\mathbf{H}^{-1}(z)\prod(z)\,dz \qquad \text{for } t>\varkappa. \tag{3.4}$$

The determinant of the matrix $\mathbf{H}(z)$ is

$$det\,\mathbf{H}(z) = (z+\alpha)\,(z+\beta) - \varrho\nu\,(1-e^{-\varkappa(z+\beta)})\,. \qquad (3.5)$$

In order to evaluate the integral in (3.4) one would like to find all the zeroes of the function in (3.5) in the complex plane. Apart from the obvious zero $z = -\beta$, they do not seem to be easily uncovered.

§ 3.3. Bellman and Cooke's approach is probably too much geared to the general situation to be efficient when the differential – difference equations are as simple as (3.1). We are able to establish more informative expressions than the one is (3.4) by the following approach, which actually uses much simpler mathematics. The theorem as well as its proof are due to Professor W. Simonsen.

Theorem: For $n\varkappa \leqq t < (n+1)\varkappa$, $n \geqq 0$, we get

$$R_a(t) = {}_1P_n(t)\,e^{r_1 t} + {}_2P_n(t)\,e^{r_2 t} \qquad (3.6)$$

where r_1 and r_2 are given by (1.4), and where ${}_1P_n(t)$ and ${}_2P_n(t)$ are certain polynomials of degree n. Here

$${}_1P_0(t) = \frac{r_2+\alpha}{r_2-r_1}\,, \qquad {}_2P_0(t) = \frac{r_1+\alpha}{r_1-r_2} \qquad \text{for } 0 \leqq t < \varkappa\,,$$

and for $n \geqq 1$ and $n\varkappa \leqq t < (n+1)\varkappa$ we have the recursion formula

$$\begin{aligned} {}_1P_n(t) = {} & \frac{\nu\varrho}{r_2-r_1}\,e^{-(r_1+\beta)\varkappa}\left\{\int_{(n-1)\varkappa}^{t-\varkappa} {}_1P_{n-1}(\tau)\,d\tau \right. \\ & \left. + \sum_{k=0}^{n-1} \frac{{}_1P_{n-1}^{(k)}(t-\varkappa)}{(r_2-r_1)^{k+1}}\right\} + {}_1P_{n-1}(n\varkappa) - \frac{{}_1P'_{n-1}(n\varkappa)}{r_2-r_1} \\ & + \frac{e^{(r_2-r_1)n\varkappa}}{r_2-r_1}\left\{\nu\varrho\,e^{-(r_2+\beta)\varkappa}\sum_{k=0}^{n-1}\frac{{}_2P_{n-1}^{(k)}[(n-1)\varkappa]}{(r_1-r_2)^{k+1}} - {}_2P'_{n-1}(n\varkappa)\right\}. \end{aligned} \qquad (3.7)$$

A recursion formula for ${}_2P_n(t)$ results if the subscripts 1 and 2 are interchanged everywhere in (3.7).

Remark: By formula (3.1)

$$R_i(t) = \{R_a'(t) + \alpha R_a(t)\}/\varrho\,, \tag{3.8}$$

which in conjunction with the theorem permits us to find a formula for $R_i(t)$ similar to (3.6).

We precede the proof of the theorem by two lemmas:

Lemma 1; Let $a_1(x)$ and $a_2(x)$ be two polynomials of degree m, and let s_1 and s_2 be two distinct real numbers. Then the differential equation

$$y''(x) - (s_1+s_2)y'(x) + s_1 s_2 y(x) = a_1(x)e^{s_1 x} + a_2(x)e^{s_2 x} \tag{3.9}$$

has the general solution

$$y(x) = b_1(x)e^{s_1 x} + b_2(x)e^{s_2 x}\,,$$

where $b_1(x)$ and $b_2(x)$ are polynomials of degree $m+1$.

Lemma 2: If $a(x)$ and $b(x)$ are polynomials where

$$a'(x) - ca(x) = b(x) \qquad \text{for } c \neq 0\,, \tag{3.10}$$

then

$$a(x) = -\sum_{k\geqq 0} b^{(k)}(x)/c^{k+1}\,.$$

Proof of the theorem: For convenience we introduce functions $y_0(t)$, $y_1(t)$, ... by the definition

$$y_n(t) = R_a(t) \qquad \text{for } n\varkappa \leqq t < (n+1)\varkappa;\ n \geqq 0\,.$$

Using (3.8) to substitute for $R_i(t)$ in the second formula of (3.1), we get

$$y_n''(t) - (r_1+r_2)y_n'(t) + r_1 r_2 y_n(t) = -\nu\varrho e^{-\beta\varkappa} y_{n-1}(t-\varkappa)\,. \tag{3.11}$$

By this result and lemma 1 one may prove by induction that

$$y_n(t) = {}_1P_n(t)\,e^{r_1 t} + {}_2P_n(t)\,e^{r_2 t} \qquad \text{for } n\varkappa \leqq t < (n+1)\varkappa \tag{3.12}$$

for $n \geqq 0$, where ${}_1P_n(t)$ and ${}_2P_n(t)$ are polynomials of degree n. This is formula (3.6). Formula (3.2) shows that ${}_1P_0(t)$ and ${}_2P_0(t)$ are given as in the theorem. To deduce recursion formulae for the polynomials, we introduce (3.12) into (3.11) and get

$$\{{}_1P_n''(t) - (r_2-r_1)\,{}_1P_n'(t) + h_1 \cdot {}_1P_{n-1}(t-\varkappa)\}\,e^{r_1 t}$$
$$+ \{{}_2P_n''(t) - (r_1-r_2)\,{}_2P_n'(t) + h_2 \cdot {}_2P_{n-1}(t-\varkappa)\}\,e^{r_2 t} \equiv 0\,,$$

with $h_i = \nu\varrho \exp\{-(\beta + r_i)\varkappa\}$ for $i = 1, 2$. Concentrating on the $\{{}_1P_n(t)\}$ we see that

$${}_1P_n''(t) - (r_2 - r_1)\,{}_1P_n'(t) = -h_1 \cdot {}_1P_{n-1}(t - \varkappa),$$

which is of the form (3.10). Thus by lemma 2

$${}_1P_n'(t) = h_1 \sum_{k=0}^{n-1} {}_1P_{n-1}^{(k)}(t-\varkappa) \,/\, (r_2 - r_1)^{k+1}.$$

By integration

$${}_1P_n(t) - {}_1P_n(n\varkappa) = \frac{h_1}{r_2 - r_1} \int_{n\varkappa}^{t} {}_1P_{n-1}(\tau - \varkappa)\, d\tau$$

$$+ h_1 \sum_{k=1}^{n-1} \frac{{}_1P_{n-1}^{(k-1)}(t-\varkappa) - {}_1P_{n-1}^{(k-1)}[(n-1)\varkappa]}{(r_2 - r_1)^{k+1}},$$

where the sum is interpreted as zero for $n = 1$. Since $R_a(t)$ and $R_a'(t)$ are continuous at least for $t \geqq \varkappa$, we get for $n \geqq 1$,

$$y_n(n\varkappa) = y_{n-1}(n\varkappa) \quad \text{and} \quad y_n'(n\varkappa) = y_{n-1}'(n\varkappa).$$

After some further calculation this gives (3.7). This ends the proof of the theorem.

§ 3.4 Several integral equations for R_a and R_i are easily established by decomposition according to first and last disablement, etc. The most promising of these appear to be the decomposition by last (if any) disablement before time t, which gives

$$R_i(t) = \int_{t-\varkappa}^{t} R_a(\tau)\, \nu Q_i(t-\tau)\, d\tau \qquad \text{for } t > \varkappa, \text{ and}$$

$$R_a(t) = Q_a(t) + \int_0^t R_a(\tau)\, \nu \int_\tau^\theta Q_i(u-\tau)\, \varrho Q_a(t-u)\, du\, d\tau$$

$$\text{with } \theta = \min(t, \tau + \varkappa).$$

Evaluation of the innermost integral gives

$$R_a(t) = e^{-\alpha t} + (\nu\varrho/\lambda) \int_0^t R_a(t-\tau)\, e^{-\alpha\tau} (e^{\lambda \min(\tau, \varkappa)} - 1)\, d\tau. \qquad (3.13)$$

Let $F(t)$ be defined by $F(0)=0$ and $F'(t)=(\nu\varrho/\lambda)e^{-\alpha t}(e^{\lambda \min(t,\varkappa)}-1)$ for $t>0$. Then for $t>\varkappa$

$$F(t) = (\nu\varrho/\beta\lambda)(1-e^{-\beta\varkappa})-(\nu\varrho/\lambda\alpha)(1-e^{-\alpha t})+(\nu\varrho/\lambda\alpha)e^{\lambda\varkappa}(e^{-\alpha\varkappa}-e^{-\alpha t}).$$

Thus $F(\infty)=(\nu\varrho/\alpha\beta)(1-e^{-\beta\varkappa})$. Since $F(0)=0$, $F'(t)>0$ for $t>0$, and $F(\infty)<1$, $F(t)$ is a defective probability distribution function. We write (3.13) in the form

$$R_a(t) = e^{-\alpha t}+\int_0^t R_a(t-\tau)\,dF(\tau) \qquad \text{for } t>0, \tag{3.14}$$

and recognize this as a renewal equation. By Feller (1966, page 183, Theorem VI. 6.1)* (3.14) has a unique solution of the form

$$R_a(t) = \sum_{m=0}^{\infty}\int_0^t e^{-\alpha(t-\tau)}\,dF^{m*}(\tau)$$

where F^{m*} is the *m-th* convolution of F with itself. Attempts at finding an expression for F^{m*} by integration are soon stopped by ugly algebra.

We finally make an attempt at utilizing Laplace transforms. We see that for $\alpha+\zeta>0$, $\int_0^\infty e^{-\zeta t}d(e^{-\alpha t}) = -\alpha/(\zeta+\alpha)$. If $\varphi(\zeta) = \int_0^\infty e^{-\zeta t}\,dF(t)$ and $\psi(\zeta) = \int_0^\infty e^{-\zeta t}\,dR_a(t)$, formula (1.4) of Feller (1966, page 442) gives

$$\psi(\zeta) = \frac{-\alpha}{(\zeta+\alpha)\{1-\varphi(\zeta)\}}.$$

Since

$$\varphi(\zeta) = (\nu\varrho/\lambda)\int_0^\infty e^{-(\alpha+\zeta)t}(e^{\lambda\min(t,\varkappa)}-1)\,dt = \frac{\nu\varrho}{(\zeta+\alpha)(\zeta+\beta)}(1-e^{-(\zeta+\beta)\varkappa}),$$

we have

$$\psi(\zeta) = -\alpha(\zeta+\beta)/\{(\zeta+\alpha)(\zeta+\beta)-\nu\varrho(1-e^{-(\zeta+\beta)\varkappa})\}. \tag{3.15}$$

Inverting this formula is much the same problem as finding the zeroes of the function in (3.5).

* Feller's proof is valid also for defective distribution functions. See also his § XIV. 1, pp. 441–443.

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5. References

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Correction Note

Correction to "Some Notes on the Qualifying Period in Disability Insurance. I. Actuarial Values" by Jan M. Hoem *.

The author is grateful to Professor W. Simonsen, who has pointed out that the formulae for ${}^{\varkappa}_{s}p^{an}_{x+t}$ and ${}^{\varkappa}_{s}p^{rn}_{x+t}$ in lines 2 and 3 from below on page 111 of the paper need the additional member

$${}_{s-\varkappa}p^{ji}_{x+t}\,(1 - {}_{\varkappa}p^{id}_{x+t+s-\varkappa} - {}_{\varkappa}\bar{p}^{i}_{x+t+s-\varkappa})$$

for $j = a$ and $j = r$, respectively. The most compact form of the two formulae then is

$${}^{\varkappa}_{s}p^{jn}_{x+t} = {}_{h}p^{ja}_{x+t}(1 - {}_{s-h}p^{ad}_{x+t+h}) + {}_{h}p^{ji}_{x+t}\,(1 - {}_{s-h}p^{id}_{x+t+h} - {}_{s-h}\bar{p}^{i}_{x+t+h})$$

for $j = a, r,$ where $h = \max(0, s-\varkappa)$.

* *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker,* 69 (1), 105–116.

Summary

Particular problems arise in connection with the estimation of the forces of disability, recovery, and mortality in a disability model with a qualifying period, because no information is gathered concerning disability periods not exceeding the qualifying period. Some of these problems are studied in the present paper. The maximum likelihood estimators are established in a case where there is no recovery and where the mortality of the disabled is (possibly unrealistically) assumed equal to that of the able insured. For the general case where recovery may occur, it turns out to be impossible to find explicit expressions for the m.l. estimators. Formulae for some strategic functions are given, however, so that estimation can in principle be carried out by numerical methods.

Zusammenfassung

In Verbindung mit der Bewertung der Invalidisierungs-, Reaktivierungs- und Sterbeintensitäten in einem Invaliditätsmodell mit Karenzzeit entstehen besondere Probleme, weil Daten bezüglich der Invaliditätsfälle, die innerhalb der Karenzzeit liegen, fehlen. In dieser Arbeit werden einige dieser Probleme untersucht. Maximum-Likelihood-Schätzfunktionen werden unter der Voraussetzung aufgestellt, dass keine Reaktivierungen stattfinden und unter der (möglicherweise unrealistischen) Annahme, die Mortalität der Invaliden sei gleich derjenigen der Aktiven. Im allgemeinen Fall, d.h. bei Zulassung der Reaktivierung, zeigt es sich als unmöglich, explizite Ausdrücke für die Maximum-Likelihood-Schätzfunktionen zu finden. Dagegen können nach den gegebenen Anleitungen Schätzwerte mit numerischen Methoden ermittelt werden.

Résumé

L'estimation des taux instantanés d'invalidité, de réactivité et de mortalité dans un modèle d'invalidité comprenant un délai de carence, soulève des problèmes propres au manque d'informations des cas d'invalidité dont la durée est inférieure au délai de carence. Quelques-uns de ces problèmes sont étudiés dans le présent article. Des estimateurs de maximum de vraisemblance sont établis dans le cas hypothétique où l'on ne tient pas compte de la réactivité et où la mortalité des invalides est supposée égale à celle des assurés actifs. Dans le cas général où la réactivité est envisagée, il s'avère impossible de déterminer l'expression explicite des estimateurs de maximum de vraisemblance. Cependant on donne les formules necessaires pour permettre de procéder à l'estimation des paramètres par des méthodes de calcul numérique.