Price equilibria and non linear risk allocations in capital markets

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1 Introduction

Efficient risk allocation plays an important role in actuarial sciences. After *Borch*'s seminal work (1960) risk sharing became a central theme for actuaries [see e.g. *Bühlmann/Jewell* (1979), *Gerber* (1979)]. Furthermore, *Bühlmann* (1980, 1984) proved the existence of a price system leading to a Pareto efficient risk allocation which is typically non linear.
In financial economics, the determination of optimal portfolios consisting of equities, bonds, etc. was the main objective for quite a long time. Of course, the framework of the "Capital Asset Pricing Model" excluded an analysis of non linear risk allocations. However, with the path breaking work of *Black/Scholes* (1973) the theory of options became a central issue. Moreover, options became important for practical purposes. In particular, techniques like portfolio insurance attracted the attention of institutional investors. In this context, *Leland* (1980) introduced the notion of a "general insurance policy" and referred to its close connection with a non linear allocation of financial risks.
This article deals with a model where the total financial risk of an economy has to be allocated to n investors. First of all, the theory of risk exchange and in particular *Bühlmann*'s results on price equilibria are applied to this model. Thereafter, the case where the utility functions of the investors belong to the HARA class is analysed in detail. In this case, the equilibrium price density is determined by an implicit formula generalizing earlier results by *Bühlmann* (1980) and *Lienhard* (1986). Moreover, the risk allocations of these price equilibria are studied. Some investors practice a generalized portfolio insurance strategy in the sense of *Leland* (1980), others choose an opposite policy. The policy choice can be related to the risk tolerance of the corresponding investors.
It should be pointed out that this paper follows the tradition of *Rubinstein* (1976), *Brennan* (1979) and *Leland* (1980) where trades can only take place at

discrete intervals of time. This contrasts with the *Black/Scholes* model where there is continuous trading.

2 The Model

2.1 Basic Structure

Investment takes place over one period. There are n investors $i = 1,\dots,n$ initially endowed with shares s_i $\left(\sum_{i=1}^{n} s_i = 1\right)$ in total financial wealth. Furthermore, it is assumed that all investors have the same expectations. Therefore, the expectations with respect to the value of total financial wealth at the end of the period can be represented by a random variable $\widetilde{W}$. Investors $i = 1,\dots,n$ evaluate their claims on financial wealth at the end of the period by utility functions

$$u_i : R \to R \qquad i = 1,\dots,n.$$

This framework allows for the application of the theory on risk exchange as it is presented in *Bühlmann* (1980, 1984). In our model the initial allocation of claims on financial wealth is given by

$$(s_1\widetilde{W},\dots,s_n\widetilde{W}).$$

We consider reallocations

$$(\widetilde{Z}_1,\dots,\widetilde{Z}_n), \quad \text{satisfying} \quad \sum_{i=1}^{n} \widetilde{Z}_i = \widetilde{W}$$

and it is our main objective to analyse price equilibria in this context.
In the next section the theory of risk exchange (*Borch* (1960), *Bühlmann* (1980, 1984) and others) is adapted to our model.

2.2 *Theory of Risk Exchange*

First of all we need some assumptions.

A 1: The random variable $\widetilde{W}$ has a probability density function

$$g : (m, M) \to R_{++} \text{ [1]} \tag{1}$$

with

$$0 \leq m < M < \infty$$

A 2: The utility functions

$$u_i : R \to R, \qquad i = 1, \ldots, n$$

are twice differentiable and satisfy

$$u_i'(x) > 0, \qquad u_i''(x) < 0.$$

Furthermore, we have to introduce the following definitions:

Definition 1 An n-tuple of random variables $(\widetilde{Z}_1, \ldots, \widetilde{Z}_n)$ is called a *feasible allocation* if it satisfies

$$\sum_{i=1}^{n} \widetilde{Z}_i = \widetilde{W} \tag{2}$$

Definition 2 A measurable function

$$\phi : (m, M) \to R_+$$

is called *a price density* if it satisfies

$$E[\phi(\widetilde{W})] := \int_m^M \phi(w) g(w)\, dw = 1 \tag{3}$$

[1] $R_+ = [0, \infty)$, $R_{++} = (0, \infty)$.

Remarks

1) Given a price density ϕ the value of total financial wealth $\widetilde{W}$ amounts to

$$P(\widetilde{W}) := E[\widetilde{W}\phi(\widetilde{W})] = \int_m^M w\phi(w)g(w)\,dw$$

2) Under the price density ϕ investor i faces the budget constraint

$$E\left[\widetilde{Z}_i\phi(\widetilde{W})\right] \leq E\left[s_i\widetilde{W}\phi(\widetilde{W})\right]$$

Definition 3 The tuple $\{\phi, (\widetilde{Z}_1^*, \dots, \widetilde{Z}_n^*)\}$ consisting of the price density ϕ and the feasible allocation $(\widetilde{Z}_1^*, \dots, \widetilde{Z}_n^*)$ is called a *price equilibrium* if for all $i = 1, \dots, n$ $\widetilde{Z}_i^*$ is the solution of

$$\max_{\widetilde{Z}_i} E\{u(\widetilde{Z}_i)\} \tag{4}$$

under

$$E\left[\widetilde{Z}_i\phi(\widetilde{W})\right] \leq E\left[s_i\widetilde{W}\phi(\widetilde{W})\right]$$

Definition 4 The feasible allocation $(\widetilde{Z}_1^*, \dots, \widetilde{Z}_n^*)$ is called *Pareto efficient* if there exists no feasible allocation $(\widetilde{Z}_1, \dots, \widetilde{Z}_n)$ satisfying

$$E\{u_i(\widetilde{Z}_i)\} \geq E\{u_i(\widetilde{Z}_i^*)\} \qquad i = 1, \dots, n$$

with strict inequality for at least one $i_0 \in \{1, \dots, n\}$.

Bühlmann (1984) provides us with the following results:

Theorem 1 A price equilibrium $\{\phi, (\widetilde{Z}_1^*, \dots, \widetilde{Z}_n^*)\}$ exists if the following assumptions are satisfied

1) A 1, A 2
2) The risk aversions

$$\varrho_i(x) = -\frac{u_i''(x)}{u_i'(x)}, \qquad i = 1, \dots, n$$

satisfy a Lipschitz condition, i.e. there exists a constant K, such that

$$|\,\varrho_i(x) - \varrho_i(x')\,| \le K|x - x'| \qquad \forall\, x, x' \tag{5}$$

Theorem 2 Assume
1) $\{\phi, (\tilde{Z}_1^*, \dots, \tilde{Z}_n^*)\}$ is a price equilibrium,
2) A 1, A 2 are satisfied.

Then, the allocation $(\tilde{Z}_1^*, \dots, \tilde{Z}_n^*)$ is Pareto efficient.

The next part contains a detailed analysis of price equilibria for the case where the utility functions belong to a special class.

3 Analysis of Price Equilibria

3.1 *Assumptions with respect to utility functions*

In the theory of risk exchange and in portfolio theory the class of utility functions, which are characterized by

$$\varrho(x) := -\frac{u''(x)}{u'(x)} = \frac{1}{b + Rx}, \qquad x > -\frac{b}{R} \tag{6}$$

attracted a lot of attention. This class is called the HARA class (**H**yperbolic **A**bsolute **R**isk **A**version). Obviously, hyperbolic absolute risk aversion is equivalent to *linear* risk tolerance, i.e.

$$\tau(x) := -\frac{u'(x)}{u''(x)} = b + Rx \tag{7}$$

The coefficient R is called *cautiousness*[2] in the literature.

[2] The following interpretation is possible: Under a high (low) cautiousness, a financial loss leads to a substantial (non substantial) decrease in risk tolerance.

From now on we make the following assumption:

A 3: The utility functions are increasing, concave and satisfy:

a) Linear risk tolerance

$$\tau_i(x) := -\frac{u_i'(x)}{u_i''(x)} = b_i + R_i x, \qquad i = 1, \dots, n$$

b) Non negative cautiousness

$$R_i \geq 0, \qquad i = 1, \dots, n$$
$$R_i > 0, \qquad \text{for some } i \in \{1, \dots, n\}$$

Remarks

1) By requiring a non negative cautiousness we exclude utility functions with a strictly increasing absolute risk aversion $\varrho(x)$ (e.g. quadratic utility functions). This is in accordance with *Arrow*'s postulate (1965)[3].

2) An easy calculation shows that A 3 allows essentially for the following utility functions:

$$R = 0 : u(x) = -e^{-\alpha x}, \qquad \text{with } \alpha = \tfrac{1}{b} \tag{8}$$

$$R = 1 : u(x) = \ln(x - a), \qquad \text{with } a = -b \tag{9}$$

$$0 < R < 1 : u(x) = -(x - a)^{-c+1}, \qquad \text{with } c = \tfrac{1}{R},\ a = -\tfrac{b}{R} \tag{10}$$

$$1 < R < \infty : u(x) = (x - a)^{-c+1}, \qquad \text{with } c = \tfrac{1}{R},\ a = -\tfrac{b}{R} \tag{11}$$

3) From now on we suppose that A 1 and A 3 are satisfied.

3.2 *Properties of the Equilibrium Price Density*

In addition to A 1, A 3 we assume

A 4: There exists a price equilibrium $\left\{\phi, (\widetilde{Z}_1^*, \dots, \widetilde{Z}_n^*)\right\}$

[3] It is well known that a strictly increasing absolute risk aversion leads to an unrealistic investment behaviour. For this reason *Arrow* (1965) postulated to exclude utility functions with this property.

Remark

Assumptions A 1 and A 3 do not guarantee the existence of a price equilibrium. However, in some cases (e.g. $R_i = 0, \quad i = 1, \ldots, n$) existence of a price equilibrium is no problem.
If $\left\{\phi, (\widetilde{Z}_1^*, \ldots, \widetilde{Z}_n^*)\right\}$ is a price equilibrium, then we can find $\gamma_1, \ldots, \gamma_n \in R_{++}$ such that

$$u_i'(\widetilde{Z}_i^*) = \gamma_i \phi(\widetilde{W}) \qquad i = 1, \ldots, n \tag{12}$$

(see e.g. *Bühlmann* (1980) or *Huang/Litzenberger* (1988)).
Without loss of generality, we assume that the investors $i = 1, \ldots, n$ are ordered such that

$$R_1 \geq R_2 \geq \cdots \geq R_n \tag{13}$$

Furthermore, let n_1 be the number of investors with a strictly positive cautiousness. Then, under A 3 and (13) the preferences can be represented by

$$u_i'(x) = (x - a_i)^{-c_i}, \qquad i = 1, \ldots, n_1 \qquad \text{with } a_i = -\tfrac{b_i}{R_i},\ c_i = \tfrac{1}{R_i} \tag{14}$$

$$u_i'(x) = e^{-\alpha_i x}, \qquad i = n_1 + 1, \ldots, n \qquad \text{with } \alpha_i = \tfrac{1}{b_i} \tag{15}$$

From (12), (14) and (15) we obtain the following necessary conditions for a price equilibrium

$$(\widetilde{Z}_i^* - a_i)^{-c_i} = (\lambda_i)^{-c_i} \phi(\widetilde{W}), \qquad i = 1, \ldots, n_1 \tag{16}$$

$$e^{-\alpha_i \widetilde{Z}_i^*} = e^{\mu_i} \cdot \phi(\widetilde{W}), \qquad i = n_1 + 1, \ldots, n \tag{17}$$

where

$$\lambda_1, \ldots, \lambda_{n_1} \in R_{++}$$

and

$$\mu_{n_1+1}, \ldots, \mu_n \in R$$

are constants depending on the initial allocation $(s_1 \widetilde{W}, \ldots, s_n \widetilde{W})$.

From (16), (17) we get

$$\tilde{Z}_i^* = a_i + \lambda_i \left[\phi(\widetilde{W})\right]^{-\frac{1}{c_i}} \qquad i = 1, \ldots, n_1 \tag{18}$$

$$\tilde{Z}_i^* = -\frac{\mu_i}{\alpha_i} - \frac{1}{\alpha_i} \ln[\phi(\widetilde{W})] \qquad i = n_1 + 1, \ldots, n \tag{19}$$

Together with the feasibility condition

$$\sum_{i=1}^{n} \tilde{Z}_i^* = \widetilde{W} \tag{20}$$

(18), (19) lead to

$$\widetilde{W} = \sum_{i=1}^{n_1} a_i - \sum_{i=n_1+1}^{n} \frac{\mu_i}{\alpha_i} + \sum_{i=1}^{n_1} \lambda_i \left[\phi(\widetilde{W})\right]^{-\frac{1}{c_i}} - \sum_{i=n_1+1}^{n} \frac{1}{\alpha_i} \ln\left[\phi(\widetilde{W})\right] \tag{21}$$

Remarks

1) Equation (21) is an equilibrium condition for the price density $\phi(\widetilde{W})$.
2) *Bühlmann* (1980) and *Lienhard* (1986) derived explicit price formulae for the case $R_1 = R_2 = \cdots = R_n$ (exponential, logarithmic and power utility functions). These results are contained as special cases in (21).

From (18), (19) we can also see that there is a functional relationship between the equilibrium allocation $(\tilde{Z}_1^*, \ldots, \tilde{Z}_n^*)$ and $\widetilde{W}$, i.e.

$$\tilde{Z}_i^* = f_i(\widetilde{W}) \qquad i = 1, \ldots, n \tag{22}$$

Taking into account (21) allows us to analyse the shape of the functions f_i. However, before doing this we discuss how investment techniques like *portfolio insurance* are related to this issue.

3.3 Generalized Portfolio Insurance

Usually, the protection of a reference portfolio by a put option is called *portfolio insurance*. Hence, if an investor holds a share s in $\widetilde{W}$ and a corresponding position in a put option his final payoff is given by a function

$$g(\widetilde{W}) = s \cdot \max(\widetilde{W}, k) \tag{23}$$

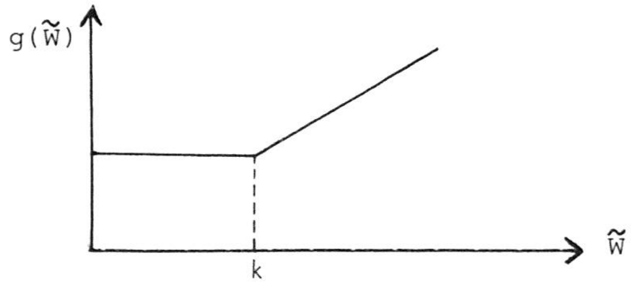

Figure 1

Obviously $g(\widetilde{W})$ is a *convex* function of $\widetilde{W}$. On the other hand, *Leland* (1980) showed that any twice continuously differentiable convex payoff function $f(\widetilde{W})$ can be approximated by a combination of the reference portfolio (in our case $\widetilde{W}$), the riskless asset and a set of corresponding put options with different strike prices. Motivated by this result he introduced the term "general insurance policy".

Definition 5 An investment policy implying a strictly convex payoff function $f(\widetilde{W})$ is called a "*general insurance policy*" (see Figure 2).

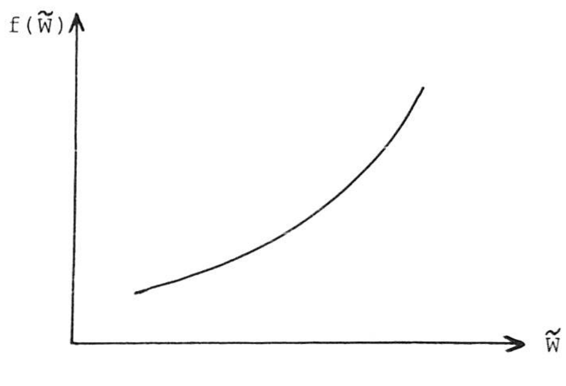

Figure 2

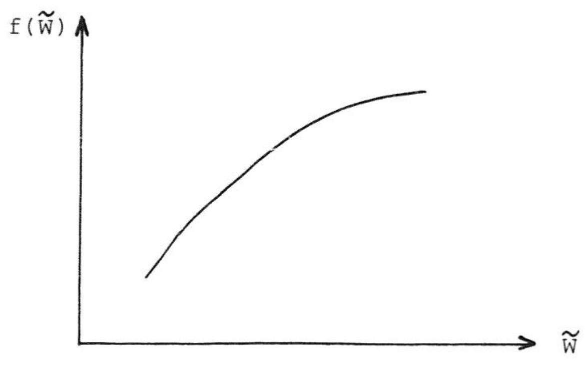

Figure 3

In Figure 3 the payoff function $f(\widetilde{W})$ is strictly concave and we have the converse of a general insurance policy.

The next section deals with the application of these notions to the analysis of price equilibria.

3.4 *Analysis of Equilibrium Allocations*

From (18), (19) and (22) we obtain

$$\widetilde{Z}_i^* = f_i(\widetilde{W}) = a_i + \lambda_i \left[\phi(\widetilde{W})\right]^{-\frac{1}{c_i}}, \qquad i = 1, \dots, n_1 \tag{18'}$$

$$\widetilde{Z}_i^* = f_i(\widetilde{W}) = -\frac{\mu_i}{\alpha_i} - \frac{1}{\alpha_i} \ln \left[\phi(\widetilde{W})\right], \qquad i = n_1 + 1, \dots, n \tag{19'}$$

Let w denote the realisation of $\widetilde{W}$. Then we get

$$f_i'(w) = -\frac{\lambda_i}{c_i} [\phi(w)]^{-\frac{1}{c_i}-1} \cdot \phi'(w), \qquad i = 1, \dots, n_1 \tag{24}$$

$$f_i'(w) = -\frac{1}{\alpha_i} \cdot \frac{\phi'(w)}{\phi(w)}, \qquad i = n_1 + 1, \dots, n \tag{25}$$

and

$$f_i''(w) = -\frac{\lambda_i}{c_i}\left\{\phi''(w)\phi(w) - \frac{c_i + 1}{c_i} [\phi'(w)]^2\right\} \cdot [\phi(w)]^{-\frac{1}{c_i}-2}, \qquad i = 1, \dots, n_1 \tag{26}$$

$$f_i''(w) = -\frac{1}{\alpha_i} \cdot \frac{\phi''(w) \cdot \phi(w) - [\phi'(w)]^2}{[\phi(w)]^2}, \qquad i = n_1 + 1, \dots, n \tag{27}$$

Furthermore, the equilibrium condition (21) leads to [4]

$$1 = -\phi'\left\{\sum_{j=1}^{n_1} \frac{\lambda_j}{c_j} \phi^{-\frac{1}{c_j}-1} + \sum_{j=n_1+1}^{n} \frac{\phi^{-1}}{\alpha_j}\right\} \tag{28}$$

and

$$\begin{aligned} 0 = &-\phi'' \cdot \phi\left\{\sum_{j=1}^{n_1} \frac{\lambda_j}{c_j} \phi^{-\frac{1}{c_j}-2} + \sum_{j=n_1+1}^{n} \frac{\phi^{-2}}{\alpha_j}\right\} \\ &+ (\phi')^2\left\{\sum_{j=1}^{n_1} \frac{\lambda_j (c_j + 1)}{c_j^2} \phi^{-\frac{1}{c_j}-2} + \sum_{j=n_1+1}^{n} \frac{\phi^{-2}}{\alpha_j}\right\}. \end{aligned} \tag{29}$$

[4] In the following formulae a simplified notation is used.

From (28), (29) one can see immediately that

$$\phi' < 0, \quad \phi'' > 0 \tag{30}$$

holds. Hence, the equilibrium price density ϕ is a decreasing convex function in the value of total final wealth.

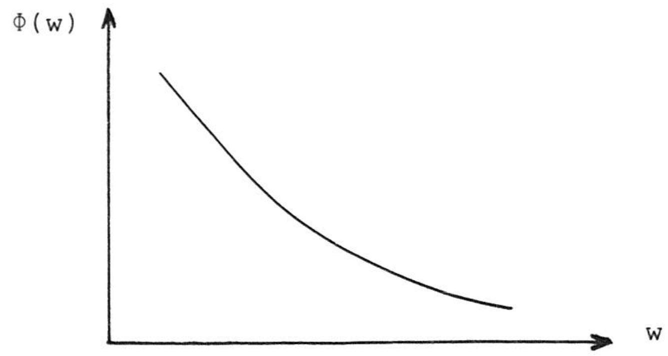

Figure 4

However, we are more interested in the shape of the equilibrium allocation $(f_1(\widetilde{W}), \ldots, f_n(\widetilde{W}))$.
From (24), (25) and (30) one obtains the well known result

$$f_i' > 0, \qquad i = 1, \ldots, n. \tag{31}$$

Moreover, (27) and (29) imply

$$f_i'' < 0, \qquad i = n_1 + 1, \ldots, n \tag{32}$$

For the investors with strictly positive cautiousness $i = 1,\dots,n_1$ the situation is somewhat more complex. (26) and (29) lead to

$$f_i'' \gtreqless 0 \Leftrightarrow \frac{c_i+1}{c_i} \gtreqless \frac{\sum_{j=1}^{n_1} \frac{\lambda_j(c_j+1)}{c_j^2}\phi^{-\frac{1}{c_j}} + \sum_{j=n_1+1}^{n} \frac{1}{\alpha_j}}{\sum_{j=1}^{n_1} \frac{\lambda_j}{c_j}\phi^{-\frac{1}{c_j}} + \sum_{j=n_1+1}^{n} \frac{1}{\alpha_j}} \qquad i = 1,\dots,n_1 \qquad (33)$$

$$f_i'' \gtreqless 0 \Leftrightarrow \frac{1}{c_i} \gtreqless \frac{\sum_{j=1}^{n_1} \frac{\lambda_j}{c_j^2}\phi^{-\frac{1}{c_j}}}{\sum_{j=1}^{n_1} \frac{\lambda_j}{c_j}\phi^{-\frac{1}{c_j}} + \sum_{j=n_1+1}^{n} \frac{1}{\alpha_j}} \qquad i = 1,\dots,n_1 \qquad (34)$$

$$f_i'' \gtreqless 0 \Leftrightarrow \sum_{j=1}^{n_1} \frac{\lambda_j}{c_j}\left(\frac{c_i}{c_j} - 1\right)\phi^{-\frac{1}{c_j}} - \sum_{j=n_1+1}^{n} \frac{1}{\alpha_j} \gtreqless 0 \qquad i = 1,\dots,n_1 \qquad (35)$$

From (13) and (35), respectively (32) we see[5]

$$f_1'' \geq 0, \qquad f_n'' \leq 0 \qquad (36)$$

i.e., the investor(s) with the highest cautiousness ($i = 1$) follows always a general insurance policy whereas the investor(s) with the lowest cautiousness ($i = n_1 + 1,\dots,n$, respectively $i = n$ if $n_1 = n$) chooses the converse strategy. Our results can be summarized as follows:

Theorem 3 Under A 1, A 3, A 4 the price equilibrium $\{\phi, (f_1(\widetilde{W}),\dots, f_n(\widetilde{W}))\}$ has the properties:

1) $\phi' < 0 \quad \phi'' > 0$
2) $f_i' > 0 \quad i = 1,\dots,n$
3) $f_i'' \geq 0 \quad \forall\, i \in \{j \mid R_j = \max R_h,\ h = 1,\dots,n\}$
   $f_i'' \leq 0 \quad \forall\, i \in \{j \mid R_j = \min R_h,\ h = 1,\dots,n\}$.

Theorem 3 contains the main result of our paper:

[5] Due to the relationship $c_i = \frac{1}{R_i}$, $i = 1,\dots,n_1$ (13) implies $c_1 \leq c_2 \leq \cdot \leq c_{n_1}$.

A high cautiousness is related to a "general insurance policy" (Figure 2) and a low cautiousness implies the converse strategy (Figure 3).

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Summary

The theory of risk exchange is applied to a model of financial risk. In particular, *Bühlmann*'s notion of a price equilibrium is used in this context. For a special class of utility functions (HARA, non-increasing absolute risk aversion) the equilibrium price density and the risk allocation are analysed in detail. The characteristics of individual investors are related to their policy choice (portfolio insurance, etc.).

Zusammenfassung

Die Theorie des Risikoaustauschs wird auf ein Modell mit Börsenrisiken angewandt. Dabei werden *Bühlmanns* Resultate über ökonomische Prämienberechnungsprinzipien verwendet. Für eine spezielle Klasse von Nutzenfunktionen (HARA, nicht-zunehmende absolute Risikoaversion) werden die Preisdichte und die Risikoallokation im Detail analysiert. Die Charakteristika der einzelnen Investoren werden dabei mit ihrer Anlagepolitik (Portfolioinsurance usw.) in Beziehung gebracht.

Résumé

L'auteur applique la théorie des échanges de risques à des risques financiers. Il utilise en particulier la notion d'équilibre des prix selon *Bühlmann*. Il analyse en détail l'allocation des risques et la densité de prix à l'équilibre dans le cas d'une classe particulière de fonctions d'utilité (HARA, aversion pour le risque absolue non-croissante). Les caractéristiques des investisseurs individuels sont mis en relation avec les choix de leur politique (assurance de portefeuilles, etc.).