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Martingales in Risk Theory

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1. The General Method

Let $\{X_t\}_0^\infty$ be a one-dimensional Markov-process (X_t = surplus of a company at time t). We are interested in T, the time of first entry into the negative half-axis (T = time of "ruin"), and X_T, the non-positive surplus at the time when ruin occurs. In many cases T and X_T are defective random variables. For notational convenience let $[T = \infty]$ be the event that ruin does not occur.
If we stop the process at T, we obtain the process $\{\tilde{X}_t\}_0^\infty$, where

$$\tilde{X}_t = \begin{cases} X_t & \text{if } T > t \\ X_T & \text{if } T \leq t. \end{cases} \tag{1}$$

Let

$$\psi_t(x) = P[T \leq t / X_0 = x] \tag{2}$$

be the probability of ruin before time t, and let

$$\psi(x) = P[T < \infty / X_0 = x] \tag{3}$$

be the probability of ultimate ruin; both are functions of the initial surplus x. Our goal is to obtain information about these two functions.
The general idea is to find an appropriate function $v(x, t)$, such that $\{v(\tilde{X}_t, \min(t, T))\}_0^\infty$ is a martingale with respect to $\{X_t\}_0^\infty$. Then the martingale property implies that for $t > 0$

$$\begin{aligned} v(x,0) &= E[v(\tilde{X}_t, \min(t, T)) / X_0 = x] \\ &= E[v(X_T, T) / X_0 = x, T \leq t]\, \psi_t(x) \\ &\quad + E[v(X_t, t) / X_0 = x, T > t][1 - \psi_t(x)]. \end{aligned} \tag{4}$$

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Making the additional assumption that $v \geq 0$, we obtain the inequalities

$$\psi_t(x) \leq \frac{v(x,0)}{E[v(X_T, T)/X_0 = x, T \leq t]} \tag{5a}$$

and

$$\psi(x) \leq \frac{v(x,0)}{E[v(X_T, T)/X_0 = x, T < \infty]} \tag{5b}$$

which are useful whenever the denominators can be estimated from below. Furthermore, equality holds in (5b), whenever

$$E[v(X_t, t)/X_0 = x, T > t][1 - \psi_t(x)] \to 0 \quad \text{for} \quad t \to \infty. \tag{6}$$

In Section 3 we shall make use of this.

Remarks

1) For example, $v(x, t) = \psi(x)$ and, for every $u > 0$, $v(x, t) = \psi_{u-t}(x)$ are functions with the desired property (if we set $\psi_t = 0$ for $t < 0$).
2) For a given function $v(x, t)$ it might be easier to verify that $\{v(X_t, t)\}_0^\infty$ is a martingale with respect to $\{X_t\}_0^\infty$. This is a sufficient condition since optional stopping does not affect the martingale property. (However, this condition is not necessary; for example $\{\psi(X_t)\}_0^\infty$ is not a martingale in general.)
3) The process $\{v(X_t, t)\}_0^\infty$ is a martingale with respect to $\{X_t\}_0^\infty$, if for all $0 \leq t < u < \infty$ and x

$$v(x, t) = E[v(X_u, u) / X_t = x]. \tag{7}$$

This implies that for all $t \geq 0$ and x

$$(\alpha_{x,t} + \frac{\partial}{\partial_t})\, v\,(x, t) = 0 \tag{8}$$

where the operator $\alpha_{x,t}$ is the generator,

$$\alpha_{x,t}\, f(x) = \lim_{h \to 0^+} \frac{E[f\,(X_{t+h}) / X_t = x)] - f(x)}{h} \tag{9}$$

Observe that equation (8) is *not* identical with the Fokker-Planck equation,

$$\left(\alpha_{x,t}-\frac{\partial}{\partial_t}\right)v(x,t)=0 \tag{10}$$

see p. 287 in [5].

2. Processes with Independent Increments

In order to apply the technique introduced in the preceding section we still need an appropriate function $v(x, t)$. In this section we solve this problem under the assumption that $\{X_t\}_0^\infty$ is a process with independent increments.
Let $Y_t = X_t - X_0$, and let us consider

$$v(x,t)=\frac{e^{-rx}}{E\left[e^{-rY_t}\right]} \tag{11}$$

for values of r for which the denominator exists. The following lemma has been used by *Meyer* (see [7], p. 180) in a different context.*

Lemma 1. $\{v(X_t, t)\}$ is a martingale with respect to $\{X_t\}_0^\infty$.
For completeness we repeat the short proof. Since

$$v(X_u,u)=v(X_t,t)\frac{e^{-r(X_u-X_t)}}{E\left[e^{-r(X_u-X_t)}\right]} \tag{12}$$

we recognize that condition (7) holds.
As a corollary, we conclude from (5a) that

$$\psi_t(x)\leq \min_r e^{-rx}\max_{0\leq s\leq t} E\left[e^{-rY_s}\right]. \tag{13a}$$

and from (5b) that

$$\psi(x)\leq \min_r e^{-rx}\max_{s\geq 0} E\left[e^{-rY_s}\right]. \tag{13b}$$

In the special case where there is a constant $R > 0$ with

$$E\left[e^{-RY_t}\right]=1 \quad \text{for all} \quad t\geq 0 \tag{14}$$

* The author is grateful to Professor Wendel for pointing out this lemma.

formula (13b) implies

$$\psi(x) \leq e^{-Rx} \tag{15}$$

which is the famous Lundberg inequality.

Examples.

1) In the classical case, Y_t is the difference between a deterministic component ct and a compound Poisson process (say with Poisson parameter α and *cdf* $F(x)$ of the jump amounts). We have

$$E\left[e^{-rY_t}\right] = \exp\{-crt + \alpha t[\varrho(r) - 1]\} \tag{16}$$

with $\varrho(r) = \int e^{rx} dF(x)$, and formula (11) reads

$$v(x, t) = \exp\{-rx + crt - \alpha t[\varrho(r) - 1]\}. \tag{17}$$

In the case where R, the positive solution of

$$cR - \alpha[\varrho(R) - 1] = 0 \tag{18}$$

exists, the inequality (13a) can be simplified. Since the maximum in (13a) is at least one, we need only consider values of $r \geq R$. But for these the maximum is assumed for $s = t$, and we obtain

$$\psi_t(x) \leq \min_{r \geq R} \exp\{-rx - crt + \alpha t[\varrho(r) - 1]\}. \tag{19}$$

If a lower bound (higher than one) is available for $E[e^{-rX_T}/T \leq t]$, this inequality may be improved by dividing the right side by this lower bound. For example, if $F(x) = 1 - e^{-x}$,

$$E\left[e^{-rX_T}/T \leq t\right] = \varrho(r) = \frac{1}{1-r} \tag{20}$$

and we get

$$\psi_t(x) \leq \min_{R \leq r < 1} (1 - r)\exp\{-rx - crt + \alpha t \frac{r}{1-r}\}. \tag{21}$$

As an illustration, let us consider the example discussed by Seal on p. 115 in [8]. Here $t = 100, x = 50, a = 1, c = 1.05$. The minimum in (21) is assumed for $r = .2$ and equals $(.8)\, e^{-6} = .002$. So we get $\psi_{100}(50) \leq .002$ which is a considerable improvement of the approximation $\psi_{100}(50) \simeq .015$ that was found in [8].

Example 2. Suppose that $\{Y_t\}_0^\infty$ is a diffusion process with constant parameters σ^2 and $\mu > 0$. Thus

$$E\left[e^{-rY_t}\right] = \exp\left\{-\mu r t + \frac{\sigma^2 r^2}{2} t\right\} \tag{22}$$

and $R = \frac{2\mu}{\sigma^2}$. Again we can restrict ourselves to values $r \geq R$ in formula (13a), in which case the maximum is assumed for $s = t$. We find that the minimum is assumed by

$$r_{\min} = \begin{cases} R & \text{if } t \geq \frac{x}{\mu} \\ \frac{x + \mu t}{\sigma^2 t} & \text{if } t < \frac{x}{\mu} \end{cases}. \tag{23}$$

Consequently, (13 a) reads now

$$\psi_t(x) \leq \begin{cases} \exp\left\{-\frac{2\mu x}{\sigma^2}\right\} & \text{if } t \geq \frac{x}{\mu} \\ \exp\left\{-\frac{(x+\mu t)^2}{2\sigma^2 t}\right\} & \text{if } t < \frac{x}{\mu} \end{cases} \tag{24}$$

In the case of a diffusion process, an explicit formula is available for $\psi_t(x)$:

$$\psi_t(x) = 1 - \Phi\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right) + \exp\left(-\frac{2\mu x}{\sigma^2}\right) \Phi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right) \tag{25}$$

where Φ denotes the standard normal *cdf* (see formula (72), p. 221, in [4], or p. 353 of [10]). This enables us to compare the upper bound in (24) with the exact values for $\psi_t(x)$. For $\mu = \sigma = x = 1$, and various values of t, we find:

(1) t	(2) upper bound	(3) exact value	$\frac{(3)}{(4)}$ in %
0	0	0	—
.09	.0014	.0003	21
.16	.015	.004	27
.25	.044	.015	34
.36	.077	.031	40
.49	.104	.048	46
.64	.122	.064	52
.81	.133	.079	59
1	.135	.090	67
2.25	.135	.123	91
4	.135	.133	99
∞	.135	.135	100

This suggests that the right side of (24), or more general of (13a), can be a rather crude estimate for $\psi_t(x)$.

3. Claim Amounts with Monotone Failure Rates

In this section we continue example 1 (compound Poisson process) of the preceding section. The existence of R is assumed, and let $v(x, t) = e^{-Rx}$. For given $K > 0$ we decompose the event $[T > t]$ according to whether $X_t > K$ or $0 \leq X_t \leq K$. Correspondingly we get the estimate

$$E\left[e^{-RX_t}/ T > t\right]\left[1-\psi_t(x)\right] \leq e^{-RK} + P[X_t \leq K]. \tag{26}$$

The last term vanishes for $t \to \infty$. Since K is arbitrary, the validity of condition (6) follows. Thus the equality holds in (5b), and we have

$$\psi(x) = \frac{e^{-Rx}}{E\left[e^{-RX_T}/T < \infty, X_0 = x\right]}. \tag{27}$$

This formula is also derived in Section 12.2 of [4].

Let us denote the denominator in (27) by D, and suppose $F(0) = 0$ (no negative "claims"). We shall estimate D under the assumption that $F(x)$ is IFR or DFR, i.e. has an increasing or decreasing failure rate (see [1] or [9]). If $F(x)$ is IFR (DFR), it follows that for every $z > 0$ and $y > 0$ with $F(y) < 1$

$$\frac{1-F(y+z)}{1-F(y)} \begin{matrix}(\geq)\\ \leq\end{matrix} 1-F(z). \tag{28}$$

For fixed $X_0 = x$, let us consider

$$P(y) = P[X_{T-0} \leq y / T < \infty]. \tag{29}$$

It follows that

$$P[-X_T \leq z / T < \infty] = \int_y \frac{F(y+z)-F(y)}{1-F(y)} \, dP(y) \tag{30}$$

where the integral has to be extended over values $y \geq 0$ for which $F(y) < 1$. Further, we recall that for any *cdf* $H(z)$ with $H(0) = 0$

$$\int_0^\infty e^{Rz} \, dH(z) = 1 + R \int_0^\infty e^{Rz} \, [1 - H(z)] \, dz \tag{31}$$

which can be verified by partial integration. If we apply this formula back and forth, we obtain from (28), (30), and finally from the definition of R, (18), the estimate

$$\begin{aligned} D &= \int_y \left(\int_0^\infty e^{Rz} \frac{F(y+dz)}{1-F(y)} \right) dP(y) \\ &\begin{matrix}(\geq)\\ \leq\end{matrix} \int_y \left(\int_0^\infty e^{Rz} \, dF(z) \right) dP(y) \\ &= \varrho(R) = 1 + R\frac{c}{a}. \end{aligned} \tag{32}$$

Using this in (27), we obtain the following result.

Theorem. If $F(x)$ is IFR (DFR), then

$$\psi(x) \begin{matrix}(\leq)\\ \geq\end{matrix} \frac{1}{1+R\frac{c}{a}} e^{-Rx}. \tag{33}$$

Remarks

1) If $F(x)$ is DFR, this improves the classical inequality (15). If, on the other hand, $F(x)$ is IFR, (15) and (33) may be combined to evaluate $\psi(x)$. In particular, if R is small (this corresponds to small security loadings) one gets an excellent estimate. Thirdly, in the case of exponential claim amounts, (33) reduces to the well known explicit expression for $\psi(x)$.
2) If we compare (33), for $x = 0$, with the identity

$$\psi(0) = \frac{\alpha\mu}{c}, \quad \text{where} \quad \mu = \int_0^\infty x \, dF(x) \tag{34}$$

(see p. 150 in [3], for example), we see that

$$R \overset{(\leq)}{\geq} \frac{1}{\mu} - \frac{\alpha}{c} \tag{35}$$

whenever $F(x)$ is IFR (DFR). If $F(x)$ is IFR, this can be used in (15) to obtain the nonparametic estimate

$$\psi(x) \leq \exp\left\{-\left(\frac{1}{\mu} - \frac{\alpha}{c}\right) x\right\} \tag{36}$$

for the probability of ruin.

4. Processes Modified by a Reflecting Barrier

To fix ideas, suppose that $\{X_t\}_0^\infty$ is the process introduced in example 1 of Section 2, with the restriction $F(0) = 0$ (the validity of the lemma below hinges essentially on the semi-continuity of the sample paths).
Let $b(t)$, $t \geq 0$, be a continuously differentiable function, and let $\{Z_t\}$ denote the process that results from $\{X_t\}$ if $\{b(t)\}$ is added as reflecting barrier. So unless a jump downwards (a claim) takes place, Z_t grows at a rate c if $Z_t < b(t)$, and at a rate $\min(b'(t), c)$ if $Z_t = b(t)$. An explanation for such a barrier would be that the company pays out premium refunds whenever the surplus reaches the barrier.
The following lemma may lead us to suitable functions $v(x, t)$.

Lemma 2. Suppose that $v(x, t)$ is a function such that $\{v(X_t, t)\}_0^\infty$ is a martingale with respect to $\{X_t\}_0^\infty$. Then $\{v(Z_t, t)\}_0^\infty$ is a martingale with respect to $\{Z_t\}_0^\infty$, if and only if

$$\left.\frac{\partial v(x, t)}{\partial x}\right|_{x=b(t)} = 0 \quad \text{whenever} \quad b'(t) < c. \tag{37}$$

The proof follows from

$$E\left[v(Z_u, u) \mid Z_t = x\right] = v(x, t) - \int_t^u \left.\frac{\partial v(y, s)}{\partial y}\right|_{y=b(s)} [c - b'(s)]^+ \, q(s)ds \tag{38}$$

valid for $t < u$, with $q(s) = P[Z_s = b(s) \mid Z_t = x]$.

To illustrate the usefulness of the lemma, let us consider

$$v(x, t) = \frac{e^{-rx}}{E\left[e^{-rY_t}\right]} + \frac{r}{s}\,\frac{e^{sx}\,e^{-(r+s)b}}{E\left[e^{sY_t}\right]} \tag{39}$$

for positive constants b, r, s with $\varrho(r) < \infty$. $\{v(X_t, t)\}$ is a martingale, because $v(x, t)$ is a linear combination of functions of the form (11). If we solve equation (37) for $b(t)$, we obtain a linear barrier, namely

$$b(t) = b + a(r, s)t, \quad \text{where}$$

$$a(r, s) = c - \alpha\,\frac{\varrho(r) - \varrho(-s)}{r+s}. \tag{40}$$

Note that $a(r, \infty) = c$, which is the classical case of no barrier. For a given linear barrier, $b(t) = b + at$, we are now able to obtain estimates for the probability of ruin. Assuming the existence of R, we have for $r \geq R, y < 0$, and $0 \leq u \leq t$

$$\frac{1}{v(y, u)} \leq E\left[e^{-rY_t}\right] = e^{-crt + \alpha t\,[\varrho(r) - 1]}. \tag{41}$$

Using this in (5a) we obtain

$$\psi_t(x) \leq \min v(x, 0)\ \exp\{-crt + \alpha t[\varrho(r) - 1]\} \tag{42a}$$

valid for $0 \leq x \leq b$, where the minimum is to be taken for $r \geq R, s$ with $a(r, s) = a$. For $t \to \infty$, we get

$$\psi(x) \leq v(x, 0) = e^{-Rx}\left[1 + \frac{R}{S}\, e^{(R+S)(b-x)}\right] \tag{42 b}$$

where S is the solution of $a(R, S) = a$. These inequalities generalize formulas (15) and (19). Inequality (42 b) appears in [6] as Theorem 2, where it was derived by different methods.

Application: The Rationale of Participating Policies

Should a company offer insurance policies on a participating or on a non-participating basis? From a pure safety point of view, this old question should be answered in favor of the former. Formulas (42) make it possible to demonstrate this numerically.

Suppose that the risk to be insured consists of a compound Poisson process (Poisson parameter $\alpha = 1$, and exponential claim amounts, $F(x) = 1 - e^{-x}$). Let us assume that the company has an initial surplus of SFr. 50, and that it plans to achieve a profit of 5% in the long run.

a) *Non-participating policies*. The proper profit margin is obtained by $c = 1.05$. The resulting probability of ultimate ruin is

$$\psi(50) = \frac{1}{c}\, e^{-50R} = .088. \tag{43}$$

(This is the example discussed on p. 115 of [8].)

b) *Participating policies*. Alternatively, the company might charge a higher premium density, say $c = 1.50$, and return 45% of the net premiums (in the long run) to the policy holders. A rational way to do this would be to introduce the dividend barrier $b(t) = 50 + .05\, t$, and to pay out premium refunds (at a rate $c - a = 1.45$) whenever the surplus coincides with the premium barrier. Here we have $R = (c - 1)/c = 1/3$, and S, the solution of $a(R, S) = .05$, equals .034. From (42 b) we obtain the estimate

$$\psi(50) \leq e^{-50/3}\left[1 + \frac{1}{.102}\right] = .00000062. \tag{44}$$

A comparison of (43) and (44) shows impressively how participating policies improve the safety of a company (at least as long as the premium refunds are allowed to depend properly on the claims experience).

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Zusammenfassung

Im Falle, wo der Einkommensprozess Markov'sch ist, gibt es eine Martingal-Technik, um Resultate über die Ruinwahrscheinlichkeiten herzuleiten. Anwendungen umfassen Prozesse mit unabhängigen Zuwächsen, insbesondere den zusammengesetzten Poissonprozess (wo weiterreichende Resultate möglich sind, wenn die Verteilungsfunktion der Schadenhöhen eine monotone Ausfallsrate besitzt), sowie die Auswirkung einer Dividendenbarriere auf die Ruinwahrscheinlichkeit.

Summary

Under the assumption that the income process is Markov, it is shown how a martingale technique can be used to derive results concerning the probabilities of ruin. Applications include processes with independent increments, in particular the compound Poisson process (where stronger results are possible whenever the distribution function of the claim amounts has a monotone failure rate), and the effect of imposing a dividend barrier on the probability of ruin.

Résumé

Sous l'hypothèse que le processus de revenu est markovien, on montre comment utiliser la technique des martingales pour dériver des résultats concernant les probabilités de ruine. On applique ensuite ceci à des processus à accroissements indépendants, en particulier au processus de Poisson composé (des résultats plus forts sont possibles dans le cas où les montants des sinistres ont une distribution du type «monotone failure rate»). Une autre application concerne l'effet d'une barrière de dividendes sur la probabilité de ruine.

Riassunto

Sotto l'ipotesi che il processo di reddito sia markoviano si mostra come si può utilizzare la tecnica dei martingali per derivare alcuni risultati concernenti la probabilità di rovina. I risultati vengono poi applicati a processi a incrementi indipendenti, in particolare al processo di Poisson composto (risultati piu forti sono possibili nel caso che gli importi dei danni hanno una distribuzione del tipo «monotone failure rate»). Un altra applicazione tratta l'effetto d'una barriera di dividendi sulla probabilità di rovina.