Transformations of claim distributions

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Transformations of Claim Distributions*

1. Introduction

Let U, V be random variables on some probability space (Ω, A, P). If V is real and integrable, the expectation of V can be obtained from one of the most useful formulas in risk theory, the iterative rule for expectations,

$$EV = EE[V|U] = \int \int v P^{V|U=u}(dv) P^U(du). \tag{1}$$

If $P^{V|U=u} = P^{V|U=u_0}$ for $u \in A$, where A is some measurable set, and u_0 is fixed, (1) reads

$$\int_{A^c} \int v P^{V|U=u}(dv) P^U(du) + \int_A \int v P^{V|U=u_0}(dv) P^U(du)$$
$$= \int_{A^c} \int v P^{V|U=u}(dv) P^U(du) + P^U(A) \int v P^{V|U=u_0}(dv). \tag{2}$$

E.g., consider a stop loss arrangement with priority P, and let U denote the total claim amount, V the first insurer's payment. Then

$$E[V|U] = \begin{cases} U & \text{if } U \leq P \\ P & \text{if } U > P \end{cases}.$$

Or, in a surplus arrangement with maximum M and random sum insured S, put $U = \left(\frac{M}{S}, X\right)$, where X denotes the related claim size, and let V be again the first insurer's payment. Then

$$E[V|U] = E\left[V\middle|\left(\frac{M}{S}, X\right)\right] = \begin{cases} \frac{M}{S} X & \text{if } \frac{M}{S} \leq 1 \\ X & \text{if } \frac{M}{S} > 1 \end{cases}.$$

As another special case, let U take values in $[0, 1]$, and assume $A = (p, 1]$ and $u_0 = p$ for some $p \in (0, 1)$. Then (2) reads

$$\int_{[0,p]} \int v P^{V|U=u}(dv) P^U(du) + P^U((p, 1]) \int v P^{V|U=p}(dv). \tag{3}$$

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This could be interpreted as the expected share of the first insurer in a reinsurance arrangement involving some «degree» u where for all $u \geq p$ the conditional distribution of the first insurer's payment $P^{V|U=u}$ equals $P^{V|U=p}$.
Let F denote the distribution function of V, $\bar{F}=1-F$ the survival function. We shall assume that $F(0-)=0$ and that

$$\mu = EV = \int xF(dx) = \int_0^\infty \bar{F}(x)dx$$

is finite and positive. Now let F^{-1} denote the pseudo-inverse of F, i.e.

$$F^{-1}(y) = \inf\{x : F(x) \geq y\}, \qquad y \in (0,1),$$

and put

$$U = F \circ V.$$

If F is continuous, U is uniformly distributed on $[0, 1]$. If, moreover, F is strictly monotone, it follows that $F(F^{-1}(u)) = u$ for all $u \in (0, 1)$, and (3) reads

$$\int_0^p F^{-1}(u)\,du + (1-p)\,F^{-1}(p). \tag{4}$$

This expression is known in reliability theory and statistics under the name of *total time on test transform (TTT transform)*, whereas the first summand in (4) (divided by μ) is called *Lorenz curve* in economics.
In the following, we shall develop some of the properties of these transformations, and display applications and examples in risk theory.

2. Lorenz curves and TTT transforms

Let F be a distribution function with $F(0-)=0$, and for $p \in (0,1)$ let

$$L_F(p) = L(p) = \int_0^p F^{-1}(u)du = \int_0^{F^{-1}(p)} (p - F(s))ds, \tag{5}$$

$$T_F(p) = T(p) = \int_0^{F^{-1}(p)} (1 - F(s))ds = L(p) + (1-p)F^{-1}(p), \tag{6}$$

where the last equalities in (5) and (6), respectively, which can be obtained by partial integration, are obvious in view of the following figure.

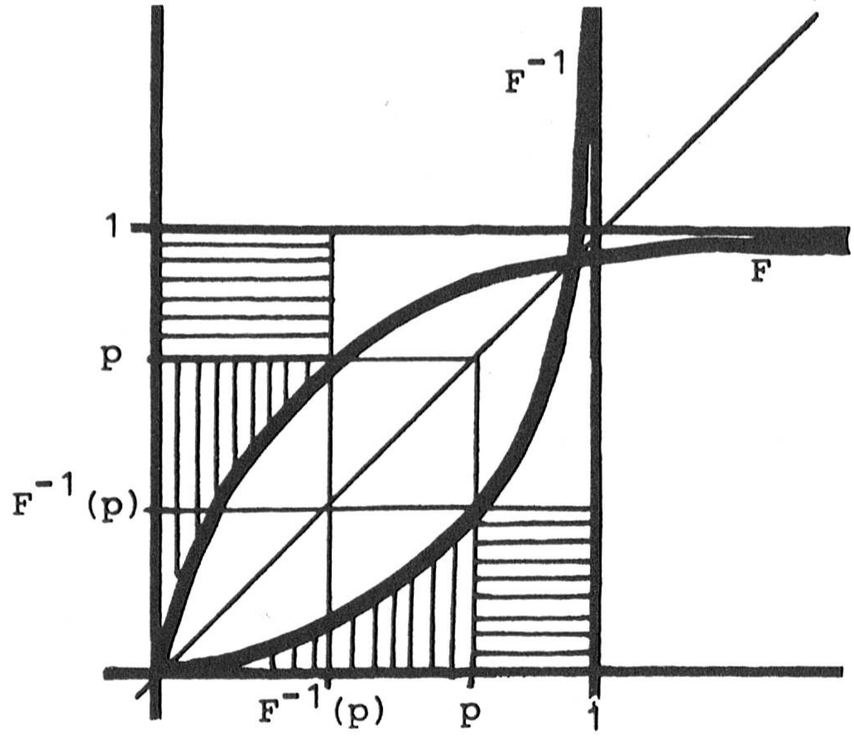

Figure 1

The transforms L and T are related to

$$S_F(t) = S(t) = \int_t^\infty (s-t)F(ds) = \int_t^\infty (1-F(s))ds, \qquad t \in (0, \infty),$$

the *stop-loss transform*. E.g.,

$$T(p) + S(F^{-1}(p)) = \mu, \qquad p \in (0, 1).$$

Obviously,

$$S(0) = T(1) = L(1) = \mu,$$

and instead of (5) and (6)

$$l_F(p) = l(p) = \frac{1}{\mu} L(p), \tag{7}$$

the *Lorenz curve*, and

$$t_F(p) = t(p) = \frac{1}{\mu} T(p) = \int_0^{F^{-1}(p)} \frac{1-F(s)}{\mu} ds, \tag{8}$$

the *scaled total time on test transform (scaled TTT transform)* can be considered. Obviously, l, L, t and T are increasing on (0, 1).

Remark. The integrand

$$s \to (1 - F(s))/\mu \tag{8a}$$

in (8) is of considerable interest in itself. In the usual compound Poisson model with claim size distribution function F, it is the density of the amount of the first surplus below the initial surplus x_0 (given that the surplus ever falls below x_0), cf. *Bowers/Gerber/Hickman/Jones/Nesbitt* [2], or it is the stationary forward recurrence time density of a stationary renewal process with lifetime distribution function F, cf. *Stoyan* [7].

In order to obtain the sample analogues to (6) and (7), we consider the ordered sample of size n

$$0 = x_{0:n} \leq x_{1:n} \leq \ldots \leq x_{n:n}$$

with the empirical distribution function F_n and obtain

$$T_{F_n}\left(\frac{i}{n}\right) = \frac{1}{n} \sum_{j=1}^{i} (n-j-1)(x_{j:n} - x_{j-1:n})$$

$$= \frac{1}{n}(x_{1:n} + \ldots + x_{i-1:n} + (n-i+1)x_{i:n}).$$

So, if F is a life distribution function, $n \cdot T_{F_n}\left(\frac{i}{n}\right)$ describes the «total time on test up to the ith failure».

Similarly,

$$l\left(\frac{i}{n}\right) = \frac{x_{1:n} + \ldots + x_{i:n}}{x_{1:n} + \ldots + x_{n:n}},$$

which, in case that F describes the distribution of income, is the fraction of the total income received by the i poorest people. This explains why the Lorenz curve has been used to illustrate income distributions in economics since it was introduced by *Lorenz* [6].

Interpretations of l and T in insurance are obvious. E.g., $l(p)$ can be thought of being the fraction of the aggregate claims caused by the 100 p% of the treaties with the lowest claim sizes, whereas T may describe the situation of a ceding company in an excess of loss situation.

The following properties of l, L, t and T are well-known or can be proved easily (cf. *Barlow/Campo* [1], *Chandra/Singpurwalla* [3], *Klefsjö* [5], e.g.).

Proposition

a) Both l and t determine F completely.

b) Both l and t are independent of scale, i.e. F and the distribution function $t \to F(a \cdot t)$, $a > 0$, have the same l and t.

c) $t(p) = p$ for all $p \in (0, 1) \Leftrightarrow l(p) = p + (1-p) \ln(1-p)$ for all $p \in (0, 1)$ $\Leftrightarrow F$ is exponential.

$t(p) = 1$ for all $p \in (0, 1) \Leftrightarrow l(p) = p$ for all $p \in (0, 1)$ $\Leftrightarrow F$ is a one point distribution.

d) $$T_{F_n}\left(\frac{i}{n}\right) = \int_0^{F_n^{-1}(\frac{i}{n})} (1 - F_n(s))\, ds$$

$$\xrightarrow[n \to \infty]{} \int_0^{F^{-1}(p)} (1 - F(s))\, ds = T_F(p) \text{ almost surely, } \quad \frac{i}{n} \to p$$

uniformly in $p \in (0, 1)$.

e) If only $x_{1:n}, \ldots, x_{i:n}$, $i < n$, are observed (the i smallest claims, say), then

$$\frac{n}{i}\, T\left(\frac{i}{n}\right)$$

is the unique minimum variance unbiased estimator of μ.

In the following figure, scaled TTT transforms for different distributions are plotted.

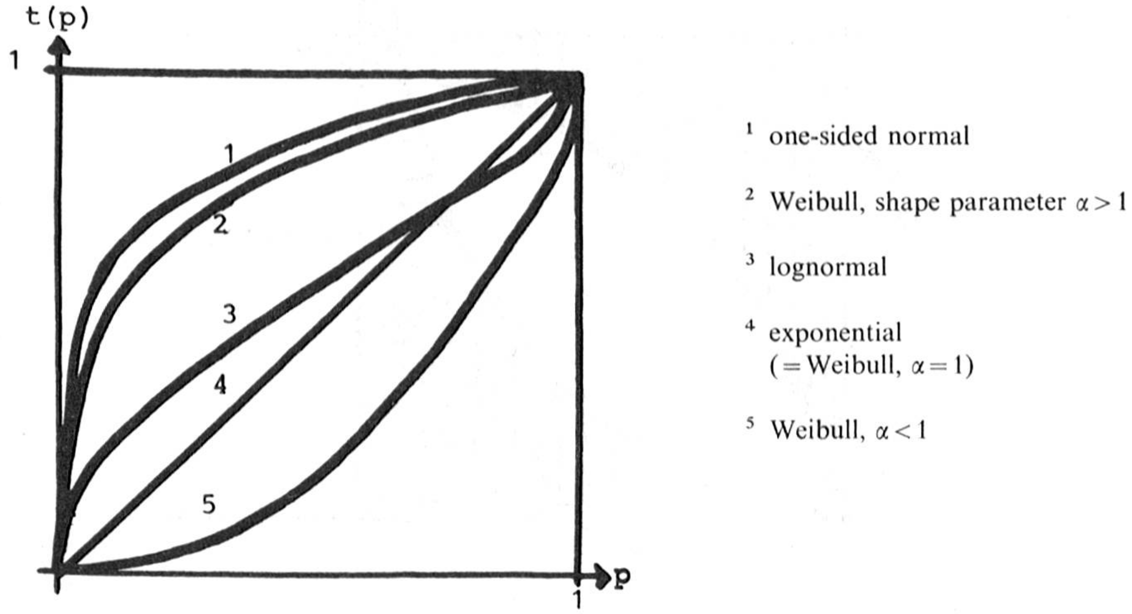

Figure 2

3. The «dual» Lorenz curve

In view of the above interpretation of the Lorenz curve, it is obvious to consider the transform

$$m_F(p) = m(p) = 1 - l(1-p), \qquad p \in (0,1),$$

which is «dual» to the Lorenz curve l in that it describes the fraction of the aggregate claims caused by the 100 p% of the treaties with the highest claim size. Clearly,

$$m(p) = \int_{F^{-1}(p)}^{\infty} (1 - F(s))\,ds + pF^{-1}(1-p) = \int_{1-p}^{1} F^{-1}(s)\,ds.$$

In the following figure, we assume $\mu = 1$ (or we consider the distribution function G given by

$$G(x) = F(\mu x),$$

yielding $l_F = l_G$ and $m_F = m_G$).

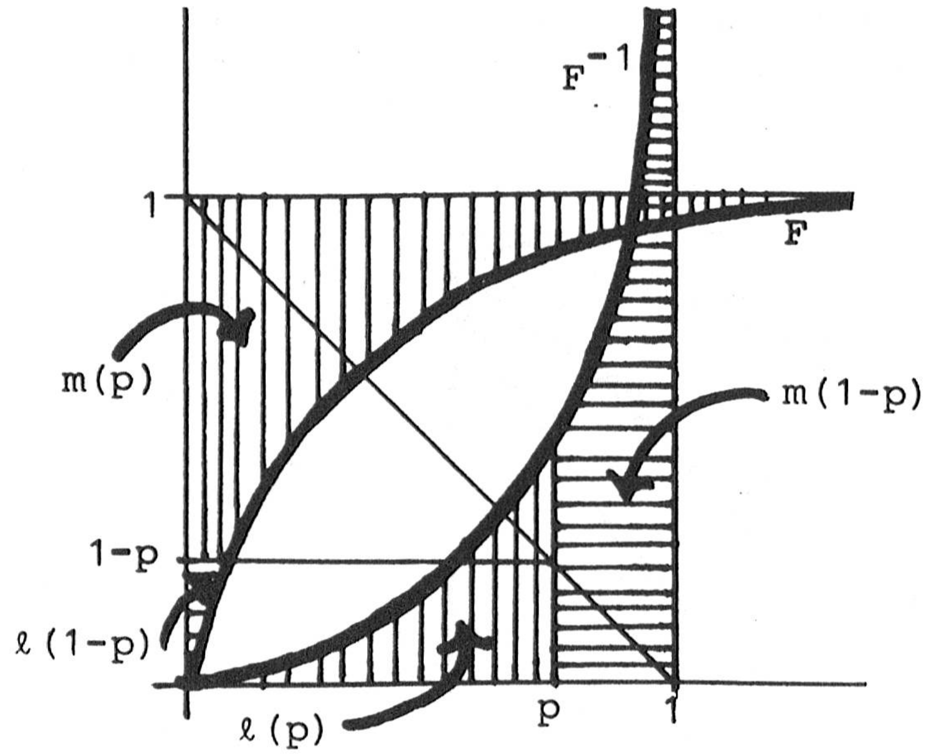

Figure 3

Examples

1. Let $F(t) = 1_{[c,\infty)}(t)$ (total mass in c). Then

$$l(p) = m(p) = p,$$

$$t(p) = 1.$$

2. Let $F(t) = \dfrac{t}{2\mu}$, $0 \le t \le 2\mu$ (uniform). Then

$$l(p) = p^2,$$

$$m(p) = 1 - (1-p)^2,$$

$$t(p) = p(2-p).$$

3. Let $F(t) = 1 - e^{-t/\mu}$, $t > 0$, (exponential). Then

$$l(p) = p + (1-p) \ln (1-p), \tag{9}$$

$$m(p) = p - p \ln p, \tag{10}$$

$$t(p) = p.$$

4. We now consider a Pareto-type distribution which is introduced most conveniently by its failure rate

$$r(t) = \frac{\beta}{\alpha + t}, \qquad t > 0.$$

Here, $\alpha > 0$ and, to make sure that the expectation is finite, $\beta > 1$, yielding $\mu = \alpha/(\beta - 1)$.
Then

$$t(p) = 1 - (1-p)^{\frac{\beta-1}{\beta}}, \tag{11}$$

$$l(p) = \beta\left(1 - (1-p)^{\frac{\beta-1}{\beta}}\right) - p(\beta - 1), \tag{12}$$

$$m(p) = \beta p^{\frac{\beta-1}{\beta}} - p(\beta - 1)$$

In the special case $\beta = 2$, we obtain

$$t(p) = 1 - \sqrt{1-p},$$

$$l(p) = 2(1 - \sqrt{1-p}) - p = (t(p))^2,$$

$$m(p) = 2\sqrt{p} - p.$$

These curves, together with (9) and (10), are displayed in the following figure.

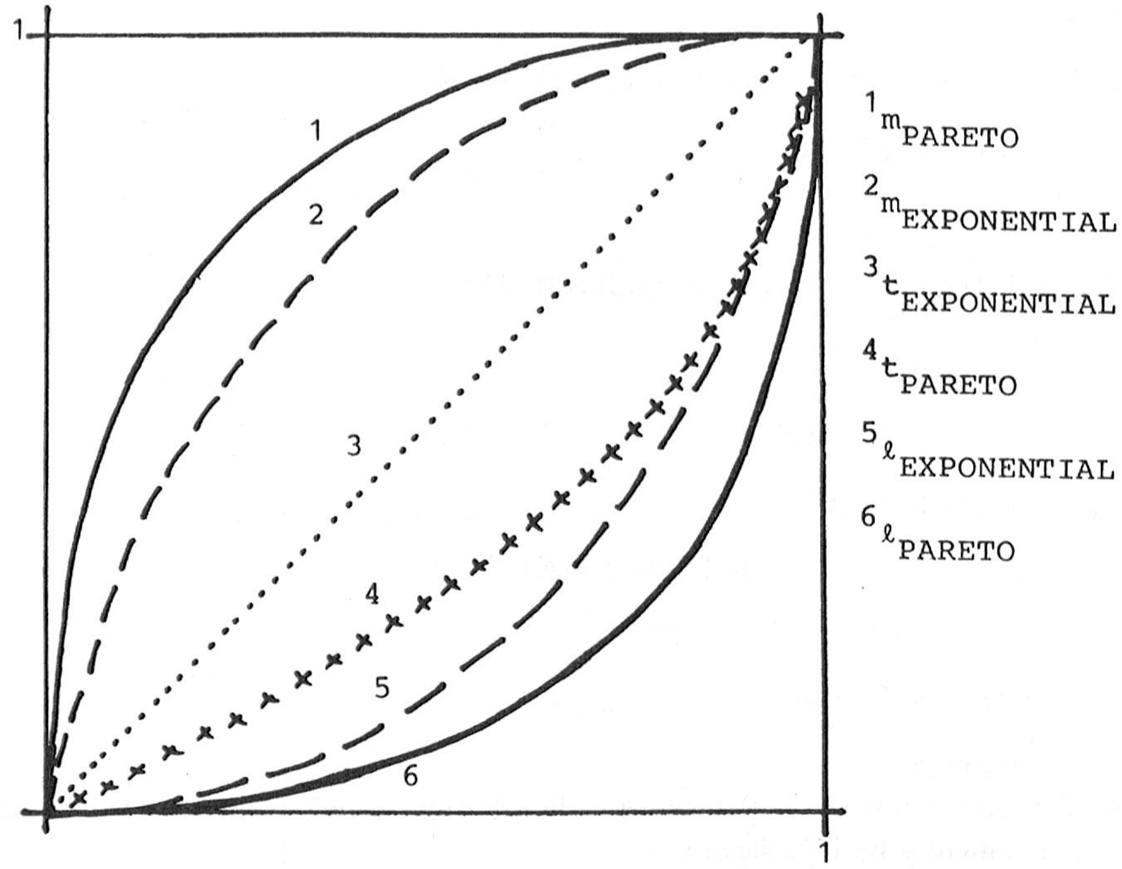

Figure 4

Since, in the description of l, low rates are related to low payments (claims or income, say) whereas in m, low rates are related to high payments, l and m provide lower and upper bounds, respectively, for curves relating rates, degrees, or fractions.

If, for instance, in excess of loss reinsurance, e.g. in the property or transportation business with fixed insurable values or sums insured, exposure rating is performed, curves of the following type arise (cf. *Gerathewohl* [4]).

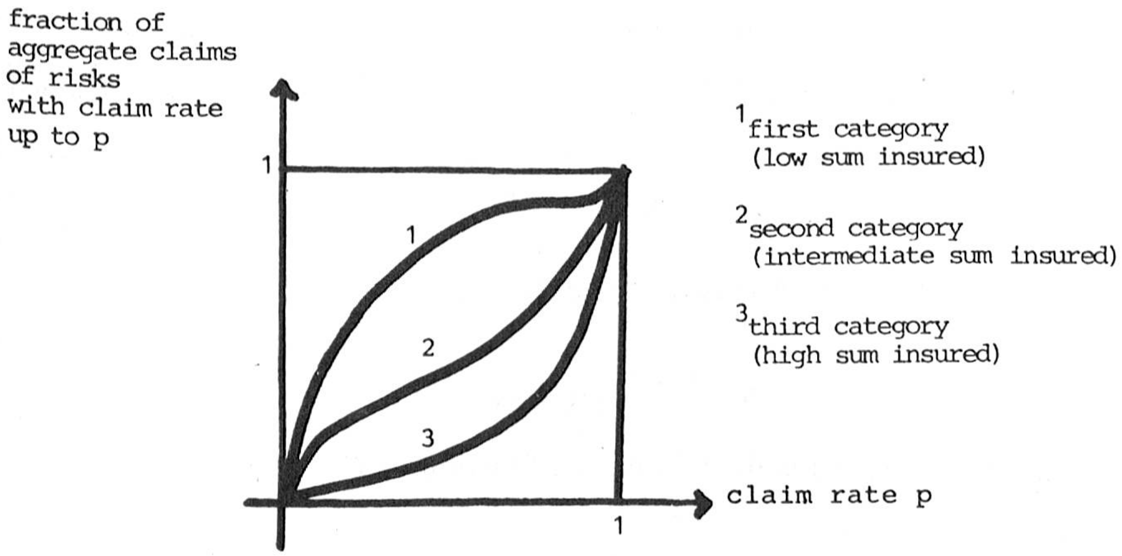

Figure 5

Here, different curves represent different risk categories (established by the sums at risk). If, for some category, p is the claim rate above which the XL treaty is exposed, the curve for that category specifies the related portion of the aggregate claims.

As indicated above, the Lorenz and the «dual» Lorenz curve, l and m, are, in a sense, lower and upper bounds for curves of that type, which, on the other hand, resemble very much the scaled TTT transforms of Figure 2.

4. Reliability and order relations

In this final section we shall point to some connections between the above concepts and reliability theory and the ordering of random variables which may be useful in risk theory. Most results are contained in *Barlow/Campo* [1], *Chandra/Singpurwalla* [3], *Klefsjö* [5], and *Taillie* [8].

In risk theory, heavy-tailed distributions are particularly important. In a sense, distributions with decreasing failure rate (DFR) belong to this class of «dangerous» distributions, cf. *Vännman* [9]. In reliability theory, the hierarchy

decreasing failure rate (DFR)

$\Rightarrow$ decreasing failure rate average (DFRA)

$\Rightarrow$ new worse than used (NWU)

$\Rightarrow$ new worse than used in expectation (NWUE)

$\Rightarrow$ harmonic new worse than used in expectation (HNWUE)

is well-known, cf. *Klefsjö* [5], e.g., and the following results can be obtained.

Proposition

1. If F is strictly monotone and has a density f,

$$\frac{d}{dp}\,T(p)=\frac{1}{r(F^{-1}(p))},\qquad p\in(0,1),$$

where $r=f/(1-F)$ is the failure rate function of F.

2. F DFR $\Rightarrow T$ convex.

3. F DFRA $\Rightarrow p\rightarrow\dfrac{T(p)}{p}$ is nondecreasing.

4. F NWUE $\Leftrightarrow t(p)\leq p,\quad p\in(0,1).$ (13)

5. F HNWUE $\Leftrightarrow l(p)\leq p+(1-p)\ln(1-p).$ (14)

Note that the right hand sides of the inequalities in (13) and (14) are the TTT transform and the Lorenz curve of the exponential distribution, respectively.

Remark. In *Chandra/Singpurwalla* [3], the distribution function

$$F(x)=1-x^{-\alpha},\qquad x\geq 1,$$

where $\alpha>1$, is considered, which defines a different kind of Pareto distribution than the one introduced above, with failure rate

$$r(x)=\frac{\alpha}{x},\qquad x\geq 1,$$

and

$$l(p)=1-(1-p)^{\frac{\alpha-1}{\alpha}},$$

$$m(p)=p^{\frac{\alpha-1}{\alpha}},$$

$$t(p)=1-\frac{1}{\alpha}(1-p)^{\frac{\alpha-1}{\alpha}}.$$

Since $r(x)=0$ for $x\in(0,1)$, this distribution is not DFR in the strict sense (one might call it «shifted DFR» as suggested by *B. Klefsjö* in a private communication), and not even HNWUE. Thus, the above proposition does not apply to F (incidentally, (13) and (14) are violated for $\alpha = 2$, say) as stated erroneously in [3].

Ordering of risks is one main topic of modern risk theory, and many different kinds of order relations on sets of nonnegative risk variables have been considered, e.g. the *stop-loss ordering*

$$X <_1 Y \Leftrightarrow S_X(t) \leq S_Y(t), \qquad t \in \mathbb{R},$$

or

$$X <_2 Y \Leftrightarrow 1 - F_X(t) \leq 1 - F_Y(t), \qquad t \in \mathbb{R}.$$

Another one would be

$$X <_3 Y \Leftrightarrow l_Y(p) \leq l_X(p), \qquad p \in (0, 1),$$

or

$$X <_4 Y \Leftrightarrow \tilde{X} <_2 \tilde{Y}, \text{ where } \tilde{X} \text{ resp. } \tilde{Y}$$

denotes a random variable with density function

$$1 - F_{X/EX} \text{ resp. } 1 - F_{Y/EY} \text{ on } (0, \infty), \text{ cf. (8a).}$$

The following proposition holds for nonnegative random variables X, Y with nonzero finite expectations EX, EY (cf. *Taillie* [8]).

Proposition

$$\frac{X}{EX} <_1 \frac{Y}{EY} \Leftrightarrow \frac{X}{EX} <_3 \frac{Y}{EY} \Leftrightarrow X <_3 Y \Leftrightarrow X <_4 Y.$$

Finally, we shall prove the following simple property of the *Lorenz ordering* $<_3$ with respect to the Pareto class. (Similarly, a *TTT ordering* could be established.)

Proposition. Let X, Y be random variables with failure rates

$$r_X(t) = \frac{\beta}{\alpha + t}, \qquad r_Y(t) = \frac{\gamma}{\alpha + t}, \qquad t > 0,$$

respectively, where $\alpha > 0$ and $\beta, \gamma > 1$. Then

$$X <_3 Y \Leftrightarrow \gamma \leq \beta.$$

Proof

$$l_X(p)=\beta[1-p-(1-p)^{\frac{\beta-1}{\beta}}]+p$$
$$=\beta[(1-p)(1-(1-p)^{-\frac{1}{\beta}})]+p,$$

and, with $\delta=1/\beta$, $a=1/(1-p)$,

$$\beta(1-(1-p)^{-\frac{1}{\beta}})$$
$$=\frac{1}{\delta}(1-a^{\delta})$$
$$=\frac{1}{\delta}\left(1-\left(1+\delta\ln a+\frac{(\delta\ln a)^2}{2}+\ldots\right)\right)$$
$$=\ln(1-p)-\frac{1}{\beta}\cdot\frac{(\ln(1-p))^2}{2}-\ldots,$$

which is increasing in β.

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Summary

In the present note, for two well-known transforms of claim (or life) distributions, viz. the Lorenz curve and the total time on test transform, applications and interpretations in risk theory are displayed.

Zusammenfassung

In der vorliegenden Arbeit werden für zwei wohlbekannte Transformationen von Schaden- (oder Lebensdauer-) Verteilungen, nämlich die Lorenz-Kurve und die sogenannte TTT-Transformierte, risikotheoretische Anwendungen und Interpretationen angegeben.

Résumé

La présente note propose des applications et des interprétations relevant de la théorie du risque pour deux transformées classiques de distributions de sinistres (ou de durée de vie), à savoir pour la courbe de Lorenz et pour la transformée TTT.