Claims Mortality and the Existence of Moments of the Claim Size Distribution

By H. Bühlmann and G. C. Taylor

Abstract

It has been conjectured by Benktander that the condition

$$\lim_{x \to \infty} x\,\mu(x) = \infty;$$

where $\mu(x)$ is the mortality of a claim size distribution, is sufficient to ensure the existence of all moments of this distribution.
This suggestions is examined, and it is shown that a *necessary and sufficient condition for existence of all moments* is:

$$\frac{\int_0^x \mu(t)\,dt}{\log x} \to \infty \quad \text{as} \quad x \to \infty.$$

This result is then used to examine the effect of the behaviour of $x\,\mu(x)$ for large x on the existence of moments. In particular Benktander's condition is shown to be sufficient but not necessary.

1. Introduction and Notation

As the present paper deals with the problem of existence of moments of a distribution (of a random variable of one real dimension), only distributions of infinite range need to be considered. Moreover, distributions whose range is infinite in both directions can be treated by decomposition. Thus, we need only here consider distributions on $(0, \infty)$. For such the following definition makes sense:
Consider a claim size distribution with p.d.f. and d.f. $f(\cdot)$ and $F(\cdot)$ respectively. Define

$$H(x) = 1 - F(x)$$

and

$$\mu(x) = f(x)/H(x).$$

It follows easily from this definition that:

$$\mu(x) = -\frac{dH}{dx}/H(x),$$

and so

$$H(x) = \exp\left[-\int_0^x \mu(t)\,dt\right]. \tag{1}$$

The function $\mu(x)$, well known as intensity of mortality in life assurance, is here called *claims mortality at size x*, as introduced by Benktander and Segerdahl (1960) and later studied by Benktander (1963, 1970). In the 1970 paper, Benktander deals with two particular classes of distribution and shows (p. 274) that the condition:

$$\lim_{x\to\infty} x\mu(x) = \infty, \tag{2}$$

is sufficient to ensure the existence of all moments. He has conjectured privately that this result holds for more general classes of distribution.

2. The Pareto Distribution

We shall find it helpful to consider here some properties of the Pareto distribution. The density function of such a distribution is

$$f(x) = \alpha c^{\alpha} x^{-(\alpha+1)}, \quad x \geq c.$$

This leads to

$$H(x) = \left(\frac{x}{c}\right)^{-\alpha}, \quad x \geq c;$$

$$\mu(x) = \alpha/x,$$

whence

$$\int_0^x \mu(t)dt = \int_c^x \alpha/t\,dt = \alpha\log\left(\frac{x}{c}\right). \tag{3}$$

Another fact that we know concerning the Pareto distribution is that all moments up to and including that of order [α] exists, where [·] denotes "greatest integer strictly less than". (Note: not as usual less than or equal!)

3. Statement and Proof of the Result

Lemma. Let F, G be two d.f.'s (with infinite range) and let μ_F, μ_G be the associated claims mortalitites. Suppose that

$$(a) \quad \int_0^\infty x^n \, dF(x) < \infty;$$

and there exists $x_0 \geq 0$ and a constant C, such that:

$$(b) \quad C + \int_{x_0}^{x} \mu_G(t) dt \geq \int_{x_0}^{x} \mu_F(t) dt \quad \text{for all} \quad x > x_0.$$

Then
$$\int_0^\infty x^n \, dG(x) < \infty.$$

Proof. By integration by parts

$$\int_0^\infty x^n \, dF(x) = -x^n [1 - F(x)] \Big|_{x=0}^{\infty} + n \int_0^\infty x^{n-1} [1 - F(x)] \, dx, \tag{4}$$

provided that all quantities on the right exist. Now, by hypothesis (a) the integral on the left exists, implying that

$$\int_N^\infty x^n \, dF(x) \to 0 \quad \text{as} \quad N \to \infty.$$

Thus,
$$\lim_{N \to \infty} N^n [1 - F(N)] \leq \lim_{N \to \infty} \int_N^\infty x^n \, dF(x) = 0$$

and it follows
$$\int_0^\infty x^n \, dF(x) = n \int_0^\infty x^{n-1} [1 - F(x)] \, dx$$

$$= n \int_0^\infty x^{n-1} \exp\Big[- \int_0^x \mu_F(t) \, dt \Big] dx$$

$$= n \Big[\int_0^{x_0} + \int_{x_0}^\infty \Big] x^{n-1} \exp\Big[- \int_0^x \mu_F(t) \, dt \Big] dx \tag{5}$$

$$\leq x_0^n + n \exp\Big[- \int_0^{x_0} \mu_F(t) \, dt \Big] \cdot \int_{x_0}^\infty x^{n-1} \exp\Big[- \int_{x_0}^x \mu_F(t) \, dt \Big] dx.$$

Without the first term x_0^n, the right hand side is also a lower bound for $\int_0^\infty x^n dF(x)$. It is then easily seen that

$$\int_0^\infty x^n dF(x) < \infty \Leftrightarrow \int_{x_0}^\infty x^{n-1} \exp\left[-\int_{x_0}^x \mu_F(t)dt\right] dx < \infty. \tag{6}$$

The lemma follows directly from (6) and (5) consecutively applied to F and G.
Corollary. Suppose that

$$(a) \quad \int_0^\infty x^n \, dF(x) < \infty;$$

and there exists $x_0 > 0$ such that:

$$(b) \quad \mu_G(x) \geq \mu_F(x) \quad \text{for all} \quad x \geq x_0.$$

Then

$$\int_0^\infty x^n \, dG(x) < \infty.$$

Theorem. All moments associated with a d.f. G exist if and only if

$$\frac{\int_0^x \mu_G(t)\, dt}{\log x} \to \infty \quad \text{as} \quad x \to \infty.$$

Proof. If. Let n be an arbitrary positive integer. By hypothesis, there exists $x_0 > 0$ such that

$$\int_0^x \mu_G(t)\, dt \geq (n+1) \log x \quad \text{for all} \quad x \geq x_0. \tag{7}$$

Hence

$$\int_0^{x_0} \mu_G(t)\, dt + \int_{x_0}^x \mu_G(t)\, dt \geq (n+1) \int_{x_0}^x \frac{1}{t} dt \quad \text{for all } x \geq x_0 \geq 1. \tag{8}$$

As the right hand side is the integral of the Pareto intensity with $\alpha = n+1$, the moments up to and including order n exist by virtue of the lemma.

Only if. Suppose that

$$\frac{\int_0^x \mu_G(t)\,dt}{\log x} \nrightarrow \infty. \qquad (9)$$

Then there exists K and an unbounded increasing sequence y_1, y_2, etc. such that

$$\int_0^{y_i} \mu_G(t)\,dt < K \log y_i, \quad i = 1, 2, \text{etc.} \qquad (10)$$

Therefore,

$$y_i^n[1 - G(y_i)] = y_i^n \exp\left[-\int_0^{y_i} \mu_G(t)\,dt\right]$$

$$> y_i^n\, y_i^{-K} \quad \text{(by 10).}$$

Thus for $n \geq K$, $x^n\,[1 - G(x)]$ does not converge to zero for $x \to \infty$.
Since existence of the n-th moment implies $x^n\,[1 - G(\lambda)] \to 0$ as $x \to \infty$ not all moments do exist.

4. The conjecture of Benktander

We are now in a position to return to Benktander's conjecture mentioned in Section 1. Let us examine the effect of the limiting behaviour of $x\mu(x)$ on the existence of the moments of the distribution. We distinguish three cases:

Case I $\quad \lim_{x \to \infty} x\mu(x) = \infty.$

Case II $\quad \lim_{x \to \infty} x\mu(x)$ does not exist.

Case III $\quad \lim_{x \to \infty} x\mu(x) = a < \infty.$

In Case I, for arbitrarily large K, we can find x_0 such that

$$\mu(x) > K/x \quad \text{for} \quad x > x_0.$$

Existence of moments follow from the corollary with $\mu_F(x) = \dfrac{K}{x}$.

In Case III, we can use precisely similar methods to show that not all moments exist.
Case II is inconclusive as far as the existence of moments is concerned. This is easily seen as follows.
Consider a mortality $\mu_{F_1}(x)$ satisfying Case I. Now construct a new mortality $\mu_{G_1}(x)$, defined by:

$$\mu_{G_1}(x) = \begin{cases} \mu_{F_1}(x), & x \text{ non-integral;} \\ 0, & x \text{ integral.} \end{cases}$$

Then $\mu_{G_1}(x)$ comes under Case II, but $F_1(x) \equiv G_1(x)$ so that all moments of G_1 exist.
Now consider $\mu_{F_2}(x)$ satisfying Case III, and construct a new mortality $\mu_{G_2}(x)$ from $\mu_{F_2}(x)$ in the same way as $\mu_{G_1}(x)$ was constructed from $\mu_{F_1}(x)$. Then $\mu_{G_2}(x)$ comes under Case II, but $F_2(x) = G_2(x)$ so that not all moments of G_2 exist.
We thus have constructed two mortality functions which are included under Case II, one of which has all moments existing and the other of which does not.

We can conclude as follows:

Case I All moments exist.

Case II Inconclusive. Perhaps all moments exist, perhaps some, perhaps none.

Case III Moments higher than some finite order do not exist.

In particular, we note from Cases I and II that condition (2), suggested by Benktander, is sufficient but not necessary for the existence of all moments.

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Prof. Dr. Hans Bühlmann
Lehrstuhl für Mathematik
an der ETH
Rämistrasse 101
8006 Zürich

Mr. Greg Taylor
School of Economic & Financial
Studies
Macquarie University
North Ryde
New South Wales 2113
Australia

Zusammenfassung

Die Autoren beweisen einen interessanten Zusammenhang zwischen dem asymptotischen Verhalten der Schadensterblichkeit und der Existenz aller Momente der Schadenverteilung.

Résumé

Les auteurs démontrent une relation intéressante entre la conduite asymptotique de la mortalité des sinistres et l'existence des moments de la loi de répartition des sinistres.

Riassunto

Gli autori dimostrano una relazione molto interessante fra il comportamento asintotico della mortalità dei sinistri e l'esistenza dei momenti della distribuzione delle probabilità.

Summary

The authors prove an interesting connection between the asymptotic behaviour of the claims mortality and the existence of all moments of the claims distribution.