A partial ordering of claim amount distributions and its relation to ruin probabilities in the Poisson model

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For a compound Poisson process S_t, $t \geqslant 0$, specified by the Poisson parameter $\lambda > 0$ and the claim amount distribution Q on $(0, \infty)$ with finite mean, and the total of premiums $c > 0$ received in the time interval $(0, 1)$, the probability of ruin, given the initial reserve $x \geqslant 0$, is

$$\psi(x) = \mathfrak{P}\{x + ct - S_t < 0 \quad \text{for some} \quad t > 0\} \tag{1}$$

where $x + ct - S_t$, $t \geqslant 0$, represents the so-called surplus process. (Observe that $\psi(x)$ does depend on λ, Q and c.)
It is the purpose of this paper to relate the ruin probabilities (1) for two surplus processes and to characterize this relation according to a suitable partial ordering of the corresponding claim amount distributions. For easier reference we at first quote some standard results on the ruin probability (1). We have
(a)

$$\psi(x) = \mathfrak{P}\left\{\sup_n \sum_{i=1}^{n} (X_i - c\,W_i) > x\right\},$$

where $X_1, X_2, \ldots$ are i.i.d. with common distribution Q, $W_1, W_2, \ldots$ are i.i.d. exponentially distributed with mean $1/\lambda$, and $X_1, X_2, \ldots$ are stochastically independent of $W_1, W_2, \ldots$.
(b) If $c \leqslant \lambda\mu(Q)$, where $\mu(Q)$ denotes the mean of Q, then $\psi(x) = 1$, $x \geqslant 0$. (Observe that $\operatorname{Var}(X_1 - c\,W_1) = \operatorname{Var}(X_1) + (c/\lambda)^2 > 0$, which gives $\mathfrak{P}\{X_1 - c\,W_1 = 0\} < 1$.)
(c)

$$\psi(0) = \min\left(1, \frac{\lambda\mu(Q)}{c}\right).$$

(This follows from (b) and formula (2.9), p. 113, in *Gerber* [2], respectively.)
(d) For $c > \lambda\mu(Q)$ we have

$$\psi(x) = (1-p) \sum_{n=1}^{\infty} p^n D^{*n}(x, \infty), \qquad x \geqslant 0, \tag{2}$$

where

$$p = \frac{\lambda\mu(Q)}{c} = \psi(0) \tag{3}$$

and where D is the distribution on $[0, \infty]$ with Lebesgue-density

$$y \to \frac{1}{\mu(Q)} Q(y, \infty)\, 1_{[0, \infty)}(y). \tag{4}$$

(This immediately follows from the renewal equation (3.7), p. 115, in *Gerber* [2] in connection with the known representation of the solution of such an equation.)

Definition: For two claim amount distributions Q_1, Q_2 on $(0, \infty)$ with finite means we write

$$Q_1 \leqslant Q_2$$

if

$$D_1(x, \infty) \leqslant D_2(x, \infty) \quad \text{for all } x \geqslant 0 \tag{5}$$

where D_i, $i = 1, 2$, are the associated distributions with Lebesgue-densities given according to (4).

Remark 1:

(i) Relation (5) represents the classical partial ordering of distributions, called relation of stochastic dominance.

(ii) $Q_1 \leqslant Q_2$ and $Q_2 \leqslant Q_1$ imply $Q_1 = Q_2$.
(This follows from the fact that the Lebesgue-densities (4) are continous from the right).

(iii) $Q_1 \leqslant Q_2$ is equivalent to

$$D_1^{*k}(x, \infty) \leqslant D_2^{*k}(x, \infty), \quad x \geqslant 0, \quad k = 1, 2, \ldots \tag{6}$$

A result on the implication of the considered partial ordering of claim amount distributions to ruin probabilities has been derived in *Taylor* [3], who shows that "Q_1 is more dangerous than Q_2" (in the sense of *Bühlmann* et al. [1]) and $\mu(Q_1) = \mu(Q_2)$ imply $Q_1 \leqslant Q_2$, and that this in turn yields

$$\psi_1(x) \leqslant \psi_2(x), \quad x \geqslant 0,$$

for the corresponding ruin probabilities, given that

$$c_i = (1+\beta)\lambda_i \mu(Q_i), \quad i = 1, 2$$

(where $\beta > 0$ is fixed).
As result of this paper we have the following complete characterization of the circumstances.

Theorem: For two claim amount distributions Q_1, Q_2 on $(0, \infty)$ with finite means,

$$Q_1 \leqslant Q_2$$

is equivalent to

$$\psi_1(x) \leqslant \psi_2(x), \quad x \geqslant 0,$$
$$\text{for all } \lambda_i,\ c_i > 0,\ i = 1, 2, \text{ with } c_2 \lambda_1 \mu(Q_1) \leqslant c_1 \lambda_2 \mu(Q_2). \tag{7}$$

Remark 2:
(i) Obviously, $\mu(Q_1) = \mu(Q_2)$ and $c_i = (1+\beta)\lambda_i \mu(Q_i)$, $i = 1, 2$, imply $c_2 \lambda_1 \mu(Q_1) = c_1 \lambda_2 \mu(Q_2)$.
(ii) The condition on the quantities λ_i, c_i, $i = 1, 2$, in (7) is necessary in the following sense:
If $\psi_1(x) \leqslant \psi_2(x)$, $x \geqslant 0$, and if there exists $x_0 \geqslant 0$ with $\psi_1(x_0) < 1$ then

$$c_2 \lambda_1 \mu(Q_1) \leqslant c_1 \lambda_2 \mu(Q_2). \tag{8}$$

To see this we note that according to (b),

$$c_1 > \lambda_1 \mu(Q_1).$$

Hence, $c_2 \leqslant \lambda_2 \mu(Q_2)$ gives (8). In the case $c_2 > \lambda_2 \mu(Q_2)$ we use $\psi_1(0) \leqslant \psi_2(0)$ and apply (c) to obtain (8).

Proof of the theorem:
(i) At first we assume that $Q_1 \leqslant Q_2$ and that λ_i, $c_i > 0$, $i = 1, 2$, fulfill (8). If

$$c_2 \leqslant \lambda_2 \mu(Q_2),$$

then (b) gives $\psi_2(x)=1$, $x\geqslant 0$, i.e. $\psi_1(x)\leqslant\psi_2(x)$, $x\geqslant 0$. Hence, we may assume in the following that

$$c_2>\lambda_2\mu(Q_2). \tag{9}$$

With

$$a_k=D_1^{*k}(x,\infty). \quad k=1,2,\ldots \tag{10}$$

(where $x\geqslant 0$ is fixed) let

$$f(p)=(1-p)\sum_{k=1}^{\infty}a_kp^k, \quad 0<p<1.$$

Then

$$\begin{aligned} f'(p) &= \sum_{k=1}^{\infty}a_kp^{k-1}[k-(k+1)p] \\ &= a_1+\sum_{k=1}^{\infty}(k+1)(a_{k+1}-a_k)p^k. \end{aligned}$$

Since $0\leqslant a_k\leqslant a_{k+1}$, $k=1,2,\ldots$, we have

$$f'(p)\geqslant 0, \quad 0<p<1. \tag{11}$$

Let

$$p_i=\frac{\lambda_i\mu(Q_i)}{c_i}, \quad i=1,2$$

From (8) and (9),

$$0<p_1\leqslant p_2<1.$$

According to (2), this and (11) imply

$$\psi_1(x)=f(p_1)\leqslant f(p_2)\leqslant\psi_2(x),$$

where the last inequality follows from $a_k\leqslant D_2^{*k}(x,\infty)$, $k=1,2,\ldots$ (see (10) and (6)).

(ii) We now assume that (7) holds true. At first observe that $p=\lambda\mu(Q)/c<1$ and (2) yield

$$p\,D(x,\infty)\leqslant\psi(x)\leqslant p\,D(x,\infty)+p^2,\qquad x\geqslant 0. \tag{12}$$

(Here we have used $D^{*k}(x,\infty)\geqslant D(x,\infty)$, $k=1,2,\ldots$, and $D^{*k}(x,\infty)\leqslant 1$, $k=2,3,\ldots$).
Exploiting (7) for

$$c_1=\lambda_2=1,\quad c_2=\frac{1}{\lambda_1}\frac{\mu(Q_2)}{\mu(Q_1)},\quad 0<\lambda_1<\frac{1}{\mu(Q_1)}$$

and by setting $p=\lambda_1\mu(Q_1)$ we obtain from (7) and (12) for all $p\in(0,1)$ and all $x\geqslant 0$,

$$p\,D_1(x\,\infty)\leqslant\psi_1(x)\leqslant\psi_2(x)\leqslant p\,D_2(x,\infty)+p^2,$$

i.e.

$$D_1(x,\infty)\leqslant D_2(x,\infty)+p,\qquad x\geqslant 0,\qquad 0<p<1$$

which gives

$$D_1(x,\infty)\leqslant D_2(x,\infty),\qquad x\geqslant 0.$$

Remark 3:
A closer look at (11) reveals that $f'(p)>0$, $0<p<1$, unless $f(p)=0$, $0<p<1$. Hence, we also have the following result:
If $Q_1\leqslant Q_2$, then $c_2\lambda_1\mu(Q_1)<c_1\lambda_2\mu(Q_2)$ and $\psi_1(x)<1$, $\psi_2(x)>0$ imply

$$\psi_1(x)<\psi_2(x).$$

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Abstract

To a claim size distribution Q with mean μ we associate the distribution with Lebesgue-density $(1/\mu)\,Q(x, \infty)$, $x > 0$. (This is the ladder-height-distribution of the renewal process corresponding to the given Poisson-process). It is shown that the classical partial ordering (stochastic dominance) of these associated distributions represents a partial ordering of the claim size distributions, and that it completely characterizes a certain relation of the corresponding probabilities of ruin.

Zusammenfassung

Einer Schadenhöhenverteilung Q mit einem Mittelwert μ ordnen wir die Verteilung mit der Lebesgue-Dichte $(1/\mu)\,Q(x, \infty)$, $x > 0$, zu. (Es handelt sich dabei um die Leiterhöhenverteilung zu dem dem Poisson-Prozess entsprechenden Erneuerungsprozess.) Gezeigt wird, dass die klassische partielle Ordnung (stochastische Dominanz) dieser zugeordneten Verteilungen eine partielle Ordnung der Schadenhöhenverteilungen darstellt und dass dadurch ein Vergleich der entsprechenden Ruinwahrscheinlichkeiten vollständig charakterisiert ist.

Résumé

L'auteur associe à une distribution Q du montant d'un sinistre, de moyenne μ, la distribution – selon Lebesgue – de densité $(1/\mu)\,Q(x, \infty)$, $x > 0$ (ce qui fournit la distribution de la hauteur d'échelle du processus de renouvellement associé au processus de Poisson considéré). Il montre ensuite que l'ordonnancement partiel classique (dominance aléatoire) de ces distributions associées représente un ordonnancement partiel des distributions des montants des sinistres, et que ce moyen caractérise entièrement une certaine relation existant entre les probabilités de ruine correspondantes.