A very practical solution to the retention problem for an excess-of-loss treaty

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1 Introduction

A quite old problem of reinsurance mathematics is the fixing of the retention for a given reinsurance cover. Already de Finetti dealt with the problem in 1940, at the 1954 International Congress of Actuaries at least three essential contributions were given (see *Bjerreskov* [1954], *Pentikäinen* [1954] and *Wilhelmsen* [1954]).
Different criteria were used for determining the adequate or optimal retention. Many authors use the *loss probability* for one year and choose the retention such that this probability is equal to a given value (see *Pentikäinen* [1954], *Russell* et al. [1980]). Some authors use the concept of *utility theory* (see *Röbbert* [1976], *Boyle* et al. [1982] and *Stuart* [1983]), others ideas of the *consumption theory* (see *Moffet* [1977]) and *reliability theory* (see *Heilmann* [1982]). Also *cost theoretical* arguments were applied (see *Heilmann* [1986] and *Schmitter* [1984]). Nevertheless most authors use models and results of a classical branch of risk theory, the so-called *ruin theory*. Worth mentioning are e.g. the publications of *Wilhelmsen* (1954), *Bühlmann* (1970), *Straub* (1978) and Andreadakis et al. (1980). They all propose to calculate the retention such that the ruin probability is equal to a given value or is minimized under some additional constraints or conditions. In the author's opinion this last approach is quite elegant, but it's results often are not practicable enough from the reinsurer's point of view. Some fundamental distributions and parameters are not known to the reinsurer and the important interest rates are not included in the basic models. These drawbacks led the author to reconsider the problem of retention determination and to develop a comparably simple, more practicable procedure, based on a relatively unknown result of the Swiss Professor *Gerber* (1971).

2 The Basic Result

We look at a collective of insurance risks, producing claims each year. The *number of claims* until time point t shall be denoted by the random variable

N_t, the corresponding *claims amounts* by the i.i.d. random variables X_1, X_2, X_3, ... We assume that N_t is independent of the claims sizes and *Poisson-distributed* with parameter $\lambda \cdot t > 0$, this means

$$\mathrm{Prob}(N_t = n) = \frac{(\lambda \cdot t)^n}{n!} \cdot \exp(-\lambda \cdot t)$$

for $n = 0, 1, 2, \ldots$. The yearly premium income is denoted by c, the mean claims size by

$$\mu = E(X_i).$$

Finally we use the symbol F for the distribution function of the individual claim size, this means

$$F(x) = \mathrm{Prob}(X_i \leq x)$$

and especially

$$E(X_i) = \int x\, F(dx).$$

We take the model like in *Gerber* (1971). This means we assume in addition to the above classical conditions that the surplus produces interest gains, continuously with the *interest rate* of $i \cdot 100\,\%$ (with $i > 0$) per period. *Ruin* of the (risk) business then means that the cumulative difference between the *initial reserve* u, the premium income minus claims payments, inclusively interest payment gains or losses, becomes negative at one time point $0 \leq t < \infty$. More formally we define the *surplus* (including interest income) up to the time point t according

$$R(t) = u \cdot \exp(i \cdot t) + \left(\frac{c}{i}\right) \cdot \left(\exp(i \cdot t) - 1\right) - \sum_{j=1}^{N_t} \exp\left(i \cdot (t - T_j)\right) \cdot X_j \tag{2.1}$$

with the occurrence time point T_j of the j-th claim (see formula (11) in *Gerber* [1971]). So we have a risk process of deflated surplusses $(R(t), t \geq 0)$ and the event *ruin* according

$$\{T < \infty\}$$

with the *time point of ruin*

$$T = \inf\{t : R(t) < 0\}.$$

For the *ruin probability*

$$\psi(u) = \text{Prob}(T < \infty).$$

Gerber (1971) showed in his Theorem 2 the practicable result

$$\lim_{(\lambda/i)\to\infty} \psi(u) = 1 - \Phi\left(\frac{u + (c - \lambda \cdot \mu)/i}{\sqrt{(\lambda/2) \cdot v^2/i}}\right) \tag{2.2}$$

with the standard normal distribution function Φ, the above defined parameters u, c, λ, μ, i and

$$v^2 = E(X_i^2).$$

Since in practive (λ/i) is usually quite large, the right hand side of (2.2) can be used as a good approximation to the ruin probability in the above model, which includes the important interest incomes.

3 The Retention Problem and a Solution

We assume that the insurer reinsures his collective by an excess-of-loss treaty with priority $P > 0$. This means that the insurer retains the total claims amount until time point $t > 0$

$$S_t = \sum_{j=1}^{N_t} Y_j$$

with the truncated claims sizes

$$Y_j = \min(X_j, P)$$

for $j = 1, 2, \ldots$. These have the mean value

$$E(Y_j) = \mu - \mu_P$$

with the mean excess claim

$$\mu_P = \int_{[P,\infty]} (x-P)\,F(dx).$$

Applying the so-called expectation principle (see *Kremer* [1985]), the *reinsurer's yearly risk premium* is given according

$$c_P = (1+l)\cdot\lambda\cdot\mu_P$$

with the loading factor l. The yearly premium remaining by the insurer obviously is

$$c_* = c - c_P. \tag{3.1}$$

Replacing in part 2 c by c_*, X_j by Y_j we get the risk process $(R_P(t), t \geq 0)$ for the insurer's collective, reinsured by an excess-of-loss treaty with priority $P > 0$. The *retention problem* now consists in how to fix or choose the priority P. As already written in the introduction, one can take a ruin probability ε for the risk process and tries to find P such that the corresponding ruin probability $\psi_P(u)$ is equal to the required value ε. Since the exact evaluation of $\psi_P(u)$ is too unhandy, one tries to find practicable approximations to $\psi_P(u)$. In the classical risk process (this means (2.1) after taking the limit $i \to 0$) previous authors took the so-called Lundberg bound (see *Bühlmann* [1970], *Straub* [1978]) as an approximation for $\psi_P(u)$. In our above model, which includes interest rates, the author proposes to take simply the result (2.2) with c_* instead of c,

$$\mu_* = \mu - \mu_P \tag{3.2}$$

instead of μ and

$$v_*^2 = v^2 - v_P^2 - 2\cdot P\cdot\mu_P \tag{3.3}$$

with

$$v_P^2 = \int_{[P,\infty]} (x-P)^2\,F(dx)$$

instead of v^2. This means to choose $P > 0$ such that

$$\Phi\left(\frac{u+(c_* - \lambda \cdot \mu_*)/i}{\sqrt{(\lambda/2)\cdot v_*^2/i}}\right) = 1-\varepsilon. \tag{3.4}$$

With the quantile $\Phi_\varepsilon = \Phi^{-1}(1-\varepsilon)$ and the definitions (3.1)–(3.3) the equation (3.4) means nothing else but

$$\left(u \cdot i + (c - \lambda \cdot \mu) - l \cdot \lambda \cdot \mu_P\right)^2 - \cdots$$

$$\cdots - \frac{\lambda}{2} \cdot i \cdot \Phi_\varepsilon^2 \cdot (v^2 - v_P^2 - 2 \cdot P \cdot \mu_P) = 0. \tag{3.5}$$

The solution to the retention problem is nothing else but the zero place of the left hand side of (3.5).
This can be determined by standard methods of numerical mathematics, e.g. by the well known *regula falsi method* (see *Stoer* [1972], chapter 5). Among more than one zero places one can take the largest one.

4 Practical Advice

The above method is suited for the situation where the reinsurer has to propose a suitable retention for an individual excess-of-loss contract, which an insurer likes to conclude. The reinsurer only has to know u, c, λ and F. Since he chooses ε, i and l by himself and can calculate μ, v^2, μ_P, v_P^2 for given P, he can determine the solution of (3.5) on his own computer. Unfortunately in practice F is not known, the reinsurer only is informed about the claims amounts exceeding a given limit P_O, smaller than the suitable priority. Furthermore λ is not known exactly. Clearly it is no problem to get estimates $\widehat{\lambda}$, $\widehat{\mu}$ and $\widehat{v}^2$ for λ, μ and v^2 from the insurer. So it only remains to compute estimates for μ_P and v_P^2 given P. For this the known past claims amounts have to be adjusted for inflation-, IBNER- and trend-effects in advance, which in practice is already standard. We denote the adjusted claims amounts by Z_j, $j = 1, 2, \ldots$. Then as the estimators $\widehat{\mu}_P$, $\widehat{v}_P^2$ for μ_P, v_P^2 one can use

$$\widehat{\mu}_P = \left(\frac{1}{N_P}\right) \cdot \sum_{j:Z_j>P} (Z_j - P)$$

$$\widehat{v}_P^2 = \left(\frac{1}{N_P}\right) \cdot \sum_{j:Z_j>P} (Z_j - P)^2$$

with the number N_P of Z_j, $j = 1, 2, \ldots$ exceeding the given P. These estimators have to be inserted into (3.5), yielding as the zero place of the left hand side the adequate priority P. In case that there is no solution above P_0, one has to reduce the limit P_0. In case that there are more than one zero places above P_0, one can take the largest one.

5 Outlook

The above method is based on the practical, but special result (2.2) of *Gerber* (1971). In the past years theoretical generalizations of that result were developed by *Harrison* (1977), *Schnieper* (1983) and *Braun* (1986). These might be used to derive more refined results on the above retention problem. In the author's opinion the above method is suitable enough for practical determination of the retention or priority. It can easily be programmed in the language BASIC for each modern micro- computer.

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References

Andreadakis, M. / Waters, H. R. (1980): The effect of reinsurance on the degree of risk associated with an insurer's portfolio. Astin Bulletin.

Bjerreskov, S. (1954): On the principles of choice of reinsurance method and for the fixing of net retention for an insurance company. Transactions of the international congress of actuaries.

Boyle, P. P. / Mao, J. (1982): Optimal risk retention under partial insurance. Insurance: Mathematics and Economics.

Braun, H. (1986): Weak convergence of assets processes with stochastic interest return. Scandinavian Actuarial Journal.

Bühlmann, H. (1970): Mathematical Methods in Risk Theory. Berlin/New York.

De Finetti, B. (1940): Il problema dei pieni. Giornale dell'Istituto degli Atturai.

Gerber, H. U. (1971): Der Einfluss von Zins auf die Ruinwahrscheinlichkeit. Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker.

Harrison, J. M. (1977): Ruin problems with compounding assets. Stochastic processes and their applications.

Heilmann, W. R. (1982): An approach to the retention problem based on claim experience. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.

Heilmann, W. R. (1986): On optimal deductibles. In: *M. Goovaerts* et al. (eds.): Insurance and Risk Theory. D. Reidel Publishing Company.

Kremer, E. (1985): Einführung in die Versicherungsmathematik. Göttingen/Zürich.

Moffet, D. (1977): Optimal deductible and consumption theory. Journal of Risk and Insurance.

Pentikäinen, T. (1954): On the reinsurance of an insurance company. Transactions of the international congress of actuaries.

Röbbert, M. (1976): Bewertung von rückversicherten Lebensversicherungsbeständen. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.

Russel, J. / Patrik, G. / Stanard, J. (1980): A practical procedure for evaluating excess-of-loss working cover reinsurance retentions. Prudential reinsurance company.

Schmitter, H. (1984): Untere Grenzen für Selbstbehalte von Schadenexzedenten. Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker.

Schnieper, R. (1983): Risk processes with stochastic discounting. Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker.

Stoer, J. (1972): Einführung in die Numerische Mathematik I. Berlin/New York.

Straub, E. (1978): How to fix retention. Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker.

Stuart, C. (1983): Insurance: The case of homeowner insurance in Sweden. Insurance: Mathematics and Economics.

Waters, H. R. (1979): Excess of loss reinsurance limits. Scandinavian Actuarial Journal.

Wilhelmsen, L. (1954): On the stipulation of maximum net retentions in insurance companies. Transactions of the international congress of actuaries.

Summary

The author reconsiders the problem of how to fix the retention for an excess-of-loss treaty. A practicable solution is given, which can be used very well by the reinsurer.

Zusammenfassung

Der Verfasser greift das Problem der Bestimmung des Selbstbehaltes eines Schadenexzedenten-Vertrages auf. Eine praktikable Lösung wird hergeleitet, welche sehr gut vom Rückversicherer verwendet werden kann.

Résumeé

L'auteur considère la question de la détermination du plein de conservation dans le cadre d'un traîté en "Excess-of-loss". Il propose une solution qui peut fort bien être utilisée par un réassureur.