B. Wissenschaftliche Mitteilungen

On Iterative Premium Calculation Principles

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1. Some premium calculation principles

In the following a risk will be any nonnegative random variable of bounded range. A premium calculation principle, call it H, is a rule that assigns a nonnegative number P (the premium) to any given risk S. Symbolically, we write

$$P = H(S). \tag{1}$$

Most of the following examples can be found in [3], p. 85–87.

a) *The Net Premium Principle*

$$P = E[S]. \tag{2}$$

b) *The Standard Deviation Principle*

$$P = E[S] + \alpha \sqrt{\operatorname{Var}[S]}, \; \alpha > 0. \tag{3}$$

c) *The Variance Principle*

$$P = E[S] + \beta \quad \operatorname{Var}[S], \; \beta > 0. \tag{4}$$

d) *The Principle of Zero Utility*. Let $u(x), -\infty < x < +\infty$, be a twice differentiable function with $u'(x) > 0$. Then P is defined by the equation

$$E[u(P-S)] = u(0). \tag{5}$$

Since H is invariant under linear transformations of $u(x)$, we can assume that

$$u(0) = 0, u'(0) = 1. \tag{6}$$

In the case of an exponential utility function,

$$u(x) = \frac{1}{a}(1-e^{-ax}), \; a \neq 0, \tag{7}$$

we can solve (5) explicitly and find that

$$P = \frac{1}{a} \ln E[e^{aS}]. \tag{8}$$

e) *The Mean Value Principle* (see [5], Chapter 3). Let $v(x)$, $0 \leqslant x < \infty$, be a continuous, strictly increasing function. Then we define

$$P = v^{-1} \circ E[v(S)]. \tag{9}$$

Here v^{-1} is the inverse function of v, and we make use of the convenient notation $f \circ \cdot := f(\cdot)$.

f) *The Maximal Loss Principle*. For p $\geqslant 0$, $q = 1 - p \geqslant 0$, we set

$$P = p\, E[S] + q \operatorname{Max}[S]. \tag{10}$$

Here Max $[S]$ denotes the right end point of the range of S.

Remarks. 1) In each of these examples, the definition of H can be extended in a natural way to cover also random variables of unbounded range. In that case, the premium is infinite for some risks, which are therefore uninsurable.
2) If it is desired that $H(S) \geqslant E[S]$ for all S, one wants to assume concavity of the function $u(x)$ in example d) and convexity of the function $v(x)$ in example e) above (Jensen's inequality).

2. Iterative Premium Calculation Principles

Consider two risks X, Y with known joint distribution. It is well known that

$$E[X] = E\big[E[X \mid Y]\big] \tag{11}$$

which is occasionally called the iterative rule for expected values. This suggests the following definition (see [3], p. 91).
A premium calculation principle is said to be iterative, if $H(X \mid Y)$ is a risk (boundedness and measurability!) and

$$H(X) = H\big(H(X \mid Y)\big) \tag{12}$$

for any pair of risks X, Y.

Thus formula (11) tells us that the Net Premium Principle is iterative. While the principles in examples b) and c) are not iterative, the Mean Value Principle is iterative. The verification of the latter is based on formula (11) and goes as follows:

$$
\begin{aligned}
H(X) &= v^{-1} \circ E[v(X)] \\
&= v^{-1} \circ E\big[E[v(X) \mid Y]\big] \\
&= v^{-1} \circ E\big[v \circ v^{-1} \circ E[v(X) \mid Y]\big] \\
&= v^{-1} \circ E[v \circ H(X \mid Y)] \\
&= H\big(H(X \mid Y)\big) .
\end{aligned}
\tag{13}
$$

Since formula (8) is of the form (9), we see that the Principle of Zero Utility is iterative if the utility function is exponential. Theorem 1 will show us that the converse is true in essence. Finally, the reader may wish to verify that the Maximal Loss Principle is not iterative unless $p = 1$ or $q = 1$.

3. Characterization of exponential utility

Theorem 1. The Principle of Zero Utility is iterative, if and only if the underlying utility function is linear or exponential.
Proof. It remains to be shown that iterativity implies $u(x) = x$ or that formula (7) holds.

For $z > 0$ and $0 \leqslant q \leqslant 1$, let S_{zq} denote the Bernoulli risk with face amount z,

$$
P[S_{zq} = 0] = 1 - q, \qquad P(S_{zq} = z) = q \tag{14}
$$

and let $P_z(q)$ be the premium for S_{zq}. Thus equation (5) reads

$$
(1-q)\, u\big(P_z(q)\big) + q\, u\big(P_z(q) - z\big) = 0 . \tag{15}
$$

Because of the standardizing conditions (6), linearizing leads to

$$P_z'(0) = -u(-z). \tag{16}$$

Now consider any random variable Y with $cdf\,F(x)$, such that $0 \leqslant Y \leqslant 1$. Then we construct a risk X indirectly: For given $Y = q$, let X be S_{zq}. Therefore $H(X \mid Y = q) = P_z(q)$. But of course X is S_{zm} where $m = E[Y]$, and hence $H(X) = P_z(m)$. Thus iterativity means in this case that

$$\int_0^1 u\Big(P_z(m) - P_z(q)\Big)\, dF(q) = 0. \tag{17}$$

Here $F(x)$ has to satisfy

$$\int_0^1 dF(x) = 1, \quad \int_0^1 x\, dF(x) = m \tag{18}$$

but is otherwise arbitrary. We conclude that the integrand is a linear function in q (proof: Assume that $F(x)$ has a density and write the density and the integrand as a series of orthogonal polynomials),

$$u\Big(P_z(m) - P_z(q)\Big) = a(z, m) + b(z, m)q. \tag{19}$$

Differentiating twice with respect to q, we get

$$P_z'(q)^2\, u''\Big(P_z(m) - P_z(q)\Big) - P_z''(q)\, u'\Big(P_z(m) - P_z(q)\Big) = 0 \tag{20}$$

valid for all m, q between 0 and 1. By setting $m = q$, we obtain

$$u''(0)\, P_z'(q)^2 - P_z''(q) = 0 \tag{21}$$

which is a differential equation of Riccati's type. Because of the boundary conditions

$$P_z(0) = 0, \quad P_z(1) = z \tag{22}$$

the solution is

$$P_z(q) = \begin{cases} qz & \text{if } a = 0 \\ \frac{1}{a}\, ln(1 - q + q\, e^{az}) & \text{if } a \neq 0 \end{cases} \tag{23}$$

where $a = -u''(0)$. Therefore

$$P_z'(0) = \begin{cases} z & \text{if } a = 0 \\ \frac{1}{a}(e^{az} - 1) & \text{if } a \neq 0. \end{cases} \tag{24}$$

Substituting this in equation (16), we get

$$u(x) = \begin{cases} x & \text{if } a = 0 \\ \frac{1}{a}(1 - e^{-ax}) & \text{if } a \neq 0 \end{cases} \tag{25}$$

valid for $x < 0$. Using this and formula (23) in equation (15), we see that formula (25) is also valid for $x > 0$. Q.E.D.

Remarks. 1) The same class of utility functions has recently been characterized by means of the additivity property [4].

2) From Theorem 1 and the iterativity of the Mean Value Principle we get immediately a result that corresponds to a theorem in utility theory (see, for example, [7] p. 85):

Corollary. If a premium calculation principle is at the same time a Principle of Zero Utility and a Mean Value Principle, the underlying utility function is linear or exponential.

4. Characterization of the Mean Value Principle

In this section we consider premium calculation principles that satisfy the following regularity condition:

Condition C. For fixed $a > 0$, $P_a(q)$ is a continous, strictly increasing function ($0 \leqslant q \leqslant 1$) with $P_a(0) = 0$, $P_a(1) = a$.
This corresponds to the "continuity axiom" in utility theory, while iterativity corresponds to the "substitution axiom" (see [7], p. 73–77, also [8], and [2]). The following result parallels the existence theorem in utility theory.

Theorem 2. A premium calculation principle satisfying Condition C is iterative if and only if it is the Mean Value Principle.
We conduct the proof of the nontrivial part in 4 steps.
(i) Let $a > 1$. We define $v(x)$ for $0 \leqslant x \leqslant a$ as follows: If Q_a denotes the solution of $P_a(Q_a) = 1$, $v(x)$ is the function that satisfies

$$v\left(P_a(q)\right) = \frac{q}{Q_a}, \; 0 \leqslant q \leqslant 1. \tag{25}$$

(ii) If S is a risk with $0 \leqslant S \leqslant a$, then formula (9) holds: Let us construct a risk X which, for given S, is S_{aq} where $q = Q_a\, v(S)$. From the definition of $v(x)$ it follows that $H(X \mid S) = S$, and therefore

$$H\left(H(X \mid S)\right) = H(S). \tag{26}$$

On the other hand, X is of course S_{am}, where $m = Q_a E[v(S)]$. Thus

$$H(X) = P_a(m) = v^{-1}\left(\frac{m}{Q_a}\right) = v^{-1} \circ E[v(S)]. \tag{27}$$

Because of iterativity, the expressions in formulas (26) and (27) are equivalent which proves the validity of formula (9).
(iii) Now we show that $v\left(P_1(q)\right) = q$ for $0 \leqslant q \leqslant 1$: Consider a risk X, which for given S_{1q} is S_{ar} where $r = Q_a\, S_{1q}$. Thus $H(X \mid S_{1q}) = 0$ if $S_{1q} = 0$ and $H(X \mid S_{1q}) = 1$ if $S_{1q} = 1$. Therefore

$$H\left(H(X \mid S_{1q})\right) = P_1(q). \tag{28}$$

But X is S_{am} where $m = qQ_a$. Thus

$$H(X) = P_a(m) = v^{-1}\left(\frac{m}{Q_a}\right) = v^{-1}(q). \tag{29}$$

Because of iterativity, the expressions (28) and (29) are equivalent.
(iv) It remains to be shown that the definition of $v(x)$ can be extended consistently for all x. For $b > a$, we define a function $w(x)$, $0 \leqslant x \leqslant b$, by the equation

$$w\left(P_b(q)\right) = \frac{q}{Q_b} \tag{25'}$$

and want to show that $w(x) = v(x)$ for $0 \leqslant x \leqslant a$. For any risk S, with $0 \leqslant S \leqslant a$, we have

$$H(S) = v^{-1} \circ E[v(S)] = w^{-1} \circ E[w(S)]. \tag{30}$$

In the case where $S = S_{xq}$, $0 \leqslant x \leqslant a$, this reads

$$v^{-1}\left(q\,v(x)\right) = w^{-1}\left(q\,w(x)\right). \tag{31}$$

From (iii) we know that $v^{-1}(y) = w^{-1}(y)$ for $0 \leqslant y \leqslant 1$. If q is close enough to 0, this is applicable in (31), and hence $v(x) = w(x)$. Q.E.D.
Remark. Example f) with $q = 1$, $H(S) = \text{Max}\,(S)$, shows that Condition C cannot be dropped in Theorem 2.

5. Where is iterativity useful?

Iterative premium calculation principles are handy, whenever the actuary is faced with conditional random variables. This occurs in credibility theory as well as in the context of random numbers of claims. For each case we give a simple illustration assuming that formula (8) is used for calculating the premiums.

1) Suppose that S is normal with known variance σ^2 but unknown mean θ. Thus plays the role of a risk parameter. Assuming that also follwos a normal distribution say with mean μ and variance v^2, we obtain (in the terminology of credibility theory [3]):

a) Risk premium

$$P(\theta) = H(S \mid \theta) = \theta + \frac{a}{2}\sigma^2. \tag{32}$$

b) Iterative premium

$$P^* = H\left(P(\theta)\right) = \mu + \frac{a}{2}(\sigma^2 + v^2). \tag{33}$$

c) Collective premium

$$P = H(S).$$

But, since the underlying premium calculation principle is iterative, we must have $P = P^*$, and no further calculations are necessary!

2) Suppose that S is compound Poisson:

$$S = X_1 + X_2 + \cdots + X_Y \tag{34}$$

where $P(Y = k) = e^{-\lambda} \frac{\lambda^k}{k!} (k = 0, 1, 2, \cdots)$. Assuming that the individual claim amounts are normally distributed with mean μ and variance σ^2, we get

$$H(S \mid Y) = Y\left(\mu + \frac{a}{2}\sigma^2\right) \tag{35}$$

$$H\left(H(S \mid Y)\right) = \frac{\lambda}{a}\left[\exp\left(a\mu + \frac{a^2}{2}\sigma^2\right) - 1\right]. \tag{36}$$

Because of iterativity, this is $H(S)$, the premium for S. Similar calculations can be done for stop loss coverage (see [9]), where

$$L = \text{Max}\,(O, S - m) \tag{37}$$

is to be insured (retention limit m), and for largest claim cover (see [1], [6]) where

$$M = \text{Max}\,(X_1, X_2, \cdots, X_Y) \tag{38}$$

is to be insured.

6. Connection with Straub's concept of iterativity

In [9] *Straub* discusses somewhat different concepts of premium calculation principles and of iterativity. There, a premium calculation principle, call it $\mathfrak{H}$, is a rule such that for any risk X and for any given random variables $Y_1, \cdots, Y_n$

$$\mathfrak{H}\,[X;\, Y_1, \cdots, Y_n] \tag{39}$$

is a function of $Y_1, \cdots, Y_n$. Such a premium calculation principle is said to be iterative, if $\mathfrak{H}\,[X;\, Y_1, Y_2]$ is a risk and

$$\mathfrak{H}\left[\mathfrak{H}\,[X;\, Y_1, Y_2]; Y_2\right] = \mathfrak{H}\,[X;\, Y_2] \tag{40}$$

for any risk X and for any pair of random variables Y_1, Y_2.
In this sense the principle $\mathfrak{H}$ defined by the formula

$$\mathfrak{H}\,[X;\, Y_1, \cdots, Y_n] = E\,[X \mid Y_1, \cdots, Y_n] + \beta\, E\left[\text{Var}\,[X \mid Y_1, \cdots, Y_n]\right] \tag{41}$$

$(\beta > 0)$ is iterative (see [9]).

A principle H (in the sense of the preceding sections) induces a principle $\mathfrak{H}$ (in the above sense) in a natural way: Just set

$$\mathfrak{H}[X; Y_1, \cdots, Y_n] = H(X \mid Y_1, \cdots, Y_n). \tag{42}$$

Under a mild additional condition, iterativity is thereby preserved:
Theorem 3. Suppose that H is iterative in the sense of formula (12), and that $H(X \mid Y_1, Y_2)$ is a risk for any risk X and for any pair of random variables Y_1, Y_2. Then the induced principle $\mathfrak{H}$ is iterative in the sense of formula (40).

$$\begin{aligned} \text{Proof: } & \mathfrak{H}\left[\mathfrak{H}[X; Y_1, Y_2]; Y_2\right] && (43)\\ &= H\left(H(X \mid Y_1, Y_2) \mid Y_2\right) \\ &= H(X \mid Y_2) \\ &= \mathfrak{H}[X; Y_2]. && \text{Q.E.D.} \end{aligned}$$

In the opposite direction, a principle $\mathfrak{H}$ can be used to generate a principle H: set

$$H(X) = \mathfrak{H}[X; 1]. \tag{44}$$

However now, iterativity in the sense of (40) does not imply iterativity in the sense of (12). For example, the principle $\mathfrak{H}$ defined by formula (41) is iterative. It generates the Variance Principle, formula (4), which is not iterative in the sense of formula (12).

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Zusammenfassung

Es wird versucht, die Frage zu beantworten, die kürzlich von E. Straub (siehe [9], Seite 153) gestellt worden ist. Es wird gezeigt, dass das Nullnutzenprinzip iterativ ist genau dann, wenn die zugrunde liegende Nutzenfunktion entweder linear oder exponentiell ist (Satz 1). Überdies ist ein Prinzip, das einer gewissen Regularitätsbedingung genügt, dann und nur dann iterativ, falls es ein Mittelwertprinzip ist (Satz 2).

Résumé

L'auteur essaie de répondre à la question récemment soulevée par E. Straub (cf. [9], p. 153). Il nous montre que le principe d'utilité nulle est suffisamment itératif quand la fonction de l'utilité de bases est linéaire ou exponentielle (1er théorème). D'autre part le principe qui répond à certaines conditions de régularité est itératif et ne l'est seulement que si c'est un principe de valeur moyenne (2e théorème).

Riassunto

In questo studio si cerca di rispondere a una domanda posta recentemente da E. Straub (cfr. [9], pag. 153). Si dimostra che il «principio di utilità nulla» è iterativo quando e solo quando la funzione di utilità presa come base è lineare o esponenziale (Teorema 1). Inoltre un principio che soddisfa a una certa condizione di regolarità è iterativo quando e solo quando è un «principio di valor medio» (Teorema 2).

Summary

This is an attempt to answer a question that has been recently raised by E. Straub (see [9], p. 153). It is shown that the Principle of Zero Utility is iterative if and only if the underlying utility function is linear or exponential (Theorem 1). Furthermore, the class of all iterative premium calculation principles satisfying a certain regularity condition is identical with the class of the Mean Value Principles (Theorem 2).