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Numerical calculation of the probability of ruin in the Poisson/Exponential case

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Consider a nonlife insurance company operating through an interval of time $(0, t)$. The probability distribution function of the independent intervals between successive claims is

$$A(\tau) = 1 - e^{-\lambda\tau} \qquad 0 < \tau < \infty; \quad \lambda > 0 \tag{1}$$

and the probability distribution function of individual claim size is assumed to be

$$B(y) = 1 - e^{-\mu y} \qquad 0 < y < \infty; \quad \mu > 0\,. \tag{2}$$

The size of a claim is thus independent of its epoch of occurrence and of the number of claims that have already occurred. This situation may conveniently be called the Poisson/exponential model since the assumption about claim occurrences implies that the probability distribution of the number of claims in an interval (τ_1, τ_2) is Poisson with parameter $\lambda(\tau_2 - \tau_1)$.

The Poisson assumption for claims is not unreasonable for a multiple line company; if claims are occurring randomly and independently in each line of business the overall number of claims follows the Poisson distribution. On the other hand the use of (2) as the distribution function of claim sizes is more unrealistic; the tail of the claim distribution is known to be much longer for a given mean claim μ^{-1} than (2) can accommodate. Nevertheless it may be defended as a first approximation for which, as we shall see, the mathematics are beautifully explicit.

The distribution of aggregate claims

Under the assumed circumstances the distribution function of $X(t)$, the aggregate claims in the interval $(0,t)$, written $F(x,t)$, may be derived as follows. Write $B^{n*}(x)$ for the distribution function of aggregate claims stemming from exactly n claims. Then

$$P\{X(t) \leqq x\} = F(x,t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} B^{n*}(x), \qquad 0 \leqq x < \infty, \tag{3}$$

since the probability of n claims in the interval is given by the Poisson distribution with parameter λt. Now the distribution function of the sum of n claim sizes may be obtained iteratively from

$$B^{n*}(y) = \int_0^y B^{(n-1)*}(y-z)\, dB^{1*}(z) \qquad n = 1, 2, 3, \ldots \tag{4}$$

$$B^{1*}(y) = B(y) \qquad\qquad B^{0*}(y) = \begin{cases} 0 & y < 0 \\ 1 & y \geqq 0 \end{cases}$$

the integrand of (4) being the probability of suffering a first claim of size z followed by $n-1$ claims the aggregate of which does not exceed $y-z$.

It is well-known that when $B(\cdot)$ assumes the form (2) relation (4) specializes to

$$B^{n*}(y) = \frac{\mu^n}{\Gamma(n)} \int_0^y e^{-\mu z} z^{n-1}\, dz = \frac{\gamma(n, \mu y)}{\Gamma(n)} \qquad n = 1,2,3, \ldots \tag{5}$$

and this can be substituted into (3). There is, however, a more convenient formula for numerical calculations.

On differentiating (3) with respect to x (noting the discontinuity at $x = 0$ where $F(0,t) = e^{-\lambda t} \equiv f(0, t)$ by definition)

$$\frac{\partial}{\partial x} F(x, t) = f(x, t) = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{\mu^n}{\Gamma(n)} e^{-\mu x} x^{n-1} \qquad x > 0$$

$$= \lambda \mu t\, e^{-\lambda t - \mu x} \sum_{n=1}^{\infty} \frac{(\lambda \mu t x)^{n-1}}{n!\,(n-1)!}\,.$$

Define

$$J(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2}, \tag{6}$$

then

$$J'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!\,(n-1)!},$$

and the foregoing expression for $f(x,t)$ becomes in an obvious notation $(x>0)$

$$f_{\mu\lambda}(x,t) = \lambda\mu t\, e^{-\lambda t-\mu x} J'(\lambda\mu t x) = \frac{t}{x} f_{\lambda\mu}(t,x). \tag{7}$$

We mention that $J'(\cdot)$ is related to the modified Bessel function of unit order (Olver, 1967) by the formula

$$J'(x) = x^{-\frac{1}{2}} I_1(2\sqrt{x}).$$

It is convenient to derive the Laplace-Stieltjes transform (moment generating function) of $F(\cdot,t)$, namely

$$f_t^*(s) \equiv F(0,t) + \int_0^{\infty} e^{-sx} f(x,t)\, dx$$

$$= e^{-\lambda t} + \lambda\mu t\, e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda\mu t)^{n-1}}{n!\,(n-1)!} \int_0^{\infty} e^{-(s+\mu)x} x^{n-1}\, dx$$

$$= e^{-\lambda t} + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda\mu t)^n}{n!\,(s+\mu)^n} = \exp\{-\lambda t s/(s+\mu)\}. \tag{8}$$

The probability of ruin before epoch *t*

We now suppose that the insurance company receives a continuous stream of premiums from its customers. The aggregate premium collected in the interval $(0,t)$ is equal to the expected claim outgo, namely

$$E\{X(t)\} = -\frac{\partial}{\partial s} f_t^*(s)\Big|_{s=0} = \frac{\lambda t}{\mu} \quad \text{from (8)}$$

plus a risk loading ηt. The premium intensity is thus $\lambda/\mu + \eta$ payable uniformly throughout the interval $(0, t)$. It is usual and convenient to write $\lambda = 1$ so that the unit of time is the expected interval between claim occurrences, and also $\mu = 1$ so that the average individual claim size is the monetary unit. The premium intensity throughout the interval is then $1 + \eta$.

The insurance company is supposed to pay premiums as they are received into a risk reserve whose value at the commencement of the interval is $R(0) = w$. All claims are paid from this reserve as they occur and interest on the reserve is paid out as a dividend to the stockholders. The reserve at epoch τ is thus

$$R(\tau) = w + (1+\eta)\,\tau - X(\tau)\,, \tag{9}$$

and if it becomes negative (technical) ruin is said to have occurred and the insurance company is supposed to borrow capital to pay claims until premiums accumulate to repay the capital and once again produce a non-negative value of $R(\cdot)$.

We now consider $U(w, t)$, the probability that the risk reserve does not become negative in the interval $(0, t)$, which may be written $(w \geqq 0)$

$$\begin{aligned} U(w,t) &= P\{R(\tau) \geqq 0,\ \ 0<\tau<t \,|\, R(0) = w\} \\ &= P\{\inf_{\tau<t}[w + (1+\eta)\,\tau - X(\tau)] \geqq 0\} \\ &= P\{\sup_{\tau<t}[X(\tau) - (1+\eta)\,\tau] \leqq w\}\,. \end{aligned} \tag{10}$$

Relation (10) shows that $U(w,t)$ is a non-decreasing, non-negative function of w for fixed t and assumes the value unity as $w \to \infty$. However w is not a specific value of a random variable W and $U(w\ t)$ is only a pseudo probability distribution function over the non-negative values of w.

Let us first consider $U(0,t)$, the probability that ruin does not occur in the interval $(0,t)$ given that the risk business started the interval with a zero risk reserve. Suppose that the aggregate claims in the interval amount to z where z must be less than the total premium paid, namely $(1+\eta)\,t$. The probability of this event is $f(z,t)\,dz$.

Now if ruin has not occurred during the interval preceding epoch t the premium received in any interval $(0, \tau)$, namely $(1+\eta)\tau$, must have exceeded the aggregate claim outgo of that interval. Since the interval $(0, t)$ produced a claim outgo of z the excess income during the period is $(1+\eta)t - z$ and, intuitively, the probability that the income throughout the interval has been larger than the claims is proportionate to the size of $(1+\eta)t - z$ in relation to the total premiums $(1+\eta)t$. We may guess, then, that the probability of non-ruin in the interval $(0, t)$, knowing that the aggregate claims amounted to z at the end of the interval, is $\{(1+\eta)t - z\}/(1+\eta)t$.

The foregoing result is proved as an extension of "the ballot theorem" (Feller, 1968, Ch. III) in which n claims of unit amount occur in m equal intervals of time during each of which a unit premium has been paid. The excess income of the period is $m - n$ and the probability that the income has remained in excess of the aggregate claims at the end of each of the m intervals is $(m-n)/m$ (Feller, *l. c.*). The extension is first to premiums of $1+\eta$ per interval and then to the continuous case in which the intervals between n claims and the claim amounts are independent realizations of two random variables which, in our case, are exponentially distributed with unit expectation (Finetti, 1970, Ch. VIII. 6). The result is found not to involve the number of claims but only their aggregate amount and is thus true generally[1]).

Summing over all permissible values of z we have

$$U(0,t) = \int_0^{(1+\eta)t} \left\{1 - \frac{z}{(1+\eta)t}\right\} f(z,t)\, dz \tag{11}$$

$$= F(\overline{1+\eta} \cdot t, t) - \frac{1+\eta}{t} \int_0^t z f(\overline{1+\eta} \cdot z, t)\, dz$$

$$= F(\overline{1+\eta} \cdot t, t) - (1+\eta) \int_0^t e^{(t-z)\eta} f(\overline{1+\eta} \cdot t, z)\, dz \tag{12}$$

by means of (7).

[1]) We will see later that $U(0,t)$ is the probability of an empty single-server queue at epoch t. A non-combinatorial inductive proof of our result may then be obtained, for example, from Prabhu (1965, Sec. 2.3) or Tackács (1962*a*, Lemma 1).

The important specialization that (12) represents in comparison with (11) is that in this Poisson/exponential case a fixed quantity w may now be added to the risk reserve at time zero thereby increasing the fixed claim outgo throughout (12) by a similar quantity. Thus ($w \geqq 0$)

$$U(w,t) = F(w+\overline{1+\eta}\cdot t.\, t) - (1+\eta)\int_0^t e^{(t-z)\eta} f(w+\overline{1+\eta}\cdot t, z)\, dz \tag{13}$$

$$= e^{-t} + e^{-t}\int_0^t e^{-(1+\eta)z} J'\big(z(w+\overline{1+\eta}\cdot t)\big)\big\{(w+\overline{1+\eta}\cdot t)\, e^{-wz/t} - (1+\eta)\, ze^{-w}\big\} dz \tag{14}$$

by means of (7).

We may use (13) to find an asymptotic value for $U(w,t)$ as $t \to \infty$, conveniently written $U(w,\infty)$. The first term on the right of (13) tends to unity and using (7) we have

$$U(w,\infty) = 1 - \lim_{t\to\infty}(1+\eta)\, e^{-w-t}\int_0^t ze^{-(1+\eta)z} J'\big(z\cdot \overline{w+(1+\eta)t}\big)\, dz$$

$$= 1 - \lim_{t\to\infty}\frac{e^{-\eta w/(1+\eta)}}{1+\eta}\, e^{-a}\int_0^{(1+\eta)a-w} ye^{-y} J'(ay)\, dy$$

with $a = t + w/(1+\eta)$

$$= 1 - \lim_{t\to\infty}\frac{e^{-\eta w/(1+\eta)}}{1+\eta}\cdot\frac{2e^{-a}}{\sqrt{a}}\int_0^{\sqrt{(1+\eta)a-w}} x^2 e^{-x^2} I_1(2\sqrt{a}\, x)\, dx\, .$$

The above integral with an infinite upper limit is equal to $e^a\sqrt{a}/2$ (Magnus *et al.*, 1966, 3 8.3) and it remains to investigate

$$\frac{2e^{-a}}{\sqrt{a}}\int_{\sqrt{(1+\eta)a-w}}^{\infty} x^2 e^{-x^2} I_1(2x\sqrt{a})\, dx =$$

$$= \frac{e^{-a}}{a^{3/4}\sqrt{\pi}}\int_{\sqrt{(1+\eta)a-w}}^{\infty} x^2 e^{-x^2}\frac{e^{2x\sqrt{a}}}{\sqrt{x}}\big[1 - 0(x\sqrt{a})^{-1}\big]\, dx\, .$$

The expression on the right is less in absolute value than

$$\frac{1}{a^{3/4}\{(1+\eta)a-w\}^{1/4}\sqrt{\pi}} \int_{\{(1+\eta)a-w\}^{1/2}}^{\infty} \{(x-\sqrt{a})^2 + 2\sqrt{a}\,(x-\sqrt{a}) + a\}\, e^{-(x-\sqrt{a})^2}$$
$$[1-0(x\sqrt{a})^{-1}]\,dx$$

which tends to zero as $a \to \infty$. Hence

$$U(w,\infty) = 1-(1+\eta)^{-1}\, e^{-\eta w/(1+\eta)}. \tag{15}$$

There is a discontinuity at $w = 0$ where

$$U(0,\infty) = \frac{\eta}{1+\eta}$$

which is true generally.

The form assumed by (15) suggests that there is no particular advantage in standardizing the pseudo random variable W. The moment generating function (Laplace-Stieltjes transform) can be derived but does not assume a simple form.

Historical remarks

The risk process in which claims occur as a Poisson process and their sizes are independent random variables was introduced by Lundberg (1903). The three opening paragraphs of Chapter I of his article suffer from severe compression and have been regarded as impenetrable, but we can rewrite them quite briefly in modern terminology and notation without any distortion of the original. Let

$$p_i = Pr\{\text{an individual claim size} = y_i\} \qquad i = 1, 2, 3, \ldots$$

and suppose that $p_i\,dt$ is the probability that a claim of size y_i is made in the time element $(t, t+dt)$. [Lundberg writes p_i' for our p_i and dp_i for our $p_i\,dt$. Cramér (1969) says that he is considering p_i as a function of t.] The probability of two (or more) claims is of order $(dt)^2$ and may be ignored. Write $f(x,t)$ for the density function of the aggregate claims made in the interval $(0,t)$, then by considering the mutually exclusive events that are produced by the occurrence of one claim or no claims, respectively, in the interval $(t, t+dt)$ we have

$$f(x, t+dt) = dt \sum_{i=1}^{\infty} p_i f(x-y_i, t) + \left\{1 - dt \sum_{i=1}^{\infty} p_i\right\} f(x,t) . \qquad (16)$$

[This relation occurs on p. 5 of Lundberg (1903).] Changing to continuous variables x and y the foregoing may be written

$$\frac{f(x,t+dt)-f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial t} = \int_{-\infty}^{\infty} \{f(x-y,t) - f(x,t)\} \, p(y) \, dy , \qquad (17)$$

and this is Lundberg's relation (3) except that he suppresses t in $f(x,t)$.

Lundberg goes on to consider the special case where $p(\cdot)$ degenerates into the unit probability at $y = 1$; (17) then becomes

$$\frac{\partial f(x,t)}{\partial t} = f(x-1,t) - f(x,t) \qquad x = 1, 2, 3, \ldots \qquad (18)$$

which is relation $(\overline{3})$ of Lundberg. In order to solve this Lundberg considers (by implication)

$$\frac{\partial e^t f(x,t)}{\partial t} = e^t \frac{\partial f(x,t)}{\partial t} + e^t f(x,t) =$$

$$= e^t f(x-1,t) \qquad \text{by means of (18)}$$

and says that the solution of (18) is

$$f(x,t) = e^{-t} \int_0^t e^s f(x,s) \, ds \qquad x = 1, 2, 3, \ldots \qquad (19)$$

Lundberg does not state how he calculates the initial value $f(0,t)$ for insertion in (19) but we may assume that he rewrote (16) as

$$f(0, t+dt) = (1-dt) \, f(0,t)$$

and obtained

$$f(0,t) = e^{-t} .$$

Repeated substitution into (19) then results in the Poisson process defined by

$$f(x,t) = e^{-t}\frac{t^x}{x!} \qquad x = 0,1,2,\ldots$$

We add that in his 1919 paper Lundberg stated that the above equation (17) is satisfied by

$$f(x,t) = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}\,p^{n^*}(x) \tag{20}$$

where

$$p^{n^*}(x) = \int_0^x p^{(n-1)^*}(x-y)\,p(y)\,dy \qquad n = 1,2,3,\ldots$$

and

$$p^{0^*}(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

The explicit formula for the particular case in which $p(x)$ is exponential, namely (7), was first provided by Ackermann (1939).

Lundberg continued his general approach to the Poisson risk process in his 1909 article. Attention is now paid to the random variable $R(\tau)$ defined by (9) except that: (i) w may be a random variable, (ii) η is supposed to be a function of $X(\tau)$, (iii) negative claims are permitted, and (iv) dividends may be paid from the risk reserve. The probability of ruin is considered and a general inequality found for it as $t \to \infty$. However the argument is not particularized to the exponential claims distribution and the result (15) was first given by Lundberg in 1926. Twenty four years later Arfwedson gave the result for the Poisson/exponential case valid for all t. In the 1950 paper he obtained (7) and (14) (his (77) and (48), respectively), the latter using rather deep mathematics and the former by writing $1+\eta = 0$ in (14). In his 1952 paper Arfwedson developed a series expansion for calculating $U(w,t)$ and produced two-decimal values of it for $w = 0, 11, 110, 1100$ and $t = 10^j$, $j = 1,2,3,4,\infty$, with $\eta = 0$ and 0.1, respectively. This numerical achievement has not hitherto been duplicated.

Formulae (7) and (14), the explicit solutions of the two central problems of risk theory, are identical with two formulae in a queueing theory model in which a single server, idle at epoch zero, handles the service demands of customers who arrive as the realization of a Poisson process with arrival intensity λ. Customers are served in order and wait in a queue if the server is busy. The lengths of customers' service times are independent of each other and of their arrival epochs and are distributed exponentially with mean μ^{-1}. The density function of the aggregate service load offered to the server through epoch t conditional on n customer arrivals is $dB^{n^*}(y)/dy$ and the unconditional density is then given by (7). However in order to translate (14) into the corresponding queueing formula we must rescale the time and monetary units by writing $w' = w/\mu$, $t' = t/\lambda$ and $\varrho^{-1} = \mu/\lambda = 1+\eta$. It turns out that the probability on the right of (10) is then the distribution function of the random variable $W(t)$, the waiting time for service of a customer assumed to join the queueing system at epoch t. These equivalences were first made explicit by Prabhu (1961).

Although $W(t)$ is an important random variable in queueing theory at least as much attention has been paid to Q (t), the length of the queue (including the customer being served) at epoch t. Of course when Q $(t) = 0$ an arriving customer has no wait for service and the probability of this empty system is $U(0,t)$. In the Poisson/exponential case (denoted M/M/1 in the queueing literature) the distributions of Q(t) and $W(t)$ are related as follows.

If there are k individuals in the system at epoch t the aggregate unsatisfied service load is made up of the remaining service demand of the customer being served plus the sum of the service times of the $k-1$ customers in the queue. Since the distribution of a partially elapsed interval whose length is distributed exponentially is also exponential (without parameter change) we have, with appropriate choice of time scale,

$$U(w,t) = \sum_{k=0}^{\infty} P_k(t)\ B^{k^*}(w) \tag{21}$$

where $P_k(t) = P\{Q(t) = k\}$ and $B^{k^*}(\cdot)$ is given by (5).

Furthermore, a relation for $P_k(t)$ in terms of $U(\cdot,\cdot)$ may be obtained as follows. If the system contains $k > 0$ customers at epoch t

suppose that the customer receiving service arrived at epoch $t-\tau$, commenced service at epoch $t-\tau+y$ (so that his waiting time was y) and continued to be served at least until epoch t (i.e., at least for a period $\tau-y$). The $k-1$ customers waiting in the queue arrived after the designated customer, namely during the interval $(t-\tau, t)$ with probability $e^{-\lambda\tau}(\lambda\tau)^{k-1}/(k-1)!$. We thus have ($k = 1, 2, 3, \ldots$)

$$P_k(t) = \int_0^t e^{-\lambda\tau} \frac{(\lambda\tau)^{k-1}}{(k-1)!} d\tau \int_0^\tau e^{\tau-y} d_y U(y, t-\tau) . \tag{22}$$

The foregoing queueing model was introduced by Erlang in 1909 when he showed that the assumption that telephone calls were uniformly distributed in any interval of time led to a Poisson distribution for the number of independent calls in a specified period. He proceeded to derive the distribution of $W(t)$ on the assumption that individual service times were constant. His arguments and mathematics are easily understood. Erlang's first use of the exponential distribution for service times was in 1917 and he there provided the asymptotic waiting time distribution (15) as a special case of a formula valid for more than one server.

The first major contribution to the so-called transient case (t finite) of queueing theory (M/M/1) was made by Ledermann and Reuter (1954) and then, more simply, by Bailey (1954) for the function $\partial P_k(t)/\partial t$ and for $G(\cdot)$, the distribution function of the length of a (busy) period during which ther server is never idle. No attempt seems to have been made to obtain $U(w,t)$ directly for $w \neq 0$ and reference is invariably made to (21) in text-books on the subject (e.g., Takács, 1962*b*; Prabhu, 1965; Cohen, 1969).

The probability of ruin before the nth claim

Another function of interest in queueing, and possibly also in risk theory, is $W_n(\cdot)$ the probability distribution function of the waiting time for service of the nth arrival at the queue. This function has played a central role in Pollaczek's idiosyncratic contributions to telephone engineering which commenced in 1930. It has received considerable attention since the early fifties because of its connections with "fluctuation theory". Lindley (1952) wrote down the following self-explanatory recurrence relation

$$W_n(w) = \int_{-\infty}^{w} W_{n-1}(w-z)\, dK(z), \qquad n = 1, 2, 3, \ldots \tag{23}$$

$$W_0(w) = \begin{cases} 0 & w < 0 \\ 1 & w \geqq 0 \end{cases}$$

where $K(\cdot)$ is the probability distribution function of Z, the difference between the service period of a customer and the interval of time between that customer and the next. In order for customer number 1 to have a possibly non-zero waiting time we assume that customer 0 starts the system at epoch zero with a random service load from which is deducted the arrival epoch of customer number 1 to provide the first realization of Z. This is in conformity with the natural procedure in the risk model where the premium for the interval until the first claim is added to w and then reduced by the first claim amount corresponding to the service load of customer 0.

Consideration shows that

$$K(z) = \int_{y=\max(0,\,-z)}^{\infty} B(z+y)\, dA(y)$$

or, in our doubly exponential case,

$$K(z) = \begin{cases} \mu e^{\lambda z}/(\lambda+\mu) & z < 0 \\ 1 - \lambda e^{-\mu z}/(\lambda+\mu) & z > 0 \end{cases} \tag{24}$$

For this M/M/1 case a series expansion has been obtained by Cohen (1969) namely ($\varrho = \lambda/\mu$)

$$W_n(w) = \sum_{j=0}^{n} \frac{\gamma(j, \mu w)}{\Gamma(j)} \sum_{k=0}^{\infty} \frac{j+k+1}{2n-j+k+1} \binom{2n-j+k+1}{n-j} \frac{\varrho^{n+1}}{(1+\varrho)^{2n-j+k+1}} \tag{25}$$

where the gamma ratio is replaced by $B^{0*}(w)$ when $j = 0$.

The last term of this series ($j = n$) is equal to

$$\frac{\gamma(n, \mu w)}{\Gamma(n)} \left(\frac{\varrho}{1+\varrho} \right)^{n+1} \sum_{k=0}^{\infty} (1+\varrho)^{-k} = \frac{\gamma(n, \mu w)}{\Gamma(n)} \left(\frac{\varrho}{1+\varrho} \right)^{n},$$

and the inner sum for the remaining j-values is $\varrho^{n+1}/(1+\varrho)^{2n-j+1}$ times

$$\sum_{k=0}^{\infty} \frac{j+k+1}{n-j} \binom{k+2n-j}{n-1-j} (1+\varrho)^{-k}$$

$$= \frac{1}{n-j} \sum_{k=0}^{\infty} (j+k+1)(1+\varrho)^{-k} \sum_{l=0}^{n-1-j} \binom{k}{l} \binom{2n-j}{n-1-j-l}$$

on making use of the relation (Riordan, 1968)

$$\binom{k+p}{m} \equiv \sum_{l=0}^{m} \binom{k}{l} \binom{p}{m-l}.$$

Noting that

$$\sum_{k=0}^{\infty} \binom{k}{l} (1+\varrho)^{-k} = (1+\varrho)\,\varrho^{-l-1} \qquad \varrho > 0,\ l = 0, 1, 2, \ldots$$

the required sum becomes

$$\frac{1}{n-j} \sum_{l=0}^{n-1-j} \binom{2n-j}{n-1-j-l} \sum_{k=0}^{\infty} (j+k+1) \binom{k}{l} (1+\varrho)^{-k}$$

$$= \frac{1+\varrho}{(n-j)\,\varrho} \sum_{l=0}^{n-1-j} \binom{2n-j}{n-1-j-l} \{ j\varrho^{-l} + (l+1)(1+\varrho)\,\varrho^{-l-1} \}.$$

$$= \frac{1+\varrho}{(n-j)\,\varrho^2} \sum_{l=j}^{n-1} \binom{2n-j}{n-1-l} \{ (l+1)(1+\varrho) - j \} \varrho^{-l+j}.$$

We thus have

$$W_n(w) = \sum_{j=0}^{n-1} \frac{\varrho^{n-1}}{(1+\varrho)^{2n-j}} \frac{\gamma(j, \mu w)}{(n-j)\,\Gamma(j)} \sum_{l=j}^{n-1} \binom{2n-j}{n-1-l} \{(l+1)(1+\varrho)-j\}\,\varrho^{-l+j}$$

$$+ \frac{\gamma(n, \mu w)}{\Gamma(n)} \left(\frac{\varrho}{1+\varrho}\right)^n \tag{26}$$

where $\gamma(0, \mu w)/\Gamma(0)$ is understood to be unity. The right hand side is a series of $n(n+1)/2+1$ terms.

In risk theory $W_n(w)$ is the probability of non-ruin through the nth claim starting with a risk reserve of w. Only one example occurs in the literature, namely Beard's (1971) note in which he used (23) to derive the values of $W_n(\cdot)$ for $n = 1, 2$ (and for $n = 3, 4$ when $\eta = 0$) for the Poisson/exponential case. His results agree with (25) with $\varrho = (1+\eta)^{-1}$.

Numerical calculations

The calculation of $F(x, t)$ by numerical integration of (7) poses no substantial problems, particularly since $I_1(z)$ can be approximated with better than seven significant figure accuracy by polynomials (Olver, 1967, Sec. 9.8.3/4). It is well known that if $X(t)$ is the random variable having $F(\cdot, t)$ as distribution function then $X_0 = \dfrac{X(t)-t}{\sqrt{2t}}$ is asymptotically standard Normal in distribution. This suggests that t-wise interpolation of $F(x,t)$ is likely to be more successful for constant x_0–, rather than x–, values.

We accordingly calculated $F(x,t)$ for $x_0 = -5(1)5$ and $t = 1(1)\,50(50)\,2000$. Trapezoidal quadrature with 128 panels was used first on $F_0(-5, t)$ and then on each $F_0(x_0+1, t) - F_0(x_0, t)$, $x_0 = -5(1)4$ where F_0 has been written for F when x is replaced by the corresponding standardized value x_0. Actually the Trapezoidal Rule was used with 4, 8, 16, 32, 64 and 128 panels and the Romberg extrapolate based on the six resulting values (Henrici, 1964, § 13.7) was found to agree with the 128 panel result to at least five decimal places. The following Table 1

Table 1

t	$x_0 = -4$	-3	-2	-1	0	1	2	3	4	5
					$F_0(x_0, t)$,		$x_0 = (x-t)/\sqrt{2t}$			
1	–	–	–	–	.65425	.86099	.94679	.98036	.99296	.99753
2	–	–	–	.13534	.60350	.85194	.95123	.98528	.99583	.99888
3	–	–	–	.14133	.58333	.84875	.95419	.98777	.99703	.99933
4	–	–	–	.14653	.57172	.84707	.95628	.98932	.99768	.99954
5	–	–	–	.14952	.56392	.84604	.95786	.99040	.99810	.99966
10	–	–	.00234	.15470	.54489	.84384	.96236	.99308	.99897	.99987
20	–	.00001	.00844	.15685	.53164	.84266	.96609	.99493	.99944	.99995
30	–	.00003	.01124	.15749	.52581	.84224	.96790	.99571	.99960	.99997
40	–	.00009	.01289	.15780	.52234	.84203	.96903	.99616	.99968	.99998
50	–	.00015	.01400	.15798	.51997	.84190	.96982	.99647	.99973	.99999
100	–	.00037	.01669	.15833	.51411	.84163	.97186	.99718	.99983	.99999
150	–	.00051	.01785	.15844	.51152	.84154	.97280	.99748	.99987	1.
200	–	.00060	.01853	.15850	.50998	.84149	.97337	.99765	.99989	1.
400	.00001	.00079	.01980	.15858	.50705	.84142	.97447	.99796	.99992	1.
600	.00001	.00088	.02036	.15860	.50576	.84139	.97497	.99810	.99993	1.
800	.00001	.00094	.02068	.15862	.50499	.84138	.97527	.99818	.99994	1.
1000	.00001	.00098	.02091	.15862	.50446	.84137	.97547	.99823	.99994	.99999
1500	.00002	.00104	.02125	.15863	.50364	.84136	.97578	.99830	.99994	.99999
2000	.00002	.00108	.02145	.15864	.50315	.84135	.97598	.99835	.99994	.99999
∞	.00003	.00135	.02275	.15866	.50000	.84134	.97725	.99865	.99997	1.

extracted from the computer output may thus be regarded as correct to within a unit in the fifth decimal place. It is interesting to note how quickly F_0 converges towards its asymptotic value when $x_0 = \pm 1$ but how relatively incorrect the asymptotic F_0 is at the mean $x_0 = 0$ even for fairly large t-values.

Relation (14) allows $U(w,t)$ to be calculated by approximate integration for any combination of η, w and t. The following arbitrary rule was adopted: if the Romberg extrapolate based on six approximations by the Trapezoidal Rule with successive halving of the panel width differed from the sixth quadrature by 8 or less in the fifth decimal place the extrapolate was to be judged essentially correct to five decimals.

Experiments were carried out with $\eta = 0.1$ (a 10% risk loading). Using up to 64 panels calculation of $U(w,t)$ for $w = 0(11)\,22$ and $t = 1\,(1)\,100\,(2)\,150$ resulted in five decimal accuracy for $U(0,t)$ throughout, for $U(22,t)$, $t = 9\,(1)\,66$, and for a few other isolated sets of values. In order to produce five decimal accuracy throughout the number of panels was increased to 128, and the ranges were extended to $w = 0\,(11)\,110$ and $t = 50\ (50)\ 2000$. All the resulting values of $U(w,t)$ satisfied the arbitrary accuracy criterion, very few fifth place differences being as large as 7 or 8. Finally, a further set of 128-panel quadratures produced values of $U(w,t)$ for $w = 1\ (1)\ 10$ and $t = 1\ (1)\ 50$, the extrapolate now never differing from the final quadrature by more than 5 in the fifth place.

Table 2 summarizes the results obtained in the foregoing computations. We notice that there is at most a unit difference in the third decimal place between $U\ (w,1000)$ and the corresponding asymptotic value. The final line of the Table provides Segerdahl's Normal approximation to $U(w,1000)$ (see Seal, 1969, p. 115). While the values are, for $w \geqq 22$, slight improvements over the asymptotic results they do not justify the extra calculations involved.

The upper part of the Table is interesting because it shows how quickly $U\ (w,t)$ tends to unity for given small t. The step at $w = 0$ is larger than one-half when $t = 1$ showing that, with no initial capital and a 10% risk loading, there is only slightly better than an even chance of avoiding ruin during the interval at the end of which the first claim is expected.

Table 3 provides the corresponding values of $U(w,t)$ for the no-loading case, $\eta = 0.0$. Ruin is eventually certain whatever w is but for

Table 2

$U(w, t),\ \eta = 0.1$

t	$w = 0$	1	2	3	4	5	6	7	8	9	10
1	.53660	.76194	.88029	.94085	.97121	.98616	.99342	.99690	.99855	.99933	.99969
2	.40714	.64543	.79433	.88367	.93560	.96499	.98127	.99012	.99486	.99735	.99865
3	.34479	.57402	.73154	.83524	.90118	.94191	.96645	.98093	.98932	.99409	.99677
4	.30669	.52472	.68359	.79471	.86979	.91907	.95061	.97035	.98246	.98977	.99410
5	.28040	.48811	.64558	.76049	.84164	.89734	.93464	.95906	.97474	.98463	.99077
6	.26088	.45957	.61455	.73125	.81646	.87701	.91901	.94752	.96649	.97890	.98688
7	.24566	.43653	.58863	.70596	.79389	.85812	.90397	.93600	.95797	.97277	.98258
8	.23337	.41745	.56658	.68384	.77359	.84064	.88962	.92469	.94934	.96637	.97796
9	.22319	.40130	.54753	.66430	.75524	.82444	.87601	.91369	.94074	.95983	.97311
10	.21457	.38742	.53087	.64690	.73857	.80943	.86312	.90305	.93224	.95323	.96810
20	.16816	.30939	.43267	.53879	.62889	.70438	.76683	.81785	.85904	.89191	.91785
30	.14798	.27393	.38578	.48419	.57000	.64413	.70760	.76147	.80678	.84458	.87584
40	.13621	.25289	.35738	.45033	.53247	.60458	.66744	.72188	.76871	.80872	.84269
50	.12836	.23872	.33804	.42696	.50618	.57639	.63827	.69253	.73985	.78090	.81631

t	$w = 0$	11	22	33	44	55	66	77	88	99	110
50	.12836	.84671	.98438	.99904	.99996	1.	1.	1.	1.	1.	1.
100	.11001	.77244	.95621	.99373	.99933	.99994	1.	1.	1.	1.	1.
150	.10282	.73611	.93517	.98695	.99786	.99971	.99997	1.	1.	1.	1.
200	.09902	.71512	.92050	.98080	.99602	.99929	.99989	.99998	.99999	1.	1.
400	.09343	.68177	.89287	.96584	.98979	.99716	.99927	.99982	99994	.99998	.99999
600	.09191	.67215	.88372	.95977	.98652	.99565	.99865	.99960	.99986	.99996	.99998
800	.09136	.66853	.88009	.95715	.98494	.99482	.99826	.99943	.99980	.99993	.99997
1000	.09112	.66698	.87848	.95593	.98416	.99437	.99802	.99932	.99976	.99991	.99996
1500	.09095	.66582	.87725	.95496	.98350	.99397	.99780	.99920	.99970	.99989	.99996
2000	.09092	.66562	.87702	.95478	.98337	.99387	.99773	.99914	.99966	.99985	.99993
∞	.09091	.66556	.87697	.95474	.98335	.99387	.99775	.99917	.99970	.99989	.99996
1000	.09091	.66556	.87698	.95492	.98376	.99431	.99807	.99936	.99979	.99994	.99998

Table 3

$U(w, t), \ \eta = 0.0$

t	$w = 0$	1	2	3	4	5	6	7	8	9	10
1	.52378	.75406	.87580	.93842	.96993	.98551	.99309	.99674	.99848	.99929	.99967
2	.38575	.62804	.78207	.87572	.93072	.96212	.97963	.98921	.99436	.99708	.99851
3	.31871	.54911	.71164	.82083	.89140	.93557	.96249	.97853	.98789	.99327	.99630
4	.27757	.49389	.65684	.77381	.85457	.90852	.94357	.96580	.97960	.98802	.99304
5	.24910	.45252	.61280	.73344	.82085	.88216	.92399	.95183	.96996	.98154	.98881
6	.22789	.42005	.57647	.69846	.79019	.85703	.90442	.93721	.95941	.97414	.98376
7	.21131	.39368	.54586	.66786	.76234	.83332	.88524	.92233	.94825	.96603	.97800
8	.19789	.37172	.51963	.64085	.73699	.81105	.86666.	.90745	.93675	.95740	.97169
9	.18674	.35307	.49683	.61680	.71384	.79020	.84879	.89276	.92508	.94840	.96493
10	.17729	.33697	.47678	.59522	.69263	.77066	.83168	.87837	.91338	.93916	.95782
20	.12576	.24501	.35614	.45764	.54860	.62869	.69804	.75713	.80675	.84783	.88137
30	.10279	.20198	.29653	.38535	.46767	.54295	.61092	.67157	.72505	.77168	.81191
40	.08907	.17577	.25939	.33912	.41433	.48454	.54942	.60878	.66260	.71093	.75395
50	.07969	.15768	.23343	.30632	.37584	.44158	.50321	.56053	.61341	.66181	.70578

t	$w = 0$	11	22	33	44	55	66	77	88	99	110
50	.07969	.74543	.96330	.99701	.99985	1.	1.	1.	1.	1.	1.
100	.05638	.59118	.87760	.97439	.99617	.99958	.99997	1.	1.	1.	1.
150	.04605	.50370	.80017	.93780	.98492	.99712	.99956	.99995	1.	1.	1.
200	.03988	.44602	.73716	.89789	.96746	.99145	.99813	.99966	.99995	.99999	1.
400	.02821	.32649	.57755	.76124	.87868	.94462	.97728	.99161	.99721	.99916	.99977
600	.02303	.26976	.48941	.66709	.79802	.88612	.94037	.97100	.98690	.99450	.99786
800	.01995	.23503	.43204	.59985	.73294	.83137	.89935	.94323	.96975	.98477	.99275
1000	.01784	.21098	.39098	.54918	.68043	.78330	.85956	.91305	.94859	.97098	.98436
1500	.01457	.17310	.32433	.46300	.58528	.68902	.77373	.84033	.89077	.92758	.95348
2000	.01262	.15028	.28314	.40761	.52083	.62083	.70663	.77815	.83608	.88171	.91663
∞	always zero for finite w										

small t-values the values of $U(w,t)$ are not much smaller than the corresponding values for $\eta = 0.1$ particularly when w is about 10. This conforms with intuition. It is mentioned that the whole of Table 3 was produced in 90 seconds of computation on an IBM 7094.

Large values of n pose a problem in the numerical computation of $W_n(w)$ and, after putting $\mu = 1$ and $\varrho^{-1} = 1 + \eta$, we rewrite (26) in the form

$$W_n(w) = \sum_{j=0}^{n-1} \left(\frac{1+\eta}{1+\eta/2}\right)^{2n-j} (1+\eta)^{-n} P(j,w)\, a_j \sum_{l=j}^{n-1} b_{jl} + P(n,w)\ (2+\eta)^{-n}$$

where

$$a_j = \left(\frac{1}{2}\right)^{2n-j} \binom{2n-j}{n-1-j} \qquad j = 0,1,2,\dots\ n-1$$

$$b_{jl} = \frac{(n-1-j)!}{(n-1-l)!}\ \frac{(n+1)!}{(n+1+l-j)!} \cdot \frac{(l+1)\ (2+\eta)/(1+\eta)-j}{n-j}\ (1+\eta)^{l+1-j}$$

and $$P(j+1,w) = P(j,w) - w^j e^{-w}/j! \quad j = 0,1,2,\dots\ n-1\,.$$

In these expressions a_j decreases from $a_0 \sim 1/\sqrt{(n\pi)}$ to $a_{n-1} = 1/2^{n+1}$ which may be very small. The initial value of b_{jl} is

$$b_{jj} = \frac{j+2+\eta}{n-j} \quad \text{and the final value is}$$

$$b_{j,\,n-1} = \frac{(2+\eta)n-(1+\eta)j}{n-j}\ (1+\eta)^{n-j-1} \Big/ \binom{2n-j}{n-1-j}$$

the latter being very small for the lower values of j. We thus watch for excessively small values of the terms of the inner sum when j is small and stop calculations when increasing j causes the outer factors to evanesce.

Table 4 provides the values of $W_n(w)$ corresponding to the values of $U(w,t)$ given in the top part of Table 2. Bearing in mind that

$$\lim_{t\to\infty} U(w,t) = \lim_{n\to\infty} W_n(w)\,,$$

we are not surprised to see that the two distribution functions are already quite close for $t = n = 50$.

Table 4

$W_n(w),\ \eta = 0{,}1$

n	$w = 0$	1	2	3	4	5	6	7	8	9	10
1	.52381	.82482	.93555	.97629	.99128	.99679	.99882	.99957	.99984	.99994	.99998
2	.40503	.69770	.85810	.93651	.97249	.98835	.99515	.99801	.99919	.99967.	.99987
3	.34578	.61443	.79024	.89247	.94730	.97505	.98851	.99482	.99771	.99900	.99957
4	.30883	.55687	.73500	.85047	.91957	.95840	.97917	.98985	.99517	.99774	.99896
5	.28302	.51461	.69034	.81268	.89182	.93997	.96782	.98326	.99151	.99579	.99795
6	.26371	.48206	.65379	.77935	.86535	.92096	.95518	.97534	.98680	.99310	.99647
7	.24857	.45607	.62338	.75012	.84070	.90212	.94184	.96647	.98120	.98971	.99450
8	.23630	.43473	.59766	.72440	.81801	.88390	.92827	.95697	.97488	.98569	.99204
9	.22610	.41681	.57560	.70167	.79722	.86653	.91477	.94710	.96803	.98114	.98913
10	.21744	.40150	.55644	.68145	.77819	.85010	.90156	.93708	.96080	.97616	.98582
20	.17052	.31702	.44618	.55793	.65276	.73165	.79598	.84741	.88773	.91871	.94207
30	.14996	.27939	.39510	.49739	.58673	.66381	.72948	.78472	.83057	.86815	.89854
40	.13793	.25723	.36459	.46045	.54534	.61987	.68472	.74063	.78839	.82880	.86266
50	.12988	.24236	.34396	.43521	.51665	.58887	.65248	.70812	.75645	.79812	.83377

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Zusammenfassung

In einer wenig bekannten Arbeit leitete Arfwedson (1950) einen expliziten Ausdruck für die Wahrscheinlichkeit her, dass eine Nichtlebensversicherungs-Gesellschaft innerhalb von t Jahren nicht ruiniert wird, wenn die Anfangsrisikoreserve w ist, die Schäden Poisson – und ihre Beträge exponentiell – verteilt sind. Dies entspricht der Verteilungsfunktion der Wartezeit bis zur Bedienung für einen Kunden, der, im Fall M/M/1, sich einer einfachbedienten Warteschlange im Zeitpunkt t anschliesst. Diese Funktion wird in einer einfachen Weise hergeleitet, und es werden numerische Werte geliefert für

(i) $w = 0\ (1)\ 10$ und $t = 1\ (1)\ 10\ (10)\ 50$,

(ii) $w = 0\ (11)\ 110$ und $t = 50\ (50)\ 200\ (200)\ 1000\ (500)\ 2000$

in den beiden Fällen $\varrho^{-1} = 1 + \eta = 1.0$ und 1.1.

Ein neuer Ausdruck wird für die numerische Berechnung der Wahrscheinlichkeit des Nichtruins durch den n-ten Schaden (oder die Verteilungsfunktion der Wartezeit des sich der Warteschlange anschliessenden n-ten Kunden) hergeleitet, und dieser wird ausgewertet für

$$w = 0\ (1)\ 10 \quad \text{und} \quad n = 1\ (1)\ 10\ (10)\ 50 \quad \text{mit} \quad \varrho^{-1} = 1 + \eta = 1.1.$$

Summary

In a little-known paper Arfwedson (1950) derived an explicit expression for the probability that a nonlife insurance company will not be ruined within t years when the initial risk reserve is w and claims and their amounts are Poisson/exponential. Equivalently this is the distribution function of the waiting time for service of a customer joining a single-server queue at epoch t in the M/M/1 case. This function is derived in a simple way and numerical values are provided for

(i) $w = 0\ (1)\ 10$ and $t = 1\ (1)\ 10\ (10)\ 50$, and

(ii) $w = 0\ (11)\ 110$ and $t = 50 (50)\ 200\ (200)\ 1000\ (500)\ 2000$,

in the two cases $\varrho^{-1} = 1 + \eta = 1.0$ and 1.1.

A new expression is provided for the numerical calculation of the probability of nonruin through the nth claim (or the distribution function of the waiting time of the nth customer joining the queue) and this is evaluated for

$$w = 0 (1) 10 \quad \text{and} \quad n = 1\ (1)\ 10\ (10)\ 50 \quad \text{with} \quad \varrho^{-1} = 1 + \eta = 1.1.$$

Résumé

Dans un article pas très connu Arfwedson déduisait une expression explicite de la probabilité qu'une société d'assurances non-vie ne soit pas ruinée dans t années lorsque la réserve initiale est w et le nombre des sinistres est distribué selon la loi de Poisson et les montants des sinistres selon la loi exponentielle. Cela correspond à la fonction de distribution de la période d'attente d'un client, qui – dans le cas M/M/1 – se joint à une queue au moment t. Cette fonction peut être dérivé de façon simple et on ajoute des valeurs numériques pour

(i) $w = 0(1)10$ et $t = 1(1)10(10)50$,

(ii) $w = 0(11)110$ et $t = 50(50)200(200)1000(500)2000$,

dans les deux cas $\varrho^{-1} = 1 + \eta = 1.0$ et 1.1.

Une nouvelle expression est déduite pour le calcul numérique de la probabilité que le dommage n n'entraîne pas la ruine de la société (ou de la fonction de distribution de la période d'attente du client n qui se joint à la queue) et on utilise l'expression pour

$$w = 0(1)10 \quad \text{et} \quad \eta = 1(1)10(10)50 \quad \text{avec} \quad \varrho^{-1} = 1 + \eta = 1.1.$$

Riassunto

In un articolo non molto conosciuto Arfwedson deriva una espressione esplicita per la probabilità che una società assicurativa del ramo generale non vadi in rovina entro t anni quando la riserva iniziale sia w e il numero dei danni abbia la distribuzione di Poisson mentre gli importi di questi siano distribuiti di modo esponenziale. Questa distribuzione corrisponde alla funzione di distribuzione del tempo di atteso di un cliente che – nel caso M/M/1 – si unisce a una coda all'instante t. Questa funzione viene ottenuta in modo semplice e vengono forniti valori numerici per

(i) $w = 0(1)10$ e $t = 1(1)10\ (10)50$,

(ii) $w = 0(11)110$ e $t = 50(50)200(200)1000(500)2000$,

nei casi $\varrho^{-1} = 1 + \eta = 1.0$ e 1.1.

Una nuova espressione viene sviluppata per il calcolo numerico della probabilità che l'n-esimo danno non mandi in rovina la società (o la funzione di distribuzione del tempo di attesa dell'n-esimo cliente che si unisce alla coda) e vengono dati valori numerici per

$$w = 0(1)10 \quad \text{e} \quad \eta = 1(1)10(10)50 \quad \text{nel caso} \quad \varrho^{-1} = 1 + \eta = 1.1.$$