How to fix retention

How to Fix Retention

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0. Preliminaries

The mathematical treatment of the retention problem was dealt with by *de Finetti* [1] as early as 1940 and in subsequent years e.g. by *Dubourdieu* [2] and *Bühlmann* [3]. Yet it is fair to say that no simple concept exists which would answer practical questions like, for instance, "given two different reinsurance arrangements costing the same price, which one is more efficient?" or, vice versa, "given two reinsurances of the same efficiency, which one is cheaper?". In [4] I tried to take a first step in this direction. The present note which has developed from [4] in discussions with Hans Bühlmann, is a second attempt.

1. Gross and Net Portfolio. General Notations

Of the portfolio under consideration, we denote by

$W(s) = \text{Prob}[S \leq s]$ the distribution of sums insured,

$G(\xi) = \text{Prob}[\chi \leq \xi]$ the distribution of the claims degree,

$V(x) = \text{Prob}[X \leq x]$ the distribution of the individual claim amount,

$F(z) = \text{Prob}[Z \leq z]$ the distribution of the total of claims.

Restricting ourselves to Poisson distributed number of claims with parameter λ we have for the gross business

$$E[Z] = \lambda E[X] \quad \text{and} \quad E[Z^2] = \lambda E[X^2] + \lambda^2 E^2[X]$$

with

$$E[X^n] = E[S^n]E[\chi^n] \quad \text{for} \quad n = 1, 2, \ldots$$

if S and χ are assumed to be independent.

Clearly, for some lines of business (mainly Casualty) the sum insured S and the claims degree χ are not needed, however, in Property e. g. they are crucial and their connection with the individual claim amount X is given by

$$V(x) = \int_0^\infty G\left(\frac{x}{s}\right) dW(s).$$

We shall generally write Z, $\check{Z}$ and $\tilde{Z} = Z - \check{Z}$ for the gross, reinsured and net total of claims and correspondingly

$P = (1+\delta)E[Z]$ for the gross premiums after costs,
$\check{P} = (1+\check{\delta})E[\check{Z}]$ for the reinsurance premium,
and $\tilde{P} = P - \check{P}$ for the net premium.

Here δ and $\check{\delta}$ denote the premium loading applied by the ceding company and the reinsurer (s) respectively. If $\delta \neq \check{\delta}$, the ceding company's gross and net expected profit margin are unequal, which may be due e. g. to a profit or loss on reinsurance commissions.

2. Four Basic Reinsurance Forms

We shall only deal with quota, surplus, excess loss and stop loss reinsurance, although general results below can easily be applied to other forms and/or combinations of some basic reinsurance forms.
Under a *quota* with retention α $(0 \leq \alpha \leq 1)$ of each and every risk, the same percentage $1-\alpha$ is reinsured. $\tilde{X} = \alpha X$ and $\tilde{Z} = \alpha Z$.
Under a *surplus*, the ceding company retains at the most a certain amount m of each risk, called "one line". The exceeding part is reinsured but only up to a certain multiple of the retention (e. g. 10 lines). For theoretical purposes, however, we assume the treaty capacity to be unlimited and thus

$$\tilde{X} = \begin{cases} X & \text{if} \quad S \leq m \\ \frac{m}{S} X & \text{else}. \end{cases}$$

Both quota and surplus are called *proportional* treaties since everything is proportional: the relation between net and reinsured is for each risk the same with respect to sum insured, to premiums and claims.

Contrary to this, excess and stop loss treaties are called *nonproportional* because here the relation net to reinsured for sums insured, premiums and claims are either unequal or undefined.
Excess of loss reinsurance is characterized by

$$\check{X} = \begin{cases} 0 & \text{if} \quad X \leq r, \quad r = \text{retention (called "priority")} \\ X - r & \text{else.} \end{cases}$$

Clearly $\tilde{X} = X - \check{X}, \tilde{Z} = \sum_{i=1}^{K} \tilde{X}_i$ and $\check{Z} = Z - \tilde{Z}$

for K = number of gross claims.
Finally, the *stop loss* works on the total of claims Z through

$$\check{Z} = \begin{cases} 0 & \text{if} \quad Z \leq \varrho P \\ Z - \varrho P. \end{cases}$$

We write ϱP for the retention because the stop loss point – as the retention under this type of reinsurance is called – is mostly defined as a percentage of the underlying gross premium volume P.
Again, also with nonproportional treaties, reinsurance cover would be limited in practice, but for the sake of simplicity, we consider it to be unlimited.
Writing $(a-b)^+$ for max $(a-b, 0)$, we may summarize the above as follows:

	Reinsured Individual Claim	Total of Reinsured Claims
Quota Retention α	$\check{X}_k = (1-\alpha)X_k$, $k = 1, 2, ..., K$	$\check{Z} = (1-\alpha)Z$
Surplus 1 line = m	$\check{X}_k = \left(1 - \frac{m}{S_{i(k)}}\right)^+ \cdot X_k$ [1]	$\check{Z} = \sum_{k=1}^{K} \check{X}_k$
Excess loss *Priority* = r	$\check{X}_k = (X_k - r)^+$	$\check{Z} = \sum_{k=1}^{K} \check{X}_k$
Stop loss Stop loss point = ϱP	–	$\check{Z} = (Z - \varrho P)^+$

[1] Here $S_{i(k)}$ means the sum insured of the risk hit by claim number k (of amount X_k).

3. Absolute Retention Problem. General Solution

Denoting with $\tilde{Z}_j$ the total of net claims and with $\tilde{P}_j$ the net premiums in year j, we look at the probability of being ruined at the end of one of the future years (discrete version of the definition of ruin). We are thus interested in the event

$$\text{ruin} = \left\{ n\tilde{P} - \sum_{j=1}^{n} \tilde{Z}_j < -u_0 \quad \text{for some} \quad n \right\}$$

wherein $\tilde{P}_j = \tilde{P}$ independent of j for the sake of simplicity and u_0 = free initial reserves.

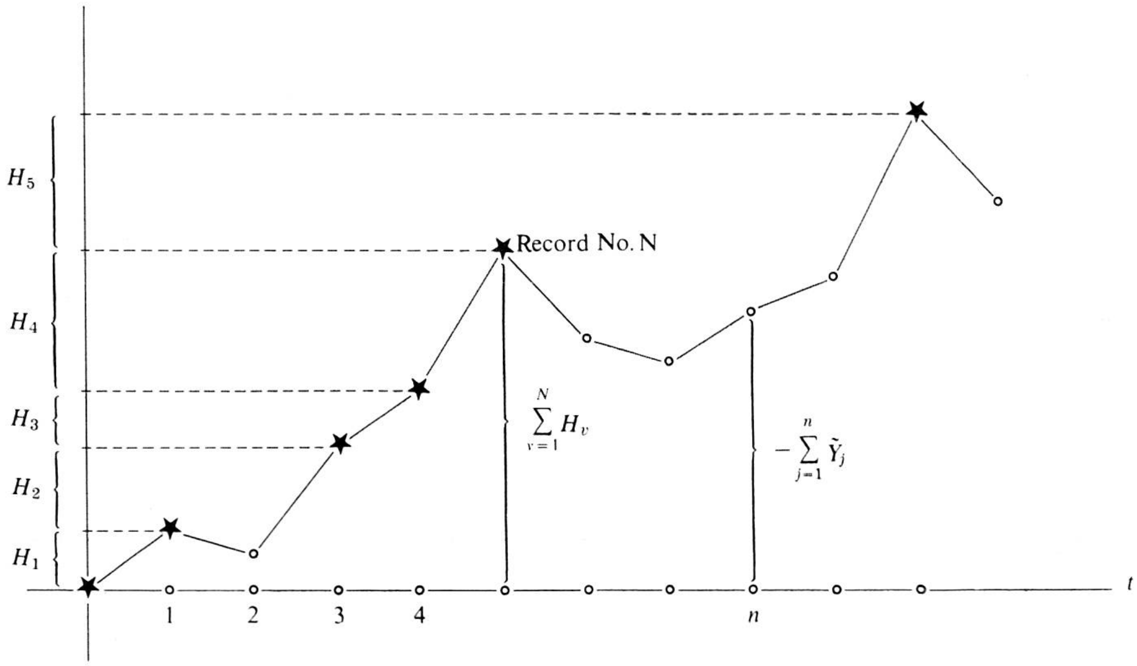
Or, in other words, ruin occurs if

$$\max_{1 \leq n < \infty} \sum_{j=1}^{n} (\tilde{Z}_j - \tilde{P}) > u_0$$

that is to say, if the sum of $-\tilde{Y}_j = \tilde{Z}_j - \tilde{P}$ exceeds u_0.

The $\tilde{Y}_j$ can be looked upon as being independent and identically distributed so that we are dealing with a random walk of the following type: On the horizontal we plot the time and on the vertical the accumulated negative results $-\tilde{Y}_j$, i.e. the sum

$$-\sum_{j=1}^{n} \tilde{Y}_j .$$



Stars (✶) denote record points, jumps H_1, H_2, ... between the records are identically distributed and from the general theory (see e.g. [3] page 146 and following) we know that the following is true

Theorem:

(i) In the equations $E[e^{-\varkappa\tilde{Y}}] = 1$ and $E[e^{\varkappa H}] = 1$ the solution κ is the same.

(ii)
$$\psi(u_0) = \text{Prob}\left[\max_{1 \le n < \infty}\left(-\sum_{j=1}^{n} \tilde{Y}_j\right) > u_0\right] \le e^{-\varkappa u_0}$$

i.e. Cramèr's inequality.

The proof of (i) is easy: Since κ is the solution of $E[e^{-\varkappa\tilde{Y}}] = 1$ and since H is a sum of independent $-Y$ variables we have

$$E[e^{\varkappa H}] = E[e^{-\varkappa\tilde{Y}_1}] \cdot E[e^{-\varkappa\tilde{Y}_2}] \dots E[e^{-\varkappa\tilde{Y}_n}] = 1.$$

As a first step of approximation, we put

$$E[e^{-\varkappa\tilde{Y}}] \sim E[1 - \kappa\tilde{Y} + \frac{\kappa^2}{2}\tilde{Y}^2] = 1$$

and thus

$$\kappa = 2\frac{E[\tilde{Y}]}{E[\tilde{Y}^2]} \quad \text{or} \quad \kappa = \frac{2E[\tilde{Y}]}{\text{Var}[\tilde{Z}] + E^2[\tilde{Y}]}.$$

Secondly, by putting equality in Cramèr's inequality and $\varepsilon = \psi(u_0)$ for the tolerated ruin probability, we obtain

$$-\frac{\ln\varepsilon}{2_{u_0}} = \frac{E[\tilde{Y}]}{\text{Var}[\tilde{Z}] + E^2[\tilde{Y}]}.$$

This equation allows us to calculate retentions, as we shall see in the next section.

4. Retentions under the Four Basic Reinsurance Forms

According to the above, all we have to do is calculate $E[\tilde{Y}]$ and Var $[\tilde{Z}]$ under a quota, surplus, excess loss and stop loss treaty. In doing so, first note that generally

$$\begin{aligned} E[\tilde{Y}] = E[P - \check{P} - \tilde{Z}] &= (1+\delta)E[Z] - (1+\check{\delta})E[\check{Z}] - E[\tilde{Z}] \\ &= \delta E[\tilde{Z}] - (\check{\delta} - \delta)E[\check{Z}]. \end{aligned}$$

Now, for a quota share we have

$$E[\tilde{Z}] = E[Z]\alpha,\ E[\check{Z}] = E[Z](1-\alpha),\ \mathrm{Var}[\tilde{Z}] = \mathrm{Var}[Z]\alpha^2.$$

In the case of a surplus, we write

$$E[\tilde{Z}] = \lambda E[\tilde{X}],\ \mathrm{Var}[\tilde{Z}] = \lambda E[\tilde{X}^2] \text{ and } E[\check{Z}] = E[Z]-E[\tilde{Z}]$$

and calculate $E[\tilde{X}]$ and $E[\tilde{X}^2]$ as follows:

We have $\tilde{V}(x) = \mathrm{Prob}[\tilde{X} \le x] = \int_0^m G\left(\frac{x}{s}\right) dW(s) + G\left(\frac{x}{m}\right)\left(1-W(m)\right)$

and thus for the $n-$th moment

$$E[\tilde{X}^n] = \int_0^\infty x^n \int_0^m g\left(\frac{x}{s}\right) dW(s)\frac{dx}{s} + \int_0^\infty x^n g\left(\frac{x}{m}\right)\left(1-W(m)\right)\frac{dx}{m}$$

$$= \int_0^m s^n \int_0^\infty \left(\frac{x}{s}\right)^n g\left(\frac{x}{s}\right)\frac{dx}{s} dW(s) + m^n \int_0^\infty \left(\frac{x}{m}\right)^n g\left(\frac{x}{m}\right)\frac{dx}{m}\left(1-W(m)\right)$$

$$= E[\chi^n]\underbrace{\left\{\int_0^m s^n dW(s) + m^n\left(1-W(m)\right)\right\}}_{= E[S^n]W^{(n)}(m) \text{ per definitionem.}} = E[\chi^n]E[S^n]W^{(n)}(m)$$

The $W^{(n)}(\cdot)$ are distributions (i.e. non-decreasing functions between 0 and 1) which are ordered in the following way:

$$W^{(n)}(m) \le W^{(k)}(m) \quad \text{for} \quad n \ge k.$$

For the proof show that for $n > k W^{(n)}(m)/W^{(k)}(m)\uparrow 1$ by verifying that the sign of its first derivate with respect to m is positive.
With this we find

$$E[\tilde{Z}] = E[Z]W^{(1)}(m) \quad \text{and} \quad \mathrm{Var}[\tilde{Z}] = \mathrm{Var}[Z]W^{(2)}(m)$$

for a surplus with retention $m =$ one line.

In the same way, we find for an excess loss with priority r

$$E[\tilde{Z}] = E[Z]V^{(1)}(r) \quad \text{and} \quad \mathrm{Var}[\tilde{Z}] = \mathrm{Var}[Z]\,V^{(2)}(r)$$

And finally, a stop loss with priority ϱP yields

$$E[\tilde{Z}^n] = \int_0^{\varrho P} z^n\,dF(z) + (\varrho P)^n \int_{\varrho P}^{\infty} dF(z) = E[Z^n]\,F^{(n)}(\varrho P)$$

and thus $\mathrm{Var}[\tilde{Z}] = \mathrm{Var}[Z]\,F^{(2)}(\varrho P) + E^2[Z]\left(F^{(2)}(\varrho P) - F^{(1)^2}(\varrho P)\right).$

Multiplying our general equation at the end of section 3 by $E[Z]$ and putting $\mathrm{v}[Z] = \mathrm{Var}[Z]/E^2[Z]$ for the coefficient of variation of the gross business leads to the following summary:

Treaty	q = value of $\frac{E[Z]}{2}\kappa = -\frac{E[Z]}{2}\frac{\ln\varepsilon}{u_0}$
Gross *No reinsurance*	$\dfrac{\delta}{\mathrm{v}[Z]+\delta^2}$
Quota Retention α	$\dfrac{\delta\alpha + (\delta - \check{\delta})(1-\alpha)}{\mathrm{v}[Z]\alpha^2 + (\text{numerator})^2}$
Surplus 1 line $= m$	$\dfrac{\delta W^{(1)}(m) + (\delta - \check{\delta})(1 - W^{(1)}(m))}{\mathrm{v}[Z]W^{(2)}(m) + (\text{numerator})^2}$
Excess loss Priority r	$\dfrac{\delta V^{(1)}(r) + (\delta - \check{\delta})(1 - V^{(1)}(r))}{\mathrm{v}[Z]V^{(2)}(r) + (\text{numerator})}$
Stop loss Priority ϱP	$\dfrac{\delta F^{(1)}(\varrho P) + (\delta - \check{\delta})(1 - F^{(1)}(\varrho P))}{\mathrm{v}[Z]F^{(2)}(\varrho P) + F^{(2)}(\varrho P) - F^{(1)^2}(\varrho P) + (\text{numerator})^2}$

Looking at this table, we notice first that all four retention formulae are of the same structure and secondly that $W^{(1)}(m)$, $V^{(1)}(r)$ and $F^{(1)}(\varrho P)$ play the role of α whereas $W^{(2)}(m)$, $V^{(2)}(r)$ and $F^{(2)}(\varrho P)$ play the role of α^2 (the stop loss being the exception to the rule. We hope, however, that the exceptional term $F^{(2)}(\varrho P) - F^{(1)^2}(\varrho P)$ is small). As expected when determining the retention under a surplus, excess loss and stop loss treaty respectively, the distributions

$W(s)$, $V(x)$ and $F(z)$ are crucial. However, the retention under a quota does not depend on the shape of any of these distributions.

If δ and $\check{\delta}$ are about the same, we get

$$\alpha = \frac{\delta}{v[Z]\beta + \delta^2} \cdot \frac{1}{q}$$

Where α equals α, $W^{(1)}(m)$, $V^{(1)}(r)$ or $F^{(1)}(\varrho P)$

$$\beta \text{ equals } 1, \quad \frac{W^{(2)}(m)}{W^{(1)^2}(m)}, \quad \frac{V^{(2)}(r)}{V^{(1)^2}(r)} \quad \text{or} \quad \frac{F^{(2)}(\varrho P)}{F^{(1)^2}(\varrho P)}$$

depending on whether we look at a quota, surplus, excess loss or stop loss treaty (and neglecting the exceptional term in the latter case)

and $\quad q = -\dfrac{E[Z]}{2} \dfrac{ln\varepsilon}{u_0} \quad$ as before.

Of course things would be much easier if β were equal or close to one for all four treaty forms. However, all that can be said about the function $\beta(m)$ in general is that $\beta(\infty) = 1$ and furthermore that $\beta(m)$ first decreases and later on increases with increasing m because of

$$\frac{d\beta(m)}{dm} = \frac{2(1-W(m))}{W^{(1)^3}(m)} \left(\frac{m}{E[S^2]} - \frac{1}{E[S]} \right).$$

We can summarize these observations by saying that if the direct insurers' and reinsurers' loadings are similar ($\delta \sim \check{\delta}$) and the retention in question is rather high ($\beta \cong 1$) then the retention may be calculated by

$$\alpha = \frac{\delta}{v[Z] + \delta^2} \cdot \frac{1}{q}$$

where α equals α, $W^{(1)}(m)$, $V^{(1)}(r)$ or $F^{(1)}(\varrho P)$ as above. With more complicated cases, this simple formula can still be used in order to calculate the initial value for an iterative computation. For certain distributions the value of m, r and ϱP can be looked up in [5].

Finally, we can formulate the above result *for practical purposes as a rule of the thumb as follows:*

(i) If δ is the premium loading, if $E[Z]$ and $\mathrm{Var}[Z]$ denote the sample mean and variance of the gross total of claims, if furthermore u_0 is the free reserve and γ ($= -\ln\varepsilon$) the security factor ($\gamma = 4.6$, e.g. corresponding to a ruin probability of about 1%) then the retention α under a quota is calculated by

$$\alpha = \frac{\delta}{\delta^2 + \mathrm{Var}[Z]/E[Z]} \cdot \frac{2u_0}{\gamma E[Z]}.$$

(ii) The retention m ($= 1$ line) under a surplus corresponding to the above quota from a stabilization point of view is found by proceeding by trial and error until

$$\frac{\text{average net sum insured}}{\text{average gross sum insured}} = \alpha.$$

(iii) For the corresponding priority under an excess of loss, try and err with statistics on individual claims until

$$\frac{\text{average net claim}}{\text{average gross claim}} = \alpha.$$

(iv) Similarly for the stop loss:

$$\frac{\text{average net total of claims}}{\text{average gross total of claims}} = \alpha.$$

5. Concluding Remarks

There are many more interesting questions with regard to retentions, e.g.

- numerical calculations on a concrete model;
- accuracy of the different proposed approximations;
- the calculation of reinsurance premiums ($\check{\delta} = ?$) based on the same type of ruin criteria as in this note;
- other reinsurances and/or combinations of the four basic forms;
- the relative retention problem (as opposed to the absolute one),
- the investigation of distributions $W^{(n)}(x)$;
- following a remark made by Mr. Amsler, one could as well approximately solve

  $\log E[e^{-\varkappa Y}] = 0$ instead of $E[e^{-\varkappa \tilde{Y}}] = 1$, yielding e.g.

$$\alpha = \frac{\gamma}{\mathrm{v}[Z]} \cdot \frac{1}{q} \quad \text{instead of} \quad \alpha = \frac{\delta}{\mathrm{v}[Z] + \delta^2} \cdot \frac{1}{q}.$$

  This idea is certainly worth following up.

Some of the above problems will be treated by a few Yugoslavian students in their written actuarial diploma, but the reader is nevertheless invited to work on them and/or to think of other challenging questions in this context.

Literature

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Zusammenfassung

Basierend auf dem Ruinkriterium und der Cramèrschen Ungleichung werden Selbstbehalte unter Quoten, Summenexzedenten, Schadenexzedenten und Stop-Loss-Verträgen berechnet.

Résumé

Basés sur un critère de ruine et l'inégalité de Cramèr, les pleins de conservation sont calculés sous les traités en quote-part, en excédent de somme, en excédent de sinistres et stop loss.

Riassuto

Sulla base di un criterio di rovina e la disuguaglianza di Cramèr si calcolano dei pieni per le riassicurazioni trattato in quota, eccedente di somma, eccesso sinistri e stop loss.

Summary

Based on the ruin criterion and Cramèr's inequality, retentions are calculated under quota, surplus, excess loss and stop loss treaties.