On stop-loss premiums and negative claim amounts

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In risk theory, one often encounters the following situation. Let N be the number of claims in an insurance portfolio during some fixed period, and let Y_i be the amount of the i-th of these claims. Then

$$X = \sum_{i=1}^{N} Y_i$$

ist the total claim amount. (We silently interpret $\sum_{i=1}^{0} = 0$.) It is assumed that $Y_1, Y_2, \ldots$ are independent and identically distributed and independent of N, and that N is Poisson distributed with parameter λ. We further assume that the common distribution of the Y_i's is arithmetic; for simplicity we assume that its span is equal to one.
Let

$$h(y) = Pr(Y_i = y)$$
$$f(y) = Pr(X = y)$$

for all integers y, and let

$$m_1 = \max\{x:\ h(x) > 0\}$$
$$m_2 = \max(0, -\min\{x:\ h(x) > 0\}).$$

We assume that $m_1 \geqslant 0$.
If $m_2 = 0$, $f(x)$ may be computed by the recursive algorithm described by *Panjer* (1981); if $h(0) > 0$, with the modification mentioned by *Sundt/Jewell* (1981).
For the case $m_2 > 0$, *Sundt/Jewell* (1981) suggested the following procedure. Let

$$Y_i^+ = \max(0, Y_i) \qquad Y_i^- = \max(0, -Y_i)$$

$$X^+ = \sum_{i=1}^{N} Y_i^+ \qquad X_i^- = \sum_{i=1}^{N} Y_i^- .$$

Then

$$X = X^+ - X^-.$$

It can easily be shown (by e.g. characteristic functions) that X^+ and X^- are independent, and thus

$$F(x) = \sum_{z=\max(0,-x)}^{\infty} f^+(x+z)f^-(z), \tag{1}$$

where f^+ and f^- denote the discrete densities of X^+ and X^- respectively. The densities f^+ and f^- may be computed by the recursive algorithm. Hence f may be computed by two applications of the recursive algorithm and one convolution.

The problem with the procedure outlined in the previous paragraph, is that it involves the infinite sum in (1). Therefore Hürlimann (1985) proposes to approximate $f(x)$ by

$$f^*(x) = \begin{cases} \sum_{z=\max(0,-x)}^{T} f^+(x+z)f^-(z) & (x \geqslant -T) \\ 0. & (x < -T) \end{cases} \tag{2}$$

From this approximation Hürlimann computes stop-loss premiums for the portfolio, and he gives an upper bound for the error produced by the use of f^*. For the deduction of the error bound, he assumes that

$$f^-(x) \leqslant f^-(T). \qquad (\forall x \geqslant T)$$

This assumption seems a bit awkward as one would have to compute $f^-(x)$ for $x \geqslant T$ to check that it is satisfied. It seems that one would have to compute $f^-(x)$ up to a z satisfying

$$F^-(z) \geqslant 1 - f^-(T),$$

where F^- denotes the cumulative distribution of X^-. The main purpose of the present note is to present a procedure that avoids this difficulty.

The stop-loss premium of X with retention t is defined as

$$SL_X(t) = E\max(0, X-t), \tag{3}$$

and we may of course also write $SL_X(t)$ as

$$SL_X(t) = EX - t + E\max(0, t-X). \tag{4}$$

To find upper and lower bounds for $SL_X(t)$, let

$$X^{-\prime} = \min(X^-, T) \qquad X' = X^+ - X^{-\prime}.$$

We clearly have

$$X^{-\prime} \leqslant X^- \qquad\qquad X' \geqslant X,$$

and from this and (3) and (4) we obtain

$$\underline{SL}_X(t) \leqslant SL_X(t) \leqslant \overline{SL}_X(t)$$

with

$$\underline{SL}_X(t) = EX - t + Emax(0, t - X')$$
$$\overline{SL}_X(t) = Emax(0, X' - t).$$

Furthermore we have

$$D_X(t) = \overline{SL}_X(t) - \underline{SL}_X(t) = EX' - EX = EX^- - EX^{-\prime}$$
$$= E[X^- - \min(X^-, T)] = Emax(0, X^- - T),$$

that is,

$$D_X = SL_{X^-}(T),$$

where we have dropped the argument t as we see that $D_X(t)$ is independent of t. We note that D_X goes to zero when T goes to infinity. In most practical applications the distribution of X^- will have a light tail relative to the right tail of the distribution of X, and it is therefore believed that for reasonably large T, $\underline{SL}_X(t)$ and $\overline{SL}_X(t)$ will give acceptable approximations to $SL_X(t)$.
For the computations we need the disrete density f' of X'. Let $f^{-\prime}$ denote the discrete density of $X^{-\prime}$. We have

$$f^{-\prime}(x) = \begin{cases} f^-(x) & (x < T) \\ 1 - F(T-1) & (x = T) \\ 0, & (x > T) \end{cases}$$

which gives

$$f'(x) = \sum_{z=\max(0,-x)}^{\infty} f^+(x+z) f^{-\prime}(z)$$

$$f'(x)=\begin{cases}\sum_{z=\max(0,-x)}^{T-1} f^{+}(x+z)f^{-}(z)+f^{+}(x+T)(1-F(T-1)) & (x\geqslant -T)\\ 0. & (x<-T)\end{cases} \qquad (5)$$

For all integers $t\geqslant -T$ we have

$$\underline{SL}_X(t)=EX-t+\sum_{x=-T}^{t-1}(t-x)f'(x).$$

From this we see that $\underline{SL}_X(t)$ may be computed recursively by

$$\underline{SL}_X(-T)\ =EX+T$$

$$\underline{SL}_X(t+1)=\underline{SL}_X(t)-1+F'(t),$$

where F' denotes the cumulative distribution of X'. For the starting value we use that

$$EX=\lambda EY_1=\lambda\sum_{y=-m_2}^{m_1} yh(y).$$

When we have found $\underline{SL}_X(t)$, we compute $\overline{SL}_X(t)$ most easily by

$$\overline{SL}_X(t)=\underline{SL}_X(t)+D_X.$$

We also want bounds for $F(t)$ and $f(t)$.
For $F(t)$ we have

$$F(t)=SL_X(t+1)-SL_X(t)+1\leqslant\overline{SL}_X(t+1)-\underline{SL}_X(t)+1$$
$$=\underline{SL}_X(t+1)+D_X-\underline{SL}_X(t)+1=F'(t)+D_X.$$

On the other hand, as $X'\geqslant X$, we have $F'(t)\leqslant F(t)$, and thus

$$F'(t)\leqslant F(t)\leqslant F'(t)+D_X.$$

For $f(t)$ we have

$$f(t)=F(t)-F(t-1)\leqslant F'(t)+D_X-F'(t-1)=f'(t)+D_X.$$

From (1) and (2) we see that $f^*(t)\leqslant f(t)$, and from (2) amd (5) we get

$$f^*(t) = f'(t) - f^+(t+T)(1 - F^-(T)).$$

Thus

$$f'(t) - f^+(t+T)(1 - F^-(T)) \leq f(t) \leq f'(t) + D_X.$$

With a slight abuse of notation, let us introduce $D_X(T)$ for D_X to stress the dependence on T. Then we have the recursion

$$D_X(0) = EX^-$$

$$D_X(T+1) = D_X(T) - 1 + F^-(T).$$

For the starting value we use that

$$EX^- = \lambda E Y_1^- = \lambda \sum_{y=1}^{m_2} y h(-y).$$

In practical applications it could be convenient to perform the recursion before determining the value of T, and determine the value such that $D_X(T)$ is sufficiently small.

For the numerical example presented by *Hürlimann* (1985), we found the values of $D_X(T)$ displayed in Table 1. As in that example the span of the severity distribution is different from 1, we let T be the number of points used in the distribution of the aggregate negative claim whereas $D_X(T)$ is the stop-loss premium of this claim.

Table 1

T	$D_X(T)$
0	5000
10	619
20	39.1
30	2.08
40	0.107
50	0.00419
60	$1.45 \cdot 10^{-4}$
70	$4.82 \cdot 10^{-6}$
80	$1.42 \cdot 10^{-7}$
90	$3.78 \cdot 10^{-9}$
100	$7.83 \cdot 10^{-11}$

In his computations Hürlimann uses $T = 100$, for which our bounds would give an accuracy of aboout $8 \cdot 10^{-11}$. As the stop-loss premiums displayed by

Hürlimann were rounded to the nearest integer, we see that $T = 40$ would have been more than sufficient. We also believe that a great T would increase numerical inaccuracies like rounding errors, etc.
We conclude this note by mentioning the extensive study on the computation of compound distributions with two-sided serverities performed by *Milidiu* (1985).

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Abstract

The present note gives upper and lower bounds for the stop-loss premium when the total claim amount is compound Poisson distributed with arithmetic severity distribution allowing negative severities. We also give bounds for the cumulative distribution and the discrete density of the total claim amount.

Zusammenfassung

Der vorliegende Artikel liefert obere und untere Schranken für Stop-Loss-Prämien bei zusammengesetzt Poisson-verteiltem Gesamtschaden mit arithmetischer Einzelschadenverteilung, wobei auch negative Risikosummen zugelassen sind. Zudem werden Schranken angegeben für die Verteilungsfunktion sowie die diskrete Dichte des Gesamtschadens.

Résumé

La présente note propose des bornes supérieures et inférieures pour les primes stop-loss lorsque la charge totale des sinistres est de type Poisson-composé, admettant des valeurs positives et négatives ordonnées arithmétiquement. L'auteur signale également des bornes pour les probabilités attachées aux diverses valeurs possibles de la charge totale et pour les probabilités cumulées.