An analytical approach to the generalized Poisson process in case of claim distributions with infinite skewness

An Analytical Approach to the Generalized Poisson Process in Case of Claim Distributions With Infinite Skewness

By M.J. Goovaerts and L. D'Hooge and P. Van Goethem

1. Introduction

The concept that a claim on an insurance company consists of two independent events, the occurrence of the claim and its amount, is a well-known fact. Let $p_n(t)$ denote the probability of n claims occurring in a given portfolio of contracts within a specified period of length t. If Y, the size of the individual claim, is a positive random variable that is independent of the random variable number-of-claims and its distribution function $F(y)$ does not involve t, then,

$$p(X \leq x) \equiv F(x,t) = \sum_{n=0}^{\infty} p_n(t)\, F^{n*}(x). \tag{1.1}$$

Seal [1] gives a review on some practical applications of (1.1). A distribution that has been used successfully on fire insurance [2] and automobile claims [3] is the Pareto, namely

$$1-F(y) = \left(\frac{y}{y_0}\right)^{-\alpha}, y_0 < y < \infty. \tag{1.2}$$

An awkward feature of this distribution is that the j^{th} moment about the origin becomes infinite when $j > \alpha$

$$E(Y^j) = y_0^j\left(\frac{\alpha}{\alpha-j}\right), j < \alpha. \tag{1.3}$$

The aim of the present paper consists in treating the case where the moments of $F(y)$ are not all convergent. We want to present approximate formulae for (1.1) in case the classical approximation formulae, such as the Edgeworth-, the Esscher expansion, the normal power approximation, are not valid anymore because of the divergencies of the moments $E(Y^j)$ from a certain order j on.
In what follows we will consider the case that both $E(Y)$ and $\sigma^2(Y)$ exist, but $E(Y^3)$ diverges.
In what follows we consider the asymptotic behaviour or the for large values of y convergent series development of the distribution density $f_Y(y)$:

$$f_Y(y) = \frac{a_0}{y^\sigma} + \frac{a_1}{y^{\sigma+1}} + \frac{a_2}{y^{\sigma+2}} + \ldots \forall y \geqslant 1\; a_i \in \mathbf{R} \quad 0 < \sigma < 1. \tag{1.4}$$

Because $E(1)$, $E(Y)$ and $E(Y^2)$ are supposed to be convergent one necessarely has:

$a_0 = a_1 = a_2 = 0$.
Consequently we limit ourselves to the case where, for large real y

$$f_Y(y) = \frac{a_3}{y^{\sigma+3}} + \frac{a_4}{y^{\sigma+4}} + \frac{a_5}{y^{\sigma+5}} + 0\left(\frac{1}{y^{\sigma+5}}\right),\ y > 1. \tag{1.5}$$

In the sequel we'll also use the relations:

$$E(X) = tE(Y), \text{ and}$$
$$\sigma^2(X) = tE(Y^2).$$

2. Extension of the Edgeworth Expansion

Let us consider the characteristic function of Y

$$\varphi_Y(s) = \int_0^\infty e^{isy} f_Y(y)\, dy. \tag{2.1}$$

Successively the Fourier integral can be transformed as follows:

$$\varphi_Y(s) = 1 + \frac{is}{1!} E(Y) + \frac{i^2 s^2}{2!} E(Y^2) + \int_0^\infty \left(e^{isy} - 1 - \frac{isy}{1!} - \frac{i^2 s^2 y^2}{2!}\right) f_Y(y) dy \tag{2.2}$$

Next the remaining integral is manipulated as follows:

$$\int_0^\infty \left(e^{isy} - 1 - \frac{isy}{1!} - \frac{(isy)^2}{2!}\right) f_Y(y) dy = \int_0^\infty \left(e^{isy} - 1 - \frac{isy}{1!} - \frac{(isy)^2}{2!}\right)\left(f_Y(y) - \frac{a_3}{y^{\sigma+3}}\right) dy$$

$$+ a_3 \int_0^\infty \left(e^{isy} - 1 - \frac{isy}{1!} - \frac{(isy)^2}{2!}\right) \frac{1}{y^{\sigma+3}}\, dy \tag{2.3}$$

Because the integral in the l.h.s. of (2.3) converges as well as the second term in the r.h.s. of (2.3) the first term in the r.h.s. of (2.3) also converges.

Next one has:

$$\int_0^\infty \left(e^{isy}-1-\frac{isy}{1!}-\frac{(isy)^2}{2!}\right)\left(f_Y(y)-\frac{a_3}{y^{\sigma+3}}\right)dy = \int_0^\infty \left(e^{isy}-1-\frac{isy}{1!}-\frac{(isy)^2}{2!}\right.$$

$$\left.-\frac{(isy)^3}{3!}\right)\left(f_Y(y)\,\frac{a_3}{y^{\sigma+3}}\right)dy+\frac{(is)^3}{3!}\int_0^\infty y^3\left(f_Y(y)-\frac{a_3}{y^{\sigma+3}}\right)dy. \tag{2.4}$$

Applying the same technique successively as the one giving raise to (2.3) and (2.4) one finally obtains:

$$\int_0^\infty \left(e^{isy}-1-\frac{isy}{1!}-\frac{(isy)^2}{2!}\right)f_Y(y)\,dy =$$

$$\begin{aligned}
&a_3\int_0^\infty \left(e^{isy}-1-\frac{isy}{1!}-\frac{(isy)^2}{2!}\right)\frac{dy}{y^{\sigma+3}}\\
&+a_4\int_0^\infty \left(e^{isy}-1-\frac{isy}{1!}-\frac{(isy)^2}{2!}-\frac{(isy)^3}{3!}\right)\frac{dy}{y^{\sigma+4}}\\
&+a_5\int_0^\infty \left(e^{isy}-1-\frac{isy}{1!}-\frac{(isy)^2}{2!}-\frac{(isy)^3}{3!}-\frac{(isy)^4}{4!}\right)\frac{dy}{y^{\sigma+5}}\\
&+\frac{(is)^3}{3!}\int_0^\infty y^3\left(f_Y(y)-\frac{a_3}{y^{\sigma+3}}\right)dy\\
&+\frac{(is)^4}{4!}\int_0^\infty y^4\left(f_Y(y)-\frac{a_3}{y^{\sigma+3}}-\frac{a_4}{y^{\sigma+4}}\right)dy\\
&+\int_0^\infty \left(e^{isy}-1-\frac{isy}{1!}-\frac{(isy)^2}{2!}-\frac{(isy)^3}{3!}-\frac{(isy)^4}{4!}\right)\\
&\times\left(f_Y(y)-\frac{a_3}{y^{\sigma+3}}-\frac{a_4}{y^{\sigma+4}}-\frac{a_5}{y^{\sigma+5}}\right)dy.
\end{aligned} \tag{2.5}$$

Let $\frac{1}{y^{\sigma+3}} f_Y\left(\frac{1}{y}\right)$ have continuous differential coefficients of the first two orders when $0 \leq y < \infty$, then the second member of (2.5) transforms as follows, using two theorems of ref. [4]:

$$\int_0^\infty \left(e^{isy} - 1 - \frac{isy}{1!} - \frac{(isy)^2}{2!}\right) f_Y(y)\,dy =$$

$$a_3(-1)^3 \frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+3)}(-is)^{\sigma+2} + a_4(-1)^4 \frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+4)}(-is)^{\sigma+3} + a_5(-1)^5$$

$$\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+5)}(-is)^{\sigma+4} + \frac{(is)^3}{3!}\frac{\Gamma(1-\sigma)}{\Gamma(2-\sigma)}\gamma_3 + \frac{(is)^4}{4!}\frac{\Gamma(1-\sigma)}{\Gamma(3-\sigma)}\gamma_4 + 0\,((-is)^5)$$

$$|s| \ll 1 \qquad (2.6)$$

where we have put

$$\gamma_3 = \int_0^\infty dy\; y^{\sigma-1}\left(\frac{f_Y\cdot\left(\frac{1}{y}\right)}{y^{\sigma+3}}\right)^{(1)}$$

and

$$\gamma_4 = \int_0^\infty dy\; y^{\sigma-1}\left(\frac{f_Y\left(\frac{1}{y}\right)}{y^{\sigma+3}}\right)^{(2)} \qquad (2.7)$$

where (1) and (2) denote the first and second derivative.

Next we consider the characteristic function of the generalized Poisson distribution

$$\varphi_X(s) = e^{-t+t\varphi_Y(s)}.$$

We get:

$$\varphi_{\frac{X-E(X)}{\sigma(X)}}(s) = \exp\left[-\frac{s^2}{2} + t\left\{-a_3\frac{\Gamma(\sigma)\,\Gamma(1-\sigma)}{\Gamma(\sigma+3)}\left(\frac{-is}{\alpha}\right)^{\sigma+2}\frac{1}{\sqrt{t}}^{\sigma+2}\right.\right.$$

$$+a_4\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+4)}\left(\frac{-is}{\alpha}\right)^{\sigma+3}\frac{1}{\sqrt{t}}^{\sigma+3} - a_5\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+5)}\left(\frac{-is}{\alpha}\right)^{\sigma+4}\frac{1}{\sqrt{t}}^{\sigma+4}$$

$$\left.\left.+\frac{(is)^3}{3!}\frac{\Gamma(1-\sigma)}{\Gamma(2-\sigma)}\gamma_3\frac{1}{\alpha^3}\frac{1}{\sqrt{t^3}} + \frac{(is)^4}{4!}\frac{\Gamma(1-\sigma)}{\Gamma(3-\sigma)}\gamma_4\frac{1}{\alpha^4}\frac{1}{\sqrt{t^4}} + 0\left(\frac{1}{\sqrt{t^5}}\right)\right\}\right] \quad t \gg 1 \qquad (2.8)$$

where $\alpha^2 = E(Y^2)$.

Part of the exponential function is expanded, giving raise to

$$\varphi_{\frac{X-E(X)}{\sigma(X)}}(s) = e^{-\frac{s^2}{2}} \Bigg\{ \sum_{n=0}^{\left[\frac{3}{\sigma}+1\right]} \frac{(-a_3)^n}{n!} \left(\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+3)}\right)^n \frac{1}{\alpha^{n(\sigma+2)}} \times$$

$$\times \Bigg[(-is)^{\sigma+2} \frac{1}{\sqrt{t}}\, n\sigma$$

$$-\frac{\gamma_3}{3!}\frac{\Gamma(1-\sigma)}{\Gamma(2-\sigma)}\frac{1}{\alpha^3}(-is)^{n\sigma+2n+3}\frac{1}{\sqrt{t}}\, n\sigma+1$$

$$+a_4\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+4)}\frac{1}{\alpha^{\sigma+3}}(-is)^{(n+1)\sigma+2n+3}\frac{1}{\sqrt{t}^{\,(n+1)\sigma+1}}$$

$$+\frac{\gamma_4}{4!}\frac{\Gamma(1-\sigma)}{\Gamma(3-\sigma)}\frac{1}{\alpha^4}(-is)^{n\sigma+2n+4}\frac{1}{\sqrt{t}^{\,n\sigma+2}} \tag{2.9}$$

$$+\frac{1}{2}\left(\frac{\gamma_3}{3!}\frac{\Gamma(1-\sigma)}{\Gamma(2-\sigma)}\right)^2\frac{1}{\alpha^6}(-is)^{n\sigma+2n+6}\frac{1}{\sqrt{t}^{\,n\sigma+2}}$$

$$-a_5\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+5)}\frac{1}{\alpha^{\sigma+4}}(-is)^{(n+1)\sigma+2n+4}\frac{1}{\sqrt{t}^{\,(n+1)\sigma+2}}$$

$$-\frac{a_4\gamma_3}{3!}\frac{\Gamma(\sigma)(\Gamma(1-\sigma))^2}{\Gamma(\sigma+4)\Gamma(2-\sigma)}\frac{1}{\alpha^{\sigma+6}}(-is)^{(n+1)\sigma+2n+6}\frac{1}{\sqrt{t}^{\,(n+1)\sigma+2}}$$

$$+\frac{1}{2}\left(a_4\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+4)}\right)^2\frac{1}{\alpha^{2\sigma+6}}(-is)^{(n+2)\sigma+2n+6}\frac{1}{\sqrt{t}^{\,(n+2)\sigma+2}}\Bigg]$$

$$+0(\sqrt{t^3})\Bigg\}$$

where $[x]$ denotes the largest integer contained in x.

In order to obtain the distribution density of $\dfrac{X-E(X)}{\sigma(X)}$ one has to consider:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, e^{-\frac{s^2}{2}}(-is)^n\, ds. \tag{2.10}$$

(2.10) can be expressed by means of the well known integral representation of the Parabolic cylinder function [6]

$$\boldsymbol{D}_p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\pi}{2}pi} e^{\frac{z^2}{4}} \int_{-\infty}^{+\infty} x^p e^{-\frac{x^2}{2}+ixz} dx \tag{2.11}$$

$$[Re\, p > -1, \text{ for } x < 0 \arg x^p = p\pi i]$$

where $\boldsymbol{D}_p(z)$ can be expressed by means of the degenerate hypergeometric functions:

$$\boldsymbol{D}_p(z) = 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \Phi\left(\frac{p}{2}, \frac{1}{2}, \frac{z^2}{2}\right) - \frac{\sqrt{2\pi} z}{\Gamma\left(-\frac{p}{2}\right)} \Phi\left(\frac{1-p}{2}, \frac{3}{2}, \frac{z^2}{2}\right) \right\} \tag{2.12}$$

and where of course

$$\Phi(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{z^3}{3!} + \ldots \tag{2.13}$$

Consequently:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isx} e^{-\frac{s^2}{2}} (-is)^\eta \, ds = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4}} \boldsymbol{D}_\eta(-x) \tag{2.14}$$

From (2.12) and (2.13) it follows that $\boldsymbol{D}_p(-x)$ is real for all $x \in \boldsymbol{R}$. Consequently (2.9) can be cast into the form

$$f_{\frac{X-E(X)}{\sigma(X)}}(x) = \sum_{n=0}^{\left[\frac{3}{\sigma}+1\right]} \frac{(-a_3)^n}{n!} \left(\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+3)}\right)^n \frac{1}{\alpha^{n(\sigma+2)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4}}$$

$$\times \left[\boldsymbol{D}_{n(\sigma+2)}(-x) \frac{1}{\sqrt{t^{n\sigma}}} \right.$$

$$- \frac{\gamma_3}{3!} \frac{\Gamma(1-\sigma)}{\Gamma(2-\sigma)} \frac{1}{\alpha^3} \boldsymbol{D}_{n\sigma+2n+3}(-x) \frac{1}{\sqrt{t^{n\sigma+1}}}$$

$$+ a_4 \frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+4)} \frac{1}{a^{\sigma+3}} \boldsymbol{D}_{(n+1)\sigma+2n+3}(-x) \frac{1}{\sqrt{t^{(n+1)\sigma+1}}}$$

$$+\frac{\gamma_4}{4!}\frac{\Gamma(1-\sigma)}{\Gamma(3-\sigma)}\frac{1}{\alpha^4}\boldsymbol{D}_{n\sigma+2n+4}(-x)\frac{1}{\sqrt{t^{n\sigma+2}}}$$

$$+\frac{1}{2}\left(\frac{\gamma_3}{3!}\frac{\Gamma(1-\sigma)}{\Gamma(2-\sigma)}\right)^2\frac{1}{\alpha^6}\boldsymbol{D}_{n\sigma+2n+6}(-x)\frac{1}{\sqrt{t^{n\sigma+2}}}$$

$$-a_5\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+5)}\frac{1}{\alpha^{\sigma+4}}\boldsymbol{D}_{(n+1)\sigma+2n+4}(-x)\frac{1}{\sqrt{t^{(n+1)\sigma+2}}}$$

$$-\frac{a_4\gamma_3}{3!}\frac{\Gamma(\sigma)(\Gamma(1-\sigma))^2}{\Gamma(\sigma+4)\Gamma(2-\sigma)}\frac{1}{\alpha^{\sigma+6}}\boldsymbol{D}_{(n+1)\sigma+2n+6}(-x)\frac{1}{\sqrt{t}^{(n+1)\sigma+2}}$$

$$\left.+\frac{1}{2}\left(a_4\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+4)}\right)^2\frac{1}{\alpha^{2\sigma+6}}\boldsymbol{D}_{(n+2)\sigma+2n+6}(-x)\frac{1}{\sqrt{t^{(n+2)\sigma+2}}}\right]$$

$$+0\left(\frac{1}{\sqrt{t^3}}\right). \tag{2.15}$$

In order to be able to deduce the for large values of t leading terms in the expansion of the cumulative distribution function one has to consider

$$\int_{-\infty}^{x} e^{-\frac{x^2}{4}}\boldsymbol{D}_p(-x)\,dx.$$

Making use of the well-known recursion formula

$$\frac{d}{dz}\boldsymbol{D}_p(z)-\frac{1}{2}z\boldsymbol{D}_p(z)+\boldsymbol{D}_{p+1}(z)=0 \tag{2.16}$$

one easily deduces:

$$\int_{-\infty}^{x} e^{-\frac{x^2}{4}}\boldsymbol{D}_{p-1}(-x)\,dx = e^{-\frac{x^2}{4}}\boldsymbol{D}_p(-x). \tag{2.17}$$

Consequently one obtains:

$$F_{\frac{X-E(X)}{\sigma(X)}}(x)=\sum_{n=0}^{\left[\frac{3}{\sigma}+1\right]}\frac{(-a_3)^n}{n!}\left(\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+3)}\right)^n\frac{1}{\alpha^{n(\sigma+2)}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{4}}$$

$$\times\left[\boldsymbol{D}_{n(\sigma+2)+1}(-x)\frac{1}{\sqrt{t^{n\sigma}}}\right.$$

$$
\begin{aligned}
&-\frac{\gamma_3}{3\,!}\frac{\Gamma(1-\sigma)}{\Gamma(2-\sigma)}\frac{1}{\alpha^3}\boldsymbol{D}_{n\sigma+2n+4}(-x)\frac{1}{\sqrt{t^{n\sigma+1}}}\\
&+a_4\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+4)}\frac{1}{\alpha^{\sigma+3}}\boldsymbol{D}_{(n+1)\sigma+2n+4}(-x)\frac{1}{\sqrt{t^{(n+)\sigma+1}}}\\
&+\frac{\gamma_4}{4\,!}\frac{\Gamma(1-\sigma)}{\Gamma(3-\sigma)}\frac{1}{\alpha^4}\boldsymbol{D}_{n\sigma+2n+5}(-x)\frac{1}{\sqrt{t^{n\sigma+2}}}\\
&+\frac{1}{2}\left(\frac{\gamma_3}{3!}\frac{\Gamma(1-\sigma)}{\Gamma(2-\sigma)}\right)^2\frac{1}{\alpha^6}\boldsymbol{D}_{n\sigma+2n+7}(-x)\frac{1}{\sqrt{t^{n\sigma+2}}}\\
&-a_5\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+5)}\frac{1}{\alpha^{\sigma+4}}\boldsymbol{D}_{(n+1)\sigma+2n+5}(-x)\frac{1}{\sqrt{t^{(n+1)\sigma+2}}}\\
&-\frac{a_4\gamma_3}{3\,!}\frac{\Gamma(\sigma)(\Gamma(1-\sigma))^2}{\Gamma(\sigma+4)\Gamma(2-\sigma)}\frac{1}{\alpha^{\sigma+6}}\boldsymbol{D}_{(n+1)\sigma+2n+7}(-x)\frac{1}{\sqrt{t^{(n+1)\sigma+2}}}\\
&+\frac{1}{2}\left(a_4\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(\sigma+4)}\right)^2\frac{1}{\alpha^{2\sigma+6}}\boldsymbol{D}_{(n+2)\sigma+2n+7}(-x)\frac{1}{\sqrt{t^{(n+2)\sigma+2}}}\\
&+0\left(\frac{1}{\sqrt{t^3}}\right)\Bigg]
\end{aligned}
\qquad (2.18)
$$

3. Application to Some Actual Distributions of Claims Arising in Insurance Applications

Let us first consider the case of a Pareto density

$$f_Y(y) = \alpha y_0^{\alpha}\frac{1}{(y+y_0)^{1+\alpha}}$$

where $\alpha = 2+\sigma$ with $0 < \sigma < 1$.
In the present case one obtains the following expansion for (1.5)

$$f_Y(y) = \frac{\Gamma(3+\sigma)}{\Gamma(2+\sigma)}\frac{y_0^{2+\sigma}}{y^{3+\sigma}} - \frac{\Gamma(4+\sigma)}{\Gamma(2+\sigma)}\frac{y_0^{3+\sigma}}{y^{4+\sigma}} + \frac{\Gamma(5+\sigma)}{\Gamma(2+\sigma)}\frac{y_0^{4+\sigma}}{y^{5+\sigma}} + \dots$$

$$\text{where } a_3 = \frac{\Gamma(3+\sigma)}{\Gamma(2+\sigma)}y_0^{2+\sigma},\quad a_4 = -\frac{\Gamma(4+\sigma)}{\Gamma(2+\sigma)}y_0^{3+\sigma},\quad a_5 = \frac{\Gamma(5+\sigma)}{\Gamma(2+\sigma)}y_0^{4+\sigma}.$$

For the Pareto case it is well known that for the compound variable holds:

$$\begin{cases} E(X) = t\,\dfrac{y_0}{1+\sigma} \\ \\ \sigma^2(X) = t\,2\,y_0^2\dfrac{1}{\sigma}\cdot\dfrac{1}{1+\sigma}. \end{cases}$$

One easily calculates γ_3 and γ_4 from (2.7) to give:

$$\gamma_3 = -3\,!\,y_0^3\frac{\Gamma(\sigma)}{\Gamma(2+\sigma)}$$

$$\gamma_4 = 4\,!\,y_0^4\frac{\Gamma(\sigma)}{\Gamma(3+\sigma)}.$$

Hence inserting these results for a_3, a_4, a_5, γ_3 and γ_4 into the r.h.s of (2.15) and (2.18) gives an approximation for the compound Poisson process for a Pareto claim distribution.
Another distribution arising in insurance applications is given by [7]:

$$f_Y(y) = k\,y^{q_2}(y+a)^{-q_1} \qquad q_1 > q_2+1\,.$$

This distribution falls between types III and V into the region of the type VI distributions.
In case $q_1 = q_2+3+\sigma$ the following expansion for $f_Y(y)$ valid for large $y\,(y \geq a)$ is obtained

$$f_Y(y) = k\,.\left[\frac{1}{y^{3+\sigma}}+\frac{(q_2+3+\sigma)}{1\,!}\,\frac{a}{y^{4+\sigma}}+\frac{(q_2+3+\sigma)\,(q_2+2+\sigma)}{2\,!}\,\frac{a^2}{y^{5+\sigma}}+\ldots\right]$$

where $k = a^{2+\sigma}\dfrac{\Gamma(q_2+\sigma+3)}{\Gamma(q_2+1)\Gamma(\sigma+2)}$.

Hence:

$$a_3 = k, \quad a_4 = k\,a(q_2+3+\sigma), \quad a_5 = k a^2(q_2+3+\sigma)\,(q_2+2+\sigma).$$

In the present case one still has the following well known result

$$\begin{cases} E(X) = t \, . \dfrac{k}{a^{1+\sigma}} \dfrac{\Gamma(q_2+2)\Gamma(1+\sigma)}{\Gamma(q_2+\sigma+3)} \\ \sigma^2(X) = t \, . \dfrac{k}{a^{\sigma}} \dfrac{\Gamma(q_2+3)\Gamma(\sigma)}{\Gamma(q_2+\sigma+3)} . \end{cases}$$

One also easily calculates γ_3 and γ_4 to give:

$$\gamma_3 = -a^{1-\sigma} \frac{\Gamma(\sigma+1)\Gamma(q_2+3)}{\Gamma(\sigma+q_2+3)} k$$

$$\gamma_4 = a^{-\sigma} \frac{\Gamma(\sigma+1)\Gamma(q_2+4)}{\Gamma(\sigma+q_2+3)} k .$$

Conclusion

The results obtained above suggest to continue the search for results according to Esscher's approximation and in the case where the variance resp. the mean of the variable Y isn't finite.

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M.J. Goovaerts and L. D'Hooge
Instituut voor Actuariele Wetenschappen
Katholieke Universiteit Leuven
Dekenstraat 2
B-3000 Leuven
Belgium

P. Van Goethem
Departement Wiskunde
Universitaire Instelling Antwerpen
Universiteitsplein 1
B-2610 Wilrijk
Belgium

Zusammenfassung

Im vorliegenden Beitrag wird die Edgeworth-Reihenentwicklung für den Fall, dass das dritte Moment der Schadenverteilungsfunktion divergiert, hergeleitet und auf zwei in der Versicherung auftretende Verteilungen angewandt.

Résumé

Le développement en série de Edgeworth est étendu au cas où le moment du 3e ordre de la fonction de répartition des sinistres diverge et est ensuite appliqué à deux répartitions se rencontrant dans l'assurance.

Riassunto

Nel presente articolo si deriva la serie di Edgeworth nel caso che il terzo momento della funzione di distribuzione dei sinistri sia divergente e la si applica a due distribuzioni che si incontrano nell'assicurazione.

Summary

In the present contribution the extension of the Edgeworth series is expanded to the case where the third moment of the claim distribution diverges and applied to two actual distributions of claims that arise in insurance applications.