Risk processes with stochastic discounting

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Risk Processes with Stochastic Discounting

Introduction

The compound Poisson process and the random walk can both be modified to model the present value of the surplus of a risk business if a constant interest rate or, equivalently, a constant discount factor is given.
We investigate the properties of these processes when the discount factors themselves are random. The first problem thereby is the choice of the random process which generates the discount factors. It is not realistic to assume that the interest rates are a sequence of independent random variables. On the other hand, to assume that the interest rate is given by a general autoregressive process is very unconvenient as far as mathematical tractability is concerned. We show that modelling interest with a Markov process is a good compromise. The Markovian assumption seems realistic and the model is tractable.
As far as the first moment of the present value of the surplus is concerned, our model is equivalent to the assumption of a constant discount factor. It is possible to compute higher order moments and hence to approximate the distribution function of the surplus. If the individual claims are exponentially distributed, one can compute the probability of ruin exactly. In general we give an upper bound for the probability of ruin.
The main lesson which can be drawn from the results is that one cannot in general replace stochastic discounting by the constant mean interest rate. Though the procedure is legitimate in some special cases, it is very misleading if for instance the model allows the alternance of positive and negative interest rates.
The paper is based on the author's Ph. D. thesis now referred to as [5], a copy of which can be obtained from the author.

I. Discrete Time

1. The Model

We consider a sequence of random cash flows $X_1, X_2, \dots$. X_i represents for instance the premium for period $i+1$ minus the cumulative claim of period i. X_i is paid at the end of period i. We also consider a sequence of random discount factors $V_1, V_2, \ldots$. V_i is the discount factor in force during period i. The present value of a monetary unit at time t is equal to

$$W_t = V_1 \cdot V_2 \cdot \ldots \cdot V_t,$$

and the present value of the cumulative cash flows for the first n time period is

$$S_n = \sum_{t=1}^{n} W_t X_t.$$

Notice that if the discount factors are constant, $V_t = v$, we get the familiar expression

$$S_n = \sum_{t=1}^{n} v^t X_t.$$

We make the following assumptions:

- $X_1, X_2, \ldots$ are independent, identically distributed and integrable.
- $V_1, V_2, \ldots$ is a Markov chain with a finite state space $\{v_1, v_2, \ldots, v_q\}$, initial probabilities $a = (a_1, a_2, \ldots, a_q)$ and homogeneous transition probabilities given by a matrix P.

We think of each discount factor V as the product of an interest component (interest rate I) and an inflation component (inflation rate J):

$$V = (1 + I)^{-1} (1 + J).$$

Consequently we only require $o < v_i < \infty$ but we explicitly allow some of the v_i's to be smaller than 1 and some to be larger than 1. This means that the weights W_t can converge to o or to ∞ or oscillate between o and ∞ as t goes to ∞.

- The cash flows and the discount factors are stochastically independent or, to be more precise, their probability space is the product space of the two probability spaces defined above.

2. *The Expectation of S_n*

Since the discount factors and the cash flows are independent we have

$$E(S_n) = E\left(\sum_{t=1}^{n} W_t X_t\right) = E(X_1) \sum_{t=1}^{n} E(W_t).$$

On the other hand we have

$$E(W_t) = E(V_1 V_2 \cdot \ldots \cdot V_t) = \sum_{i_1, \ldots, i_t} v_{i_1} v_{i_2} \cdot \ldots \cdot v_{i_t} a_{i_1} p_{i_1 i_2} \cdot \ldots \cdot p_{i_{t-1} i_t} \qquad (1)$$

which shows that a direct computation is not possible in general since the sum is to be taken over q^t terms.

We introduce the following definitions:

- $M = (m_{ij}) = (p_{ij}\, v_j)$

is the *discounted transition matrix* corresponding to both the set of discount factors $\{v_1, v_2, \dots v_q\}$ and to the matrix of transition probabilities (p_{ij}). M is in general no longer a transition matrix but still a nonnegative matrix, i.e. all entries of M are nonnegative.

- $v = (a_1\, v_1, \dots, a_q\, v_q)$

is analogously a row vector of discounted initial probabilities corresponding to $\{v_1, \dots v_q\}$ and to the initial probabilities $(a_1, \dots a_q)$.
We shall use the following notations and conventions:

- $e = (1, 1, \dots, 1)'$

e is a column vector whose components are all equal to 1.
- I is the identity matrix.
- Matrix convergence denotes elementwise convergence.
- If Q is a nonnegative matrix then $\lambda(Q)$ denotes its dominant root in the sense of the Frobenius-Perron theorem (cf. appendix).

We can now state the following:

Theorem

(i) $E(W_t) = v\, M^{t-1}\, e.$

If $I - M$ is regular we have

$$E\left(\sum_{t=1}^{n} W_t\right) = v\,(I - M)^{-1}\,(I - M^n)\, e = v\,(I - M^n)\,(I - M)^{-1}\, e$$

(ii) $E\left(\sum_{t=1}^{\infty} W_t\right) < \infty$ for any set of initial probabilities if and only if

$$\lambda(M) < 1.$$

In this case $I - M$ is regular and

$$E\left(\sum_{t=1}^{\infty} W_t\right) = v\,(I - M)^{-1}\, e.$$

(iii) If the transition matrix P is irreducible and aperiodic we have

$$\lim_n \left(\lambda(M)^{-1}\, M\right)^n = C,$$

where C is a constant matrix.

Proof

(i) Upon rearranging the factors in (1) we get:

$$E(W_t) = \sum_{i_1, \ldots, i_t} a_{i_1} v_{i_1} p_{i_1 i_2} v_{i_2} \cdot \ldots \cdot p_{i_{t-1} i_t} v_{i_t} = v M^{t-1} e$$

which proves the first part of (i). On the other hand,

$$(I + M + \ldots + M^{n-1})(I - M) = (I - M)(I + M + \ldots + M^{n-1}) = I - M^n \qquad (2)$$

and the second part of (i) follows.

(ii) We first prove the equivalence:

$E\left(\sum_{t=1}^{\infty} W_t\right) < \infty$ for any set of initial probabilities if and only if

$$\lim_{n \to \infty} M^n = 0 .$$

If $M^n \to 0$, $I - M$ is regular, as is seen from (2). From (i) and the monotone convergence theorem, we then get

$$E\left(\sum_{t=1}^{\infty} W_t\right) = \lim_n v(I - M)^{-1}(I - M^n)\, e = v(I - M)^{-1} e,$$

which shows that the condition is sufficient. The converse follows from

$$E\left(\sum_{t=1}^{\infty} W_t\right) = \sum_{t=1}^{\infty} v M^{t-1} e .$$

It remains to show that $M^n \to 0$ when $n \to \infty$ if and only if $\lambda(M) < 1$, but this follows at once from the Jordan canonical representation of M and from the properties of the dominant root of M.

(iii) P irreducible and aperiodic translates into M indecomposable and primitive and the asymptotic behaviour of M^n follows from theorem 8.1 in Nikaido [3] (cf. appendix).

Part (iii) of the theorem states that $M^n \simeq \lambda^n(M)\, C$ for large n. Together with (ii) it shows that, as far as the first moment of the surplus is concerned, our stochastic discount factor can be replaced by a constant discount factor, the dominant root of the discounted transition matrix.

We shall now give a few numerical examples which show that to naively replace the stochastic interest rate by the mean interest rate can be very misleading.

Let $q = 2$ and consider the following transition matrix:

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

It is irreducible and aperiodic and it has the following stationary probabilities:

$$\Pi = (0.5, 0{,}5).$$

Let $(a_1, a_2) = (1, 0)$ be the initial probabilities.

Instead of defining the state space in the form $\{v_1, v_2\}$, we specify the corresponding interest rates $\{i_1, i_2\}$ with the understanding that $v_k = (1 + i_k)^{-1}$. We display the dominant root of M as well as the expected value of the surplus for the case where the cash flows X_i are all equal to 1. In parentheses we give the figures corresponding to the mean interest rate (mean with respect to the stationary transition probabilities).

$\{i_1, i_2\}$	$\lambda(M)$		$E\left(\sum_{t=1}^{\infty} W_t\right)$	
{−3%, 5%}	0.962	(0.962)	25.6	(25)
{−0, 8%}	0.964	(0.962)	29	(25)
{−3%, 5%}	0.993	(0.990)	157.1	(100)
{−4%, 5%}	0.999	(0.995)	1 100	(200)
{−6%, 7%}	1.003	(0.995)	∞	(200)

Remarks

1) The method given to compute $E(W_t)$ and $E\left(\sum_{t=1}^{n} W_t\right)$ can be generalized to an inhomogeneous Markov chain. Examples are given in [5]. However, the asymptotical results stated in the theorem only hold for a homogeneous Markov chain.
2) The condition $\lambda(M) < 1$ is sufficient for S_n to converge almost surely to a finite limit as n goes to infinity. The converse is not true as the following example shows.

Example

Let $X_i = x$ be a nonzero constant. Let $V_1, V_2, \ldots$ be independent and identically distributed with $E(\log(V_1)) < o$, $\log(E(V_1)) > o$ and $E(\log(V_1))^2 < \infty$. By definition we have $W_t = \left(\exp\left(\frac{1}{t}\sum_{i=1}^{t}\log V_i\right)\right)^t$, and the almost sure convergence of $S_n = x\sum_{t=1}^{n} W_t$ to a finite limit follows from the strong law of large numbers. On the other hand, it follows from the independence assumption that $\lambda(M) = E(V)$, which is larger than 1 by assumption.

3. Higher Moments and the Distribution Function of S_n

Let us first consider the simple case where the discount factors are constant, $V_t = v$. We have $S_n = \sum_{t=1}^{n} v^t X_t$.

If the X's are normally distributed, S_n is normally distributed too. If the X's are exponentially distributed, we have

$$P(S_n > x) = \sum_{k=1}^{n} g_k\, e^{-v^{-k}x} \qquad g_k = \frac{V^k}{\prod_{i \neq k}(v^k - v^i)}.$$

We have thereby arbitrarily assumed $E(X) = 1$. The formula is from Rényi [4]. The result can be proven by induction.

Under the assumption that X is integrable to the 3rd power, it follows from the Berry-Esseen theorem (cf. [1]) that the distribution of S_∞ converges to the normal distribution as v goes to 1. However, the approximation is bad for X exponentially distributed and v as large as 0.95.

We thus see that there is no general result for the distribution of S_∞. To put it differently, the distribution of S_∞ is going to depend on the distribution of the X's. Consequently we have to rely on approximations, for this simple model as well as for our more general model.

Whatever approximation method we choose, the Edgeworth expansion, the normal power or the Esscher method, we need the higher moments of S_n. Consequently we must make the supplementary assumption that the X's are integrable to the 4th power.

A straightforward computation shows

$$E(S_n) = E(X)\, E(\Sigma\, W_t)$$
$$\mathrm{Var}(S_n) = \mathrm{Var}(X)(E\,\Sigma\, W_t)^2 + (E\,X)^2\, \mathrm{Var}(\Sigma\, W_t).$$

Under the assumption that $E(X) = 0$ we, moreover, have

$$E(S_n^3) = E(X^3)\, E(\Sigma\, W_t^3)$$
$$E(S_n^4) = E(X^4)\, E(\Sigma\, W_t^4) + 6\,\mathrm{Var}\,(X^2)\, E(\sum_{i>j} W_i^2\, W_j^2).$$

This last assumption greatly simplifies the expressions we get for $E(S_n^3)$ and $E(S_n^4)$, but it is by no means necessary. The general results can be found in [5].

It remains to determine the value of the expressions involving the W's. For this purpose we introduce the following notation:

$$M_k = (p_{ij}\, v_j^k) \qquad k = 1, 2, \ldots$$
$$v_k = (a_1\, v_1^k, \ldots, a_q\, v_q^k) \qquad k = 1, 2, \ldots$$

M_1 and v_1 are the discounted transition matrix and the discounted initial probabilities defined in 2. Since M_k is a nonnegative matrix, it has a dominant root, which we denote by $\lambda\,(M_k)$. We can now state the following:

Theorem

1) $E(W_t^k) = v_k\, M_k^{t-1}\, e.$

If $I - M_k$ is regular we have

$$E\left(\sum_{t=1}^{n} W_t^k\right) = v_k\,(I - M_k)^{-1}\,(I - M_k^n)\, e = v_k\,(I - M_k^n)\,(I - M_k)^{-1}\, e.$$

$E\left(\sum_{t=1}^{\infty} W_t^k\right) < \infty$ for any set of initial probabilities if and only if

$\lambda\,(M_k) < 1$. In this case $I - M_k$ is regular and

$$E\left(\sum_{t=1}^{\infty} W_t^k\right) = v_k\,(I - M_k)^{-1}\, e.$$

2) If $I-M_{k+1}$ and $I-M_1$ are regular, we have

$$E\left(\sum_{1\leq i\leq j\leq n} W_i^k\, W_j^1\right) = v_{k+1}\,(I-M_{k+1}^{n-1}\,(I-M_{k+1})^{-1}\,\mathrm{M}_1\,(I-M_1)^{-1}\,e\,-$$

$$-\;v_{k+1}\left(\sum_{i=o}^{n-2} M_{k+1}^i\, M_1^{n-i}\right)(I-M_1)^{-1}\,e.$$

If $\lambda(M_{k+1})<1$, then $\lambda(M_1)<1$ and $(I-M_{k+1})$ as well as $(I-M_1)$ are regular. Moreover, we have:

$$E\left(\sum_{i<j} W_i^k\, W_j^1\right) = v_{k+1}\,(I-M_{k+1})^{-1}\,M_1\,(I-M_1)^{-1}\,e.$$

We omit the proof, since it is a straightforward generalization of the previous one.

We shall now give a few numerical examples. We assume that the transition probabilities and the initial probabilities are the same as in 2. We give the results corresponding to the mean interest rate in parentheses.

$\{i_1, i_2\}$	$\lambda(M_2)$		$E\left(\sum_{t=1}^{\infty} W_t^2\right)$		$\mathrm{Var}\left(\sum_{t=1}^{\infty} W_t\right)$
{3%, 5%}	0.925	(0.925)	12.8	(12.3)	2.1
{0%, 8%}	0.934	(0.925)	16.0	(12.3)	48.1
{−3%, 5%}	0.991	(0.980)	125.7	(49.8)	12316
{−4%, 5%}	1.004	(0.990)	∞	(99.8)	∞
{−6%, 7%}	1.019	(0.990)	∞	(99.8)	–

These examples show that as far as higher moments (and consequently the distribution function) are concerned, the stochastic discount factor can no longer be replaced by the dominant root of M. For instance, in the case {−4%, 5%} $\lambda(M)$ was smaller than 1, which would imply that $\mathrm{Var}(S_\infty)$ is finite, whereas $\lambda(M_2)$ is actually larger than 1 and the variance is infinite.

4. The Probability of Ruin

It is now more convenient to work with the future value of the surplus, which is defined as follows:

$$R_n = (x+S_n)\,(1+D_1)\,(1+D_2)\cdot\ldots\cdot(1+D_n) \qquad n=1,2,\ldots$$

$$R_o = x,$$

where x is the initial surplus and D_k the interest rate in force during period k. D_k can be positive or negative and it defines V_k through the relation $V_k = (1+D_k)^{-1}$.

We define:

$$\psi_t(d, x) = P(\min_{o \leq k \leq t} R_k < o \mid D_1 = d, R_o = x).$$

$\Psi_t(d, x)$ is the probability that ruin will occur before time t, given that the initial surplus is x and that the interest rate in force during the first time period is d.

From the theorem of total probability and the Markov property of R_n we derive the following recurrence relation for $\psi_t(d, x)$:

$$\Psi_{t+1}(d_k, x) = F\big(-(1+d_k)\, x\big) + \sum_{j=1}^{q} p_{kj} \int_{-(1+d_k)x}^{\infty} \Psi_t\big(d_j, (1+d_k)\, x + y\big)\, dF(y),$$

where $F(x) = P(X < x)$.

If we assume $X = p - Z$, where Z is an exponentially distributed claim with expectation 1 and p is a deterministic premium, we have

$$\psi_{t+1}(x) = e^{-(1+d_k)x-p}\left[1 + \sum_{j=1}^{q} p_{kj} \int_{o}^{(1+d_k)x+p} \Psi_t(d_j, u)\, e^u du\right].$$

From this recurrence relation we can compute the probability of ruin.
We shall now give a few numerical examples. We assume that the initial probabilities and the transition matrix are the same as in the previous two examples. We display $x_{0.01}$ and $x_{0.001}$, the amount of initial surplus necessary to keep the probability of ruin in the first 100 time periods equal to 0.01 and 0.001 respectively. In parentheses we give the initial surplus corresponding to a process with a constant mean interest rate.

$\{d_1, d_2\}$	$x_{0.01}$		$x_{0.001}$	
{3%, 5%}	10.3	(10)	13.9	(13.6)
{0, 8%}	11.8	(10)	16.4	(13.6)
{−3%, 5%}	26.6	(17.9)	41.5	(23.8)
{−4%, 5%}	38.9	(21.5)	66	(28.3)
{−6%, 7%}	69	(21.5)	137	(28.3)

Again we see that, when we have positive and negative interest rates, it is very misleading to replace stochastic discounting by a constant discount factor since we then severely underestimate the necessary initial surplus.
A computation of the probability of ruin with the recurrence relation given above is only possible for exponentially distributed claims. In the general case, however, it is possible to give an upper bound for

$$\Psi(d, x) = \lim_{n \to \infty} \Psi_n(d, x).$$

Theorem

If $\lambda(M) < 1$ then

$$\Psi(d, x) \leqq \frac{P(S_\infty < -x \mid D_1 = d)}{\min_k P(S_\infty < o \mid D_1 = d_k)}.$$

This is a generalisation of a similar result by Gerber [2] for a constant discount factor. It provides an upper bound for the probability of ruin, which depends only on the distribution function of S_∞.

Proof

Let $H_n = \sigma(R_o, D_1, \ldots, R_n, D_{n+1})$ be the σ-algebra generated by $\{R_o, D_1, \ldots, R_n, D_{n+1}\}$ and $M_n = P(S_\infty < -x \mid H_n)$.

Since $\lambda(M) < 1$, S_∞ almost surely exists, so that M_n is meaningful. Let $T = \inf \{n \mid R_n < o\}$. T is the time of ruin, it is equal to ∞ if ruin does not occur. By definition, M_n is a uniformly integrable martingale and it follows

$$E(M_o) = E(M_T \mid T < \infty)\, \Psi(D_1, R_o) + E(M_T \mid T = \infty)\,(1 - \Psi(D_1, R_o)).$$

On the other hand we have $E(M_T \mid T = \infty) = o$ and a straightforward computation shows that

$$E(M_T \mid T < \infty) \geqq \min_k P(S_\infty < o \mid D_1 = d_k),$$

which proves the theorem.

II. Continuous Time

1. The Model

Consider the following process:

$$S(t) = p \int_o^t e^{-\Delta(s)} \, ds - \sum_{i=1}^{N(t)} e^{-\Delta(W_i)} X_i, \quad t \geqq o$$

where

$$\Delta(s) = \int_o^s D(t) \, dt.$$

X_i represents the claim number i, W_i the time at which it occurs. $N(t)$ is the number of claims occurring before time t. $D(s)$ is the interest rate in force at time s, and p is a deterministic premium rate. Thus, if the initial surplus is o, $S(t)$ is the present value of the surplus at time t.

We make the following assumptions:

- X_1 $X_2, \ldots$ are independent, identically distributed and integrable. $F(x)$ denotes the distribution function of X.
- $\{N(t) \mid t \geqq o\}$ is a homogeneous Poisson process with rate 1, i.e., our time is the operational time. W_i denotes the time at which event number i occurs.
- To define $\{D(t) \mid t \geqq o\}$, we must introduce two auxiliary processes. First let $\{M(t) \mid t \geqq o\}$ be a homogeneous Poisson process with rate β and let V_i denote the time at which the $i-th$ event occurs ($V_o = o$). Second assume that $D_1, D_2, \ldots$ is Markov chain with finite state space $\{d_1, \ldots, d_q\}$, initial probabilities $a = (a_1, \ldots, a_q)$ and homogeneous transition probabilities given by a matrix P. We define $\{D(t) \mid t \geqq o\}$ in the following way:

  $$D(t) = D_i \qquad V_{i-1} \leqq t < V_i \qquad i = 1, 2, \ldots$$

  The events recorded by $\{M(t) \mid t \geqq o\}$ can be thought of as potential changes in the interest rate. Since we do not require that $p_{ii} \neq o$ for all i, $\{D(t) \mid t \geqq o\}$ is in general not a continuous time Markov chain.
- The common probability space of

  $$\{X_i \mid i = 1, 2, \ldots\}, \{N(t) \mid t \geqq o\} \quad \text{and} \quad \{(M(t), D(t)) \mid t \geqq o\}$$

is the product space of the three probability spaces defined above. This last assumption implies that the cumulative claim $\left\{\sum_{i=1}^{N(t)} X_i \mid t \geqq o\right\}$ is a compound Poisson process, which is independent of the interest rate $\{D(t) \mid t \geqq o\}$.

2. *The Moments of S(t)*

Lemma

Let $m(z) = E(e^{zX})$ be the moment generating function of an individual claim. The moment generating function of $S(t)$ is then

$$M(z, t) = E(e^{zS(t)}) = E\exp\left[z\,p\int_o^t e^{-\Delta(s)}\,ds + \int_o^t \left(m(-ze^{-\Delta(s)}) - 1\right) ds\right].$$

The proof can be found in [5] and is omitted here. Using the lemma we can compute the moments of $S(t)$

$$E\left(S(t)\right) = \left(E(X) - p\right) \mathrm{E}\left(\int_o^t e^{-\Delta(s)}ds\right)$$

$$\mathrm{Var}\left(S(t)\right) = E(X^2)\, E\left(\int_o^t e^{-2\Delta(s)}ds\right) + \left(E(X) - \mathrm{p}\right)^2 \mathrm{Var}\left(\int_o^t e^{-\Delta(s)}\,ds\right).$$

In the special case where $E(X) = p$ we moreover have

$$E\left(S(t)\right)^3 = E(X^3)\, E\left(\int_o^t e^{-\Delta(s)}ds\right)$$

$$E\left(S(t)\right)^4 = E(X^4)\, E\left(\int_o^t e^{-4\Delta(s)}\,ds\right) + 3\left(E(X^2)\right)^2 \left(\int_o^t e^{-\Delta(s)}ds\right)^2.$$

The general expressions for $E\left(S(t)^3\right)$ and $E\left(S(t)^4\right)$ can be found in [5]. We introduce the following notation:

$v_i^{(k)} = (1 + k\, d_i/\beta)^{-1} \qquad i = 1, \ldots, q \qquad k = 1, 2, \ldots$

$M_{(k)} = (p_{ij}\, v_j^{(k)})$

$M_{(k,1)} = (p_{ij}\, v_j^{(k)}\, v_j^{(1)})$

$v_{(k)} = (a_1\, v_1^{(k)}, \ldots, a_q\, v_q^{(k)})$

$v_{(k,1)} = (a_1\, v_1^{(k)}\, v_1^{(1)}, \ldots, a_q\, v_q^{(k)}\, v_q^{(1)})$.

$\lambda(M_{(k)})$ denotes the dominant root of the nonnegative matrix $M_{(k)}$. We can now state the following:

Theorem

If $\lambda(M_{(k)}) < 1$ then

$$E\int_o^\infty e^{-k\Delta(s)}\, ds = \frac{1}{\beta}\, v_{(k)}\, (I - M_{(k)})^{-1}\, e.$$

If $\lambda(M_{(2k)}) < 1$ then $\lambda(M_{(k)}) < 1$ and

$$E\left(\int_o^t e^{-k\Delta(s)}\, ds\right)^2 = \frac{2}{\beta^2}\left[v_{(k,2k)} + v_{(2k)}\, (I - M_{(2k)})^{-1}\, M_{(k,2k)}\right](I - M_{(k)})^{-1}\, e.$$

The proof of the theorem can be found in [5]. By plugging the results of the theorem in the expressions given above for the moments of $S(t)$, we can compute these moments for $t = \infty$.

3. The Probability of Ruin

It is more convenient to work with the future value of the surplus, which is defined as follows:

$$\begin{aligned} R(t) &= \big(x + S(t)\big)\, e^{\Delta(t)} \qquad t > o \\ R(o) &= x, \end{aligned}$$

where x is the initial surplus.

The probability of ruin is

$$\Psi(d, x) = P\big(\inf_{t<\infty} R(t) < o \mid D(o) = d,\ R(o) = x\big).$$

Theorem

If X is exponentially distributed with expectation 1, the functions $\Psi(d_k, x)$ $(k = 1, 2, \ldots, q)$, considered as functions of x, satisfy the following system of differential equations:

$$\Psi''(\mathrm{d}_k, x)\cdot(p + d_k x) + \psi'(d_k, x)\,(p - 1 + d_k + d_k\, x) + \\ + \beta \sum_{j=1}^{q} (p_{kj} - \delta_{kj})\,\big(\Psi + \Psi'(d_j, x)\big) = o,\ k = 1, \ldots, q$$

with the initial conditions:

$$\Psi'(d_k, o)\cdot p = \Psi(d_k, o) + \beta \sum_{j=1}^{q} (\delta_{kj} - p_{kj})\, \Psi(d_j, o) - 1.$$

If, moreover, $p \geqq E(X) = 1$ and $d_k > o$ for all k, we have

$$\lim_{x \to \infty} \Psi(d_k, x) = o \quad \text{for all } k,$$

and the above system of differential equations has a unique solution which satisfies the boundary conditions.
The result follows from the theorem of total probability and from the Markov property of R_t. The proof can be found in [5]. The theorem is a generalisation of a similar result by Segerdahl [6] for a constant interest rate.
Although we cannot find an analytic solution for the system of differential equations, it is possible to give a numerical solution. Exmples can be found in [5].
The following result is more general since it does not require the claims to be exponentially distributed.

Theorem

If $\lambda(M_{(1)}) < 1$ then

$$\psi(d, x) \leqq \frac{P\big(S(\infty) < -x \mid D(o) = d\big)}{\min_k P\big(S(\infty) < o \mid D(o) = d_k\big)}$$

The proof is a straightforward generalisation of the one given in the previous chapter for a similar result. Numerical illustrations can be found in [5].

Appendix

The results which follow can be found for instance in Nikaido [3]. The matrix $Q = (q_{ij})$ is nonnegative if and only if $q_{ij} \geqq o$ for all i, j. If Q is a nonnegative matrix, it follows from the Frobenius-Perron theorem that

- Q has at least one real nonnegative eigenvalue. Let $\lambda(Q)$ be the largest such eigenvalue of Q. $\lambda(Q)$ is called the dominant root of Q.
- For any eigenvalue ω of Q we have $|\omega| \leqq \lambda(Q)$. Moreover $\lambda(Q)$ satisfies the following inequalities:

$$\min_i \sum_j q_{ij} \leqq \lambda(Q) \leqq \max_i \sum_j q_{ij}.$$

In particular if M is the discounted transition matrix corresponding to a sequence of independent discount factors $V_1, V_2, \ldots$ we have

$$\lambda(M) = \min_i \sum_j p_{ij} v_j = \max_i \sum_j p_{ij} v_j = E(V).$$

The nonnegative matrix Q is called indecomposable if for all i and j there exists an n such that $q_{ij}^{(n)} > o$. (Thereby $q_{ij}^{(n)}$ denotes the $(i, j)-th$ element of Q^n.)
A state s_j has period $p \geqq 1$ if $q_{jj}^{(n)} = o$ for all n unless $n = k \cdot p$ and p is the largest such integer. If $p = 1$ or if there exists no such p, that is if $q_{jj}^{(n)} = o$ for all n, the state s_j is called primitive. The states corresponding to an indecomposable Q have all the same period. If this period is 1, Q itself is called primitive. If $M = (p_{ij} \cdot v_j)$ is a discounted transition matrix, it is indecomposable if and only if $P = (p_{ij})$ is irreducible. Furthermore M is primitive if and only if P is aperiodic. If M is indecomposable and primitive we have

$$\lim_{n \to \infty} \left(\lambda(M)^{-1} M\right)^n = C,$$

where C is a constant matrix.

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Abstract

We investigate mathematical models for the present value of the surplus of risk business in the presence of a stochastically varying interest rate. The interest rate is given by a Markov chain. We show that such an interest rate cannot in general be approximately replaced by a constant.

Zusammenfassung

Es werden mathematische Modelle untersucht für den Barwert der freien Reserven eines Versicherungsgeschäfts. Dabei wird angenommen, dass die Zinsrate durch eine Markovkette definiert wird. Es wird gezeigt, dass eine solche Zinsrate im allgemeinen nicht approximativ durch eine Konstante ersetzt werden kann.

Résumé

L'article étudie des modèles mathématiques pour la valeur actuelle de la provision de fluctuation d'un portefeuille d'assurance lorsque le taux d'escompte varie de façon aléatoire. Le taux d'escompte est défini par une chaîne de Markov. Il est montré qu'un tel taux d'escompte ne peut pas être, en général, remplacé – même pour approximation – par une constante.