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A study of a family of equivalent martingale measures to price an option with an application to the Swiss market

1 Introduction

In this paper we consider the problem of pricing a European option in the context of incomplete markets.
Let us first consider a complete market, i. e. a market where every contingent claim is attainable. Black and Scholes (1973) have shown that under some ideal conditions, it is possible to create a hedged position, consisting of a long position in the stock and a short position in the option. Moreover it is possible to maintain the hedge continuously and as a consequence the return on the hedged position becomes certain. Hence the unique rational price of a contingent claim can be obtained as if it existed in a risk-neutral world, this price being equal to the expected value of the discounted payoff according to an appropriate probability measure Q. This probability measure is equivalent to P[1], the *physical* probability measure and such that it makes the discounted price process $\{e^{-rt}S(t)\}_{t\geq 0}$ a martingale. This measure Q is called the *equivalent martingale measure.*
Let $S(t)$ denote the price of a non-dividend paying stock at time $t \geq 0$. We assume that there is a stochastic process, $\{X(t)\}_{t\geq 0}$, with stationary and independent increments, $X(0) = x_0 = \ln S(0)$, such that

$$S(t) = e^{X(t)}, \qquad t \geq 0. \tag{1}$$

We may interpret the random variable $X(t) - x_0$ as the continuously compounded rate of return over the time interval $[0, t]$. We suppose throughout this paper that there exists a risk-free asset whose rate of return is known and constant through time, and we denote this risk-free rate by r. The Fundamental Theorem of Asset Pricing tells us that the absence of arbitrage opportunities implies essentially the existence of an equivalent martingale measure. However this equivalent martingale measure is unique if and only if the market is complete. So, when considering incomplete

[1] i. e. we have $P(A) = 0 \;\Leftrightarrow\; Q(A) = 0$

models for the stock price process, there exist many equivalent martingale measures. In that sense the price of a European option is not unique.

We place ourselves in the context of incomplete markets and we are interested in studying the effect of changing the martingale measure when taking the discounted expected value of the payoff. In order to achieve this goal, we consider the pricing of a European option in an incomplete market.

We can describe a European option by a payoff function $\Pi(s) \geq 0$ and a maturity date T. At time T, the holder of the option receives the amount $\Pi(S(T))$. We know from financial theory that the price at time $0 \leq t < T$ is calculated as a discounted expected value of the payoff. The rate used to perform the discounting is the risk-free rate and expectation is taken according to an equivalent martingale measure. Thus the price of the option at time t is

$$e^{-r(T-t)} E_Q[\Pi(S(T)) \mid \mathfrak{F}_t], \qquad 0 \leq t < T. \tag{2}$$

Here $E_Q[\cdot]$ denotes expectation taken with respect to the equivalent martingale measure Q. This measure must be such that the pricing formula (2) is compatible with the observed price of the stock, i.e. we require that

$$S(t) = e^{-r(T-t)} E_Q[S(T) \mid \mathfrak{F}_t], \qquad 0 \leq t < T. \tag{3}$$

2 An incomplete model

Intuitively, to have a model for the stock price process that implies an incomplete market, more than two outcomes must be possible in an infinitesimal time interval.

As before let $S(t)$ denote the stock price at time t. Here we will assume that $\{S(t)\}_{t \geq 0}$ is a geometric compound Poisson process with two possible jump heights, i.e.

$$S(t) = e^{X(t)},$$

where

$$X(t) = x_0 + k_1 N_1(t) + k_2 N_2(t), \qquad t \geq 0. \tag{4}$$

Here, $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ are independent Poisson processes with respective parameters $\lambda(k_1)$ and $\lambda(k_2)$, k_1 and k_2 are two constants and x_0 is the initial value of the process $\{X(t)\}_{t\geq 0}$. The initial stock price is $S(0) = e^{x_0}$. It is the simplest incomplete model we can think of. To avoid the existence of arbitrage opportunities, we assume that at least one of the two constants k_1 or k_2 is positive. Otherwise, selling short the stock and investing the proceeds in the risk-free asset yields $S(0)(e^{rt} - e^{k_1 N_1(t)+k_2 N_2(t)})$, which is positive whether a jump occurs or not at any time t. In order to shorten the notation in what follows, we will write λ_i for $\lambda(k_i)$.
Let μ, σ^2, γ and η denote the first four cumulants per unit time of the process $\{X(t)\}_{t\geq 0}$. Thus

$$\begin{aligned} E[X(t)] &= k_1\lambda_1 t + k_2\lambda_2 t = \mu t\,, \\ \mathrm{Var}[X(t)] &= k_1^2\lambda_1 t + k_2^2\lambda_2 t = \sigma^2 t\,, \\ E\left[(X(t)-\mu t)^3\right] &= k_1^3\lambda_1 t + k_2^3\lambda_2 t = \gamma t \end{aligned}$$

and

$$E\left[(X(t)-\mu t)^4\right] - 3(\mathrm{Var}[X(t)])^2 = k_1^4\lambda_1 t + k_2^4\lambda_2 t = \eta t\,. \tag{5}$$

Since k_1 and k_2 are observable parameters of the process, they must remain unchanged under any equivalent measure. Only the Poisson parameters λ_1 and λ_2 can be modified. We denote those modified parameters by λ_1^* and λ_2^*. Writing down the martingale condition, we obtain

$$\begin{aligned} S(0) &= E_Q\left[e^{-rt}S(t)\right] \\ &= e^{-rt}E_Q[S(t)]\,. \end{aligned}$$

By (1), the parameters λ_1^* and λ_2^* are solutions of the equation

$$0 = -r + \lambda_1^*(e^{k_1}-1) + \lambda_2^*(e^{k_2}-1)\,. \tag{6}$$

At that point, we propose to study the following family of methods to price an option: Set (for $i = 1, 2$)

$$\lambda_i^* = \exp\left[h\left(\frac{e^{ck_i}-1}{c}\right)\right]\lambda_i\,, \qquad 0 < c < 1\,. \tag{7}$$

Taking the limit as $c \to 0$ in (7), we obtain the modified Poisson parameters

$$\lambda_i^* = e^{hk_i} \lambda_i \,, \qquad i = 1,2 \,. \tag{8}$$

For the case $c = 1$, we have

$$\lambda_i^* = e^{h(e^{k_i}-1)} \lambda_i \,, \qquad i = 1,2 \,. \tag{9}$$

For a reason that will be made clear in section 6, we are motivated to study whether pricing an option according to the two different measures corresponding to (8) and (9) leads to two significantly different prices for this one. We want to determine, for every value of c, the value of h, written $h^*(c)$, such that the process

$$\{e^{-rt} S(t)\}_{t \geq 0}$$

is a martingale with respect to the probability measure corresponding to $h^*(c)$. That is $h^*(c)$ solves (6). We obtain so the following family of implicit equations for $h^*(c)$:

$$0 = -r + e^{h\frac{e^{ck_1}-1}{c}} \lambda_1 (e^{k_1} - 1) + e^{h\frac{e^{ck_2}-1}{c}} \lambda_2 (e^{k_2} - 1) \,, \; 0 \leq c \leq 1 \,. \tag{10}$$

Now the question arises what value for c should we use, when pricing an option in this incomplete model? What is a "good" value for c? Because this question cannot be answered directly in a theoretical way, we propose to explore this question by means of real data. In that order, we examine observed prices of European calls. Hence, we have payoff functions of the form $\Pi(S(T)) = (S(T) - K)_+$, where K denote the strike price. In that case, formula (2) becomes

$$e^{-rT} E_Q \big[(S(T) - K)_+ \mid S(0) \big] \,, \tag{11}$$

or, in our model,

$$e^{-rT} \sum_{x \geq \kappa} \left(e^{x_0 + x} - K \right) q(x,T) \,, \tag{12}$$

where we have considered $t = 0$, and defined $\kappa = \ln\left(\frac{K}{S(0)}\right)$. $q(x,T)$ is the probability that $k_1 N_1(T) + k_2 N_2(T) = x$, under the equivalent martingale measure Q. The distribution of N_i is of parameter $e^{h^*(c)\left(\frac{e^{ck_i}-1}{c}\right)}\lambda_i$ for $i = 1,2$. Here c is considered to be fixed. k_1, k_2, λ_1 and λ_2 are the parameters of the *physical* probability measure and are determined by solving (5) with μ, σ^2, γ and η replaced by their estimates.

3 Esscher transforms and equivalent martingale measures

In an incomplete model, there are many equivalent martingale measures. A priori, it is not clear which martingale measure should be chosen to calculate the price of an option. Gerber and Shiu (1994) suggested that, in order to obtain a unique answer, the choice of the equivalent martingale measure could be limited to the family of Esscher transforms (see Esscher (1932)). A justification in term of *minimal relative entropy* with respect to the physical probability measure P has been given by Chan (1997).
Let $M(z,t) = E[e^{zX(t)}]$ denote the moment generating function of $X(t)$. Because $M(z,t)$ is continuous at $t = 0$, it can be proved that

$$M(z,t) = M(z,1)^t, \qquad t > 0. \tag{13}$$

The process

$$\left\{e^{hX(t)}M(h,1)^{-t}\right\}$$

is a positive martingale and they used it to define a change of probability measure. That is, it is used to define the Radon-Nikodym derivative dQ/dP, where P denotes as before the original probability measure and Q is the Esscher measure of parameter h. They call the *risk-neutral Esscher measure* the Esscher measure of parameter $h = h^*$ such that the process

$$\left\{e^{-rt}S(t)\right\}_{t\geq 0}$$

is a martingale with respect to the probability measure corresponding to h^*.
Gerber and Shiu (1994) apply the Esscher transform (parameter h) to the process $\{X(t)\}_{t\geq 0}$. Let $M(z,t;h)$ denote the moment-generating function of the modified distribution of $X(t)$. It is easily verified that

$$M(z,t;h) = \frac{M(z+h,t)}{M(h,t)}. \tag{14}$$

Writing down the martingale condition, we obtain

$$\begin{aligned} S(0) &= E_Q\left[e^{-rt}S(t)\right] \\ &= e^{-rt}E_Q[S(t)]\,. \end{aligned}$$

By (1), the parameter h^* is the solution of the equation

$$1 = e^{-rt}E_Q\left[e^{X(t)-X(0)}\right],$$

or, using (14),

$$e^{rt} = E_Q\left[e^{X(t)-X(0)}\right] = \frac{M(1+h,t)}{M(h,t)}\,. \tag{15}$$

From (13) we see that the solution does not depend on t, and we may set $t = 1$:

$$e^r = \frac{M(1+h,1)}{M(h,1)}\,, \tag{16}$$

or

$$r = \ln\left(\frac{M(1+h,1)}{M(h,1)}\right). \tag{17}$$

For the model given by (4), (14) becomes

$$\begin{aligned} M(z,t;h) &= \frac{E_P\left[e^{(z+h)X(t)}\right]}{E_P\left[e^{hX(t)}\right]} \\ &= \frac{\exp\left\{x_0 + \lambda_1 t\left(e^{(z+h)k_1} - 1\right) + \lambda_2 t\left(e^{(z+h)k_2} - 1\right)\right\}}{\exp\left\{x_0 + \lambda_1 t\left(e^{hk_1} - 1\right) + \lambda_2 t\left(e^{hk_2} - 1\right)\right\}}\,. \end{aligned}$$

After simplification, we obtain

$$M(z,t;h) = \exp\left\{\lambda_1 e^{hk_1} t\left(e^{zk_1} - 1\right) + \lambda_2 e^{hk_2} t\left(e^{zk_2} - 1\right)\right\}\,. \tag{18}$$

Hence, the Esscher transform (parameter h) of the process $\{X(t)\}_{t\geq 0}$ is again a compound Poisson process, with the same two possible jump heights k_1 and k_2, but with modified Poisson parameters $\lambda_i e^{hk_i}$, $i = 1, 2$. From (17) and (18) we see that the parameter h^* of the risk-neutral Esscher measure is implicitly defined by the equation

$$r = \lambda_1 e^{h^* k_1} \left(e^{k_1} - 1 \right) + \lambda_2 e^{h^* k_2} \left(e^{k_2} - 1 \right) . \tag{19}$$

Then the risk-neutral Esscher parameters are given by $\lambda_1^* = \lambda_1 e^{h^* k_1}$ and $\lambda_2^* = \lambda_2 e^{h^* k_2}$. Hence, we see that in section 2, the case $c = 0$ corresponds to applying the method of Esscher transforms.

4 Two other incomplete models

In order to make numerical comparisons, we present in this section two other incomplete models: the shifted gamma process and the shifted inverse Gaussian process. The modeling of the stock-price movements by means of these two models was first introduced by Gerber and Shiu (1994, section 4). For these two incomplete models, we use the method of Esscher transforms in order to get a unique answer for the price of an option. Hence, the price of an option is defined to be the discounted expectation of the payoff where expectation is taken according to the Esscher transform of parameter h^*, where $h = h^*$ is determined so that (17) is satisfied. Remember that in section 2, the case c = 0 corresponds to applying the method of Esscher transforms.

4.1 The shifted gamma process

Here it is assumed that

$$X(t) = Y(t) - \nu t ,$$

where $\{Y(t)\}$ is a gamma process with shape parameter α and scale parameter β, and the positive constant ν is a third parameter. The moment generating function of $X(t)$ is

$$M(z, t) = \left(\frac{\beta}{\beta - z} \right)^{\alpha t} e^{-\nu t z} , \qquad z < \beta . \tag{20}$$

For given values of μ, σ and γ, the three parameters are chosen to match the first three cumulants per unit time, i. e., to solve

$$E[X(1)] = \frac{\alpha}{\beta} - \nu = \mu\,,$$

$$\mathrm{Var}[X(1)] = \frac{\alpha}{\beta^2} = \sigma^2\,,$$

$$E\left[(X(1) - \mu t)^3\right] = \frac{2\alpha}{\beta^3} = \gamma\,.$$

Hence we set

$$\alpha = \frac{4\sigma^6}{\gamma^2}\,, \qquad \beta = \frac{2\sigma^2}{\gamma}\,, \qquad \nu = \frac{2\sigma^4}{\gamma} - \mu\,. \tag{21}$$

From (14) and (20), we obtain

$$M(z, t; h) = \left(\frac{\beta - h}{\beta - h - z}\right)^{\alpha t} e^{-\nu t z}\,, \qquad z < \beta - h\,, \tag{22}$$

which shows that the Esscher transform of $\{X(t)\}$ is again a shifted gamma process with unchanged values of α and ν but β replaced by

$$\beta(h) = \beta - h\,.$$

From (16), we obtain the following condition for the martingale measure

$$\beta = \beta(h^*) = \frac{1}{1 - e^{-(\nu + r)/\alpha}}\,.$$

4.2 *The shifted inverse Gaussian process*

Here, it is also assumed that

$$X(t) = Y(t) - \nu t\,,$$

but with $\{Y(t)\}$ being an inverse Gaussian process with parameters a and b. The moment generating function of $X(t)$ is

$$M(z, t) = e^{at(\sqrt{b} - \sqrt{b - z}\,) - \nu t z}\,, \qquad z < b\,. \tag{23}$$

Again, for given values of μ, σ and γ, the three parameters are chosen to match the first three cumulants per unit time, or to solve

$$E[X(1)] = \frac{a}{2b^{1/2}} - \nu = \mu\,,$$

$$\mathrm{Var}[X(1)] = \frac{a}{4b^{3/2}} = \sigma^2\,,$$

$$E\left[(X(1) - \mu t)^3\right] = \frac{3a}{8b^{5/2}} = \gamma\,.$$

Hence we set

$$a = 3\sigma^5\sqrt{\frac{6}{\gamma^3}}\,, \qquad b = \frac{3\sigma^2}{2\gamma}\,, \qquad \nu = \frac{3\sigma^4}{\gamma} - \mu\,. \tag{24}$$

From (14) and (23), we obtain

$$M(z,t;h) = \exp\left[at(\sqrt{b-h} - \sqrt{b-h-z}) - \nu tz\right]\,, \qquad z < \beta - h\,, \tag{25}$$

which shows that the Esscher transform of $\{X(t)\}$ is again a shifted inverse Gaussian process with unchanged values of a and ν but b replaced by

$$b(h) = b - h\,.$$

From (17), we obtain the following condition for the martingale measure

$$r = a\left(\sqrt{b-h^*} - \sqrt{b-h^*-1}\right) - \nu\,,$$

or equivalently

$$\sqrt{b(h^*)} - \sqrt{b(h^*)-1} = \frac{\nu + r}{a}\,,$$

which is an implicit equation for $b^* = b(h)$.

5 Numerical examples

In this section we examine the family of equivalent martingale measures given implicitly by equation (10). We are interested in examining the option's price sensitivity to the change of measure involved by a change in the parameter c.
The stock prices are obtained from the data base DATASTREAM. Those prices are closing market prices. We consider daily data. From those daily prices, we compute the continuously compounded daily rate of return according to

$$X(t) = \ln\left(\frac{S(t)}{S(t-1)}\right). \tag{26}$$

The DATASTREAM's prices are adjusted for operations like splits or increases of capital but not for the payment of dividend. We had to modify the data for the days where dividend payments occurred in order to cancel the jumps (anticipated on the market) due to dividends. DATASTREAM provides us with dividends series. Hence it is possible to correct the rates of return at the dividend payment dates.
As examples we have chosen to consider American call options on stocks ALUSUISSE R ("nominative") and SWISS BANK CO B ("porteur"). We have selected derivatives for which the volume of transactions was sufficiently high, so that the prices are real market prices. We consider times to maturity between 10 days and 158 days. We made use of observed data from the SOFFEX (Swiss Options and Financial Futures Exchange). The options at the SOFFEX are *American* options. We have considered only options on stocks for which there was no dividend payment until the date of maturity. For these options the price is identical to the price of European options.
On the Swiss Option Exchange, the expiration date is always the third Friday of the relevant month. Quoted prices for options and traded volumes have also been obtained from DATASTREAM. The options are quoted in Swiss francs with the minimum quoted price fluctuations (*ticks*) given in Table 1.
We had two kinds of daily quoted prices: *last price paid (lpp)* and *settlement price (sp)*. As a general rule, the settlement price corresponds to the last price paid, unless there was no exchange during the last hour of quotation or

Option's price					*Tick*
From	Fr.	–.10	to	Fr. 9.90	Fr. –.10
From	Fr.	10.–	to	Fr. 19.80	Fr. –.20
From	Fr.	20.–	to	Fr. 99.50	Fr. –.50
From	Fr.	100.–			Fr. 1.–

Table 1: Minimum quoted price fluctuation (tick) at the SOFFEX

the last price paid was not anymore corresponding to the current situation of the market. In those two cases, the SOFFEX determines the option prices. We have to remember this when making comparisons between observed prices and theoretical ones. For the risk-free rate, we have chosen EURO-CURRENCY (SWISS FR.) from London for one, two, six and twelve months. For time to maturity of four and five months, we have considered linear interpolation of the preceding rates.
For each case considered, we have first calculated $\widehat{\mu}$, $\widehat{\sigma}^2$, $\widehat{\gamma}$ and $\widehat{\eta}$, estimates of the first four cumulants per unit time (here one day) of the process $\{X(t)\}_{t\geq 0}$. Then we have computed $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, $\widehat{k}_1$ and $\widehat{k}_2$, estimates of the parameters of our pure jump model by solving the system given by (5). See Tables 2 to 5. We have also computed estimates of α, β and ν, the parameters of the shifted gamma process, using (21) and finally estimates of a, b and ν, the parameters of the shifted inverse Gaussian process, using (24).
To obtain twenty-one different options prices in the first model, we have computed expression (12) for $c = 0, 0.05, 0.10, \ldots, 1$. At this stage, the computations are time-consuming (we obtained up to 160,000 probability masses for each distribution given by different values of c). We then computed the option prices for the two other incomplete models (see formulas (4.1.7) and (4.2.7) given in Gerber and Shiu (1994)). See Tables A.1 to A.7 in appendix A. Here because of the high-valued parameters numerical difficulties arise. For example, in the case $T = 158$ days we had to calculate a gamma distribution function with shape parameter $\alpha = 1,951.69$ and scale parameter $\beta = 271.46$ or an inverse Gaussian distribution function with shape parameter $a = 305.93$ and scale parameter $b = 202.78$.
In the first model, we see that option prices are monotone functions of the parameter c. Whether it is an increasing or decreasing function of c depends on the case considered. In every example we observe that the differences between the prices obtained with $c = 0$ and the prices obtained with $c = 1$

i	$\widehat{\lambda}_i$	$\widehat{k}_i$
1	0.250587820	0.022434177
2	0.237835455	−0.015715156

Table 2: Estimates of daily parameters for ALUSUISSE R over the period January 4 – June 30, 1992.

i	$\widehat{\lambda}_i$	$\widehat{k}_i$
1	2.651383917	0.006081860
2	6.976036534	−0.002422898

Table 3: Estimates of daily parameters for SWISS BANK CO B over the period June 29 – November 19, 1992.

i	$\widehat{\lambda}_i$	$\widehat{k}_i$
1	0.316605487	0.015705559
2	0.204049188	−0.011705723

Table 4: Estimates of daily parameters for SWISS BANK CO B over the period January 4 – June 7, 1993.

i	$\widehat{\lambda}_i$	$\widehat{k}_i$
1	0.849021090	0.011786903
2	2.422493362	−0.004344630

Table 5: Estimates of daily parameters for SWISS BANK CO B over the period October 18, 1993 – April 18, 1994.

are very small. Figures A.1 to A.7 in appendix A show for each value of c the difference between the price obtained with that particular value of c and the price obtained with $c = 0$ in percentage of this latter. Surprisingly, we observe in every of our cases that for any given value of c, the higher the strike price, the higher this percentage in absolute value. The maximal difference computed between those two prices is of 0.245 % of the price given by $c = 0$ (see Table A.3 and Figure A.3). In fact, it appears that the range of equivalent martingale measures obtained is in some sense very narrow in the pure jump model considered.

6 A more general jump model

In this section we give a justification for section 2. Consider, for the process $\{X(t)\}_{t\geq 0}$, a more general model, specified as follows. The conditional distribution of the amount of a jump is of a discrete nature. We use the symbol $\lambda_t(\cdot)$ for the measure of the jump frequencies of the process $\{X(t)\}_{t\geq 0}$, i. e. $\lambda_t(x)\,dt$ is the probability of a jump of amount x between times t and $t+dt$. We adopt a similar notation for the process $\{S(t)\}_{t\geq 0}$, with $\lambda_t(\cdot)$ replaced by $\widetilde{\lambda}_t(\cdot)$. Because $\{S(t)\}_{t\geq 0}$, is adapted, the following equality holds:

$$\lambda_t(x) = \widetilde{\lambda}_t(S(t)(e^x - 1)) \,. \tag{27}$$

We write $\widetilde{\lambda}_t^{\alpha_t}(\cdot)$ and $\lambda_t^{h_t}(\cdot)$ to indicate that we are working with the Esscher transform (parameter α_t for $\{S(t)\}_{t\geq 0}$ and h_t for $\{X(t)\}_{t\geq 0}$). So we have

$$\widetilde{\lambda}_t^{\alpha_t}(y) = e^{\alpha_t y}\widetilde{\lambda}_t(y) = \lambda_t^{h_t}(x) = e^{h_t x}\lambda_t(x)\,, \tag{28}$$

where $y = S(t)(e^x - 1)$.
We apply now, as in Gerber and Shiu (1994), the Esscher transform to the process $\{X(t)\}_{t\geq 0}$. The condition that the process $\{e^{-rt}S(t)\}_{t\geq 0}$ is a martingale resumes to

$$-r + \sum_x (e^x - 1)e^{h_t x}\lambda_t(x) = 0\,. \tag{29}$$

Consider now applying the Esscher transform directly to the process $\{S(t)\}$. This is the Esscher method in the sense of Bühlmann (1995). Then, the martingale condition is

$$-r + \sum_y y e^{\alpha_t y}\widetilde{\lambda}_t(y) = 0\,,$$

which can be rewritten in term of the "parameters" of the process $\{X(t)\}_{t\geq 0}$:

$$-r + \sum_x (e^x - 1)e^{\alpha_t^*(e^x - 1)}\lambda_t(x) = 0\,, \tag{30}$$

with $\alpha_t^* = \alpha_t S(t)$.

In comparing conditions (29) and (30), we see that, unless $\lambda_t(\cdot)$ is concentrated on one point $\widetilde{x}$ (in which case we have to choose $\alpha_t S(t)(e^{\widetilde{x}} - 1) = h_t \widetilde{x}$), the resulting equivalent martingale measures for $\{e^{-rt}S(t)\}_{t \geq 0}$ are different. For that reason, we are motivated to study whether pricing an option according to those two different measures leads to two significantly different prices for this one. Hence, we propose the following interpolation formula for the modified jump measure:

$$\lambda_t(x) \quad \longmapsto \quad e^{h_t(\frac{e^{cx}-1}{c})}\lambda_t(x)\,, \qquad 0 < c \leq 1\,.$$

Taking the limit as $c \to 0$, we see that the above expression is the modified jump measure obtained when applying the Esscher transform (parameter h_t) to the process $\{X(t)\}_{t \geq 0}$. The other extreme case where $c = 1$ corresponds to the modified jump measure obtained when applying the Esscher transform (parameter α_t^*) directly to the process $\{S(t)\}_{t \geq 0}$.
The model given by (4) is time homogeneous, so we can leave the subscript t. We want to determine, for every value of c, the value of h, written $h^*(c)$, such that the process

$$\{e^{-rt}S(t)\}_{t \geq 0}$$

is a martingale with respect to the probability measure corresponding to $h^*(c)$. In fact, we obtain a family of implicit equations for $h^*(c)$:

$$-r + \sum_x (e^x - 1)e^{h(\frac{e^{cx}-1}{c})}\lambda(x) = 0\,, \qquad 0 < c \leq 1\,. \tag{31}$$

We can rewrite this family of implicit equations for our model (4) of the process $\{X(t)\}_{t \geq 0}$ and obtain so (10).

7 Examination of an approximation formula

In this section, we examine the linear approximation formula introduced by Gerber and Landry (1997) by means of real data. The examination makes use, as in section 5, of observed data from the SOFFEX.
They have considered models where γ, the third cumulant per unit time of the process $\{X(t)\}_{t \geq 0}$, is different from zero. In order to examine the

effect of skewness, they proposed to replace the exact density of $X(1)$ by its first order expansion and obtained a linear approximation for the price of a European option. It is remarkable that the approximation formula obtained does not depend on the underlying model, as long as option prices are calculated by the Esscher method. The interested reader is referred to their paper for further details. Here is their formula

$$e^{-r}E_Q\big[S(1)-K)_+\big] \simeq e^{-r}\int_{\kappa}^{\infty}\left(e^{x_0+x}-K\right)f_0(x)\,dx + \frac{\gamma}{\sigma^2}e^{-r}\int_{\kappa}^{\infty}\left(e^{x_0+x}-K\right)f_1(x)\,dx\,, \qquad (32)$$

where $\kappa = \ln\left(\frac{K}{S(0)}\right)$ and without loss of generality the maturity considered is of 1. Here K is the strike price, $f_0(x)$ and $f_1(x)$ are given by

$$f_0(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu^*}{\sigma}\right) \qquad (33)$$

and

$$\begin{aligned} f_1(x) = {} & \frac{1}{2}(r-\mu)\left(\frac{1}{\sigma^3}\phi''\left(\frac{x-\mu^*}{\sigma}\right)+\frac{1}{\sigma^2}\phi'\left(\frac{x-\mu^*}{\sigma}\right)\right) \\ & -\frac{1}{12}\phi'\left(\frac{x-\mu^*}{\sigma}\right)-\frac{1}{4\sigma}\phi''\left(\frac{x-\mu^*}{\sigma}\right) \\ & -\frac{1}{6\sigma^2}\phi'''\left(\frac{x-\mu^*}{\sigma}\right), \end{aligned} \qquad (34)$$

where $\mu^* = r - \frac{1}{2}\sigma^2$. Here $\phi(\cdot)$ denotes the standard normal probability density function. $f_0(x)$ is the martingale density of $X(1)$ in the classical Black-Scholes model. Hence, the approximation consists of the Black-Scholes price combined with an adjustment for skewness.

The prices obtained by this method are displayed under the heading "**Linear Approximation**" in Tables A.1 to A.7 in appendix A. In most of our cases the difference between the settlement price and the price obtained by this method is negative. For example (see Table 6) we consider a call option with strike price $K = 400$ and 64 days to maturity on a stock (SWISS BANK CO.) that is selling at present time at 374 Swiss francs. The third cumulant per unit time γ of the stock is estimated at 0.000 001 191 664 3 per day. We obtain a rate of change at $\gamma = 0$ of 3632.480. Hence the Black-Scholes price must be adjusted by 3632.4797 per γ. The first order effect of the skewness on the price of this option is given for different strike prices in Figure 1. We note that this adjustment can be positive or negative, depending on the strike price K.

Exercise Price K	Observed Prices lpp	/ sp	Black Scholes	Rate of Change at $\gamma = 0$	Linear Approximation
350	31.50	33.00	31.596	815.967	31.658
360	27.00	26.50	24.639	1549.226	24.757
370	19.00	21.00	18.671	2325.263	18.849
380	15.00	15.00	13.736	3001.996	13.965
390	11.50	11.50	9.805	3459.197	10.069
400	8.50	8.50	6.789	3632.480	7.066
425	4.00	3.40	2.377	2963.007	2.603

Table 6: Call option (on SWiSS BANK CO B) prices with $S(0) = 374$, $T = 64$ days on August 18, 1994 ($\gamma = 0.000\,001\,191\,664\,3$)

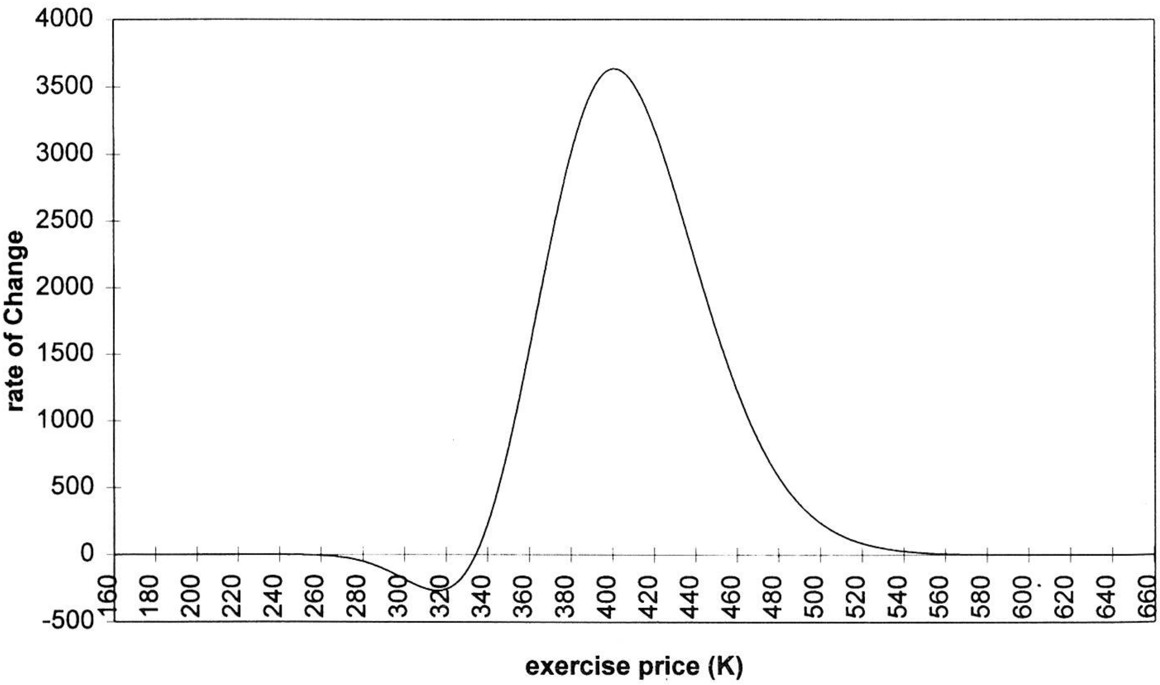

Figure 1: Adjustment of the call option price per γ (see Table 6)

8 Implied parameters and the linear approximation formula

In sections 5 and 7 we computed the Black-Scholes formula and expression (32) in taking parameter values estimated from historical data and substituting them into those two formulas. From a practical point of view, the one parameter in the Black-Scholes formula that cannot be observed directly is the volatility. By using the historical standard deviation to estimate the volatility, we assume that the past variability of the stock's returns is invariant through time. It is not obvious that volatility is constant for long periods of time and that the historical volatility is independent of the time series from which it is calculated. It is therefore difficult to measure directly the volatility in practice.

However, option prices are quoted in the market. An alternative concept, *implied volatility,* consists of estimating the volatility of stock returns implicitly reflected in current option prices. A call option price increases monotonically with volatility, so there is a one-to-one correspondence between the volatility and the option price. The idea is to invert the Black-Scholes formula from the currently observed price of a call option. In this way we obtain the market's opinion of the value of the volatility over the remaining life of the option. This method was originally proposed by Latané and Rendleman (1976). The implied volatility derived from several options written on the same stock will generally not be equal.

Now the problem is to take a suitable weighted average of the individual implied volatilities. One can think about taking the arithmetic average or even to weight each option's implied volatility according to its degree of price elasticity with respect to the volatility. Here we mention Beckers' empirical study (1981) of stock returns' future variability estimates. He suggests the use of only one call option price, the one whose price is most sensitive to σ. We measure the sensitivity of an option with respect to σ by the partial derivative of its price with respect to σ, that is

$$\begin{aligned}\frac{\partial C}{\partial \sigma} &= S(0)\sqrt{T}\Phi'\left(\frac{rT - \kappa + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ &= S(0)\sqrt{\frac{T}{2\pi}}\exp\left(-\frac{1}{2}\frac{(rT + \ln(S(0)/K) + \frac{1}{2}\sigma^2 T)^2}{\sigma^2 T}\right).\end{aligned}$$

This expression is maximal for

$$\widetilde{K} = S(0)e^{rT+\frac{1}{2}\sigma^2 T}. \tag{35}$$

Hence the call option whose strike price is the nearest to the one given by (35) will be chosen.

The parameters in the linear approximation formula (32) that cannot be observed directly are the first three cumulants per unit time of the stock price. Now the idea is to invert this approximation formula by observing the current call option's price. In this way we obtain the market's opinion of the value for the drift, the volatility and the third cumulant per unit time over the remaining life of the option. To do this, we apply the following algorithm. Choose arbitrary initial values for μ and γ, say μ_0 and γ_0 (a good idea is to choose historical estimates of them). Then compute σ_1, the value of σ that makes the approximation formula meet exactly the last observed price. Now use σ_1 and γ_0 to compute μ_1, the value of μ that makes the approximation formula meet exactly the last observed price. Repeat these steps to obtain γ_1, σ_2, μ_2, γ_2, σ_3, μ_3, γ_3, ... until convergence is observed.

Tables B.1 to B.7 in appendix B show the prices obtained using implied volatility to compute Black-Scholes prices and implied μ, σ and γ to compute the linear approximation formula. For example, consider Table B.5, which shows the prices obtained for a call option on stocks of the Swiss Bank Corporation on August 18, 1994, using implied parameters. In this particular example we found an annual implied σ of 0.29321 for the Black-Scholes formula. For the linear approximation formula, we obtained the following implied annual parameters: $\mu = -0.18889$, $\sigma = 0.29122$, $\gamma = 0.00043496$. In comparing settlement prices and the prices given by the Black-Scholes formula using implied volatility we see that settlement prices are overforecasted in almost every of our cases. Except for the prices given by Table B.6, we observe that using implied parameters with both formulas leads to differences of identical signs. For both formula and for a particular choice of $S(0)$ and T, the largest difference is obtained for the more out-of-the-money call option. In Table 7 are displayed the mean absolute differences between the theoretical prices and the settlement prices and the sum of absolute differences for all of our cases. Both for the Black-Scholes formula and the linear approximation we remark that the mean absolute spread decreases significantly while using implied parameters. However this effect is the strongest for the Black-Scholes formula. Examining all our cases together, we remark that 23 times out of 37 the Black-Scholes formula with volatility estimated implicitly leads to better results than using the Black-Scholes formula with volatility estimated historically. We remark also that 26 times out of 37 using implied parameters in the linear approximation formula leads to better results than without using implied parameters. Figure B.1 in appendix B shows valuation errors in percent of the settlement prices for the Black-Scholes and linear approximation formulas, using for both formula implied parameters. Figure B.2 in appendix B shows valuation errors in percent of the settlement prices for the linear approximation formula with and without the use of implied parameters. *Moneyness* is defined as

$$\frac{S(0)}{K} - 1\,.$$

Options whose *absolute moneyness* $\left|\frac{S(0)}{K} - 1\right|$, is greater than ten percent are not taken into account. These options have little trading activity and price quotes are generally not supported by actual trades.

	Mean Absolute Differences From sp (in % of sp)	Sum of Absolute Differences from sp (in SFR.)
Black-Scholes	15.52	79.66
BS-iv	10.35	37.67
Linear approx.	15.62	77.06
LA-i	12.18	48.10

Table 7: Valuation errors statistics

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Appendix A

		$K = 475$	492	500	525
Observed	lpp	49.00	42.50	35.50	22.00
Prices	sp	49.00	35.50	33.50	21.50
Black Scholes		49.382	38.390	33.798	21.871
Linear Approximation		48.163	37.071	32.465	20.622
Shifted Gamma		48.172	37.072	32.463	20.619
Shifted Inverse Gaussian		48.183	37.083	32.474	20.628
c = 0		48.9185	37.688	33.020	21.006
0.05		48.9176	37.687	33.019	21.005
0.10		48.9167	37.686	33.018	21.004
0.15		48.9157	37.685	33.017	21.003
0.20		48.9148	37.684	33.016	21.001
0.25		48.9138	37.683	33.015	21.000
0.30		48.9129	37.682	33.014	20.999
0.35		48.9120	37.681	33.012	20.998
0.40		48.9110	37.680	33.011	20.997
0.45		48.9101	37.678	33.010	20.996
0.50		48.9091	37.677	33.009	20.994
0.55		48.9082	37.676	33.008	20.993
0.60		48.9073	37.675	33.007	20.992
0.65		48.9063	37.674	33.006	20.991
0.70		48.9054	37.673	33.004	20.990
0.75		48.9044	37.672	33.003	20.989
0.80		48.9035	37.671	33.002	20.987
0.85		48.9026	37.670	33.001	20.986
0.90		48.9016	37.669	33.000	20.985
0.95		48.9007	37.668	32.999	20.984
$c = 1$		48.8998	37.666	32.998	20.983

Table A.1: Call option (on ALUSUISSE R) prices with $S(0) = 508$ and $T = 94$ days on July 13, 1993

		$K = 240$	260	280	300
Observed	lpp	26.50	9.00	2.40	1.00
Prices	sp	25.50	8.50	2.40	1.00
Black Scholes		25.063	9.164	1.714	0.148
Linear Approximation		25.058	9.222	1.808	0.178
Shifted Gamma		25.057	9.222	1.808	0.179
Shifted Inverse Gaussian		25.057	9.222	1.809	0.180
$c = 0$		25.180539	9.267172	1.818309	0.180996
0.05		25.180544	9.267181	1.818315	0.180997
0.10		25.180544	9.267187	1.818319	0.180998
0.15		25.180547	9.267195	1.818325	0.180999
0.20		25.180548	9.267201	1.818330	0.181001
0.25		25.180549	9.267207	1.818335	0.181002
0.30		25.180551	9.267215	1.818340	0.181003
0.35		25.180555	9.267223	1.818346	0.181004
0.40		25.180556	9.267230	1.818351	0.181005
0.45		25.180558	9.267237	1.818357	0.181007
0.50		25.180560	9.267244	1.818362	0.181008
0.55		25.180562	9.267252	1.818367	0.181009
0.60		25.180565	9.267260	1.818373	0.181010
0.65		25.180566	9.267266	1.818378	0.181011
0.70		25.180569	9.267274	1.818383	0.181013
0.75		25.180571	9.267281	1.818389	0.181014
0.80		25.180573	9.267289	1.818394	0.181015
0.85		25.180575	9.267296	1.818399	0.181016
0.90		25.180578	9.267303	1.818405	0.181018
0.95		25.180579	9.267310	1.818410	0.181019
$c = 1$		25.180581	9.267317	1.818415	0.181020

Table A.2: Call option (on SWISS BANK CO B) prices with $S(0) = 263.5$ and $T = 28$ days on November 20, 1992

		$K = 280$	300	320	340
Observed	lpp	50.00	26.00	13.00	4.00
Prices	sp	45.50	26.00	13.00	4.00
Black Scholes		46.441	27.283	11.764	3.278
Linear Approximation		46.385	26.884	10.962	2.715
Shifted Gamma		46.822	27.353	11.276	2.850
Shifted Inverse Gaussian		46.557	27.090	11.090	2.775
$c = 0$		46.823845	27.3226	11.2974	2.8653
0.05		46.823828	27.3224	11.2970	2.8650
0.10		46.823826	27.3223	11.2966	2.8646
0.15		46.823815	27.3222	11.2962	2.8643
0.20		46.823804	27.3220	11.2958	2.8639
0.25		46.823795	27.3219	11.2954	2.8636
0.30		46.823786	27.3217	11.2950	2.8632
0.35		46.823778	27.3216	11.2946	2.8629
0.40		46.823766	27.3215	11.2941	2.8625
0.45		46.823759	27.3213	11.2937	2.8622
0.50		46.823749	27.3212	11.2933	2.8618
0.55		46.823737	27.3211	11.2929	2.8615
0.60		46.823728	27.3209	11.2925	2.8611
0.65		46.823718	27.3208	11.2921	2.8608
0.70		46.823710	27.3206	11.2917	2.8604
0.75		46.823699	27.3205	11.2913	2.8601
0.80		46.823690	27.3204	11.2909	2.8597
0.85		46.823680	27.3202	11.2905	2.8594
0.90		46.823670	27.3201	11.2901	2.8590
0.95		46.823661	27.3200	11.2896	2.8587
$c = 1$		46.823651	27.3198	11.2892	2.8583

Table A.3: Call option (on SWISS BANK CO B) prices with $S(0) = 325$ and $T = 37$ days on January 13, 1993

	$K = 370$	380	390	400	425
Observed Prices lpp	19.00	14.00	9.50	6.50	1.40
sp	19.00	14.00	9.50	6.50	1.40
Black Scholes	20.346	14.736	10.271	6.884	2.132
Linear Approximation	20.447	14.901	10.491	7.132	2.344
Shifted Gamma	20.460	14.913	10.498	7.138	2.349
Shifted Inverse Gaussian	20.460	14.913	10.498	7.139	2.350
$c = 0$	20.57149	14.99394	10.55606	7.17867	2.36462
0.05	20.57152	14.99397	10.55608	7.17869	2.36464
0.10	20.57155	14.99400	10.55612	7.17872	2.36465
0.15	20.57157	14.99403	10.55614	7.17875	2.36467
0.20	20.57160	14.99406	10.55617	7.17877	2.36468
0.25	20.57163	14.99409	10.55620	7.17880	2.36470
0.30	20.57165	14.99412	10.55623	7.17883	2.36472
0.35	20.57168	14.99415	10.55626	7.17885	2.36473
0.40	20.57171	14.99418	10.55629	7.17888	2.36475
0.45	20.57173	14.99420	10.55632	7.17890	2.36476
0.50	20.57176	14.99423	10.55634	7.17893	2.36478
0.55	20.57179	14.99426	10.55637	7.17896	2.36479
0.60	20.57182	14.99429	10.55640	7.17898	2.36481
0.65	20.57184	14.99432	10.55643	7.17901	2.36483
0.70	20.57187	14.99435	10.55646	7.17903	2.36484
0.75	20.57190	14.99438	10.55649	7.17906	2.36486
0.80	20.57192	14.99441	10.55652	7.17909	2.36487
0.85	20.57195	14.99443	10.55654	7.17911	2.36489
0.90	20.57198	14.99447	10.55657	7.17914	2.36490
0.95	20.57201	14.99449	10.55660	7.17917	2.36492
$c = 1$	20.57203	14.99452	10.55663	7.17919	2.36494

Table A.4: Call option (on SWISS BANK CO B) prices with $S(0) = 380$ and $T = 50$ days on September 9, 1994

	$K = 350$	360	370	380	390	400	425
Observed Prices *lpp*	31.50	27.00	19.00	15.00	11.50	8.50	4.00
sp	33.00	26.50	21.00	15.00	11.50	8.50	3.40
Black Scholes	31.596	24.639	18.671	13.736	9.805	6.789	2.377
Linear Approximation	31.658	24.757	18.849	13.965	10.069	7.066	2.603
Shifted Gamma	32.745	25.737	19.703	14.683	10.651	7.523	2.820
Shifted Inverse Gaussian	31.656	24.755	18.847	13.963	10.067	7.066	2.606
$c = 0$	31.88106	24.93058	18.98086	14.06309	10.13975	7.11750	2.62631
0.05	31.88109	24.93061	18.98089	14.06312	10.13978	7.11753	2.62633
0.10	31.88111	24.93064	18.98092	14.06316	10.13981	7.11756	2.62634
0.15	31.88114	24.93067	18.98095	14.06319	10.13984	7.11758	2.62636
0.20	31.88116	24.93070	18.98099	14.06322	10.13987	7.11761	2.62638
0.25	31.88118	24.93073	18.98102	14.06325	10.13990	7.11764	2.62640
0.30	31.88121	24.93076	18.98105	14.06329	10.13994	7.11767	2.62642
0.35	31.88124	24.93079	18.98108	14.06332	10.13997	7.11770	2.62643
0.40	31.88126	24.93081	18.98111	14.06335	10.14000	7.11772	2.62645
0.45	31.88128	24.93084	18.98114	14.06339	10.14003	7.11775	2.62647
0.50	31.88131	24.93087	18.98118	14.06342	10.14006	7.11778	2.62649
0.55	31.88133	24.93090	18.98121	14.06345	10.14009	7.11781	2.62650
0.60	31.88136	24.93093	18.98124	14.06348	10.14012	7.11784	2.62652
0.65	31.88138	24.93096	18.98127	14.06352	10.14015	7.11787	2.62654
0.70	31.88141	24.93099	18.98130	14.06355	10.14019	7.11789	2.62656
0.75	31.88143	24.93102	18.98134	14.06358	10.14022	7.11792	2.62657
0.80	31.88146	24.93105	18.98137	14.06361	10.14025	7.11795	2.62659
0.85	31.88148	24.93108	18.98140	14.06365	10.14028	7.11798	2.62661
0.90	31.88151	24.93110	18.98143	14.06368	10.14031	7.11801	2.62663
0.95	31.88153	24.93113	18.98146	14.06371	10.14034	7.11803	2.62664
$c = 1$	31.88155	24.93116	18.98149	14.06374	10.14037	7.11806	2.62666

Table A.5: Call option (on SWISS BANK CO B) prices with $S(0) = 374$ and $T = 64$ days on August 18, 1994

	$K = 350$	360	370	380	400	425	448
Observed Prices lpp	36.50	36.00	33.00	25.00	18.50	10.50	6.00
sp	41.50	36.00	30.50	25.50	14.00	10.50	6.00
Black Scholes	37.094	30.714	25.076	20.185	12.541	6.410	3.223
Linear Approximation	37.256	30.926	25.336	20.486	12.888	6.744	3.493
Shifted Gamma	37.255	30.925	25.335	20.485	12.887	6.744	3.497
Shifted Inverse Gaussian	37.255	30.925	25.335	20.485	12.888	6.745	3.498
$c = 0$	37.77262	31.35551	25.68760	20.77044	13.06862	6.84087	3.54795
0.05	37.77265	31.35555	25.68764	20.77048	13.06866	6.84091	3.54797
0.10	37.77270	31.35560	25.68769	20.77053	13.06871	6.84094	3.54800
0.15	37.77273	31.35563	25.68774	20.77058	13.06875	6.84098	3.54802
0.20	37.77277	31.35568	25.68778	20.77062	13.06879	6.84101	3.54804
0.25	37.77280	31.35572	25.68782	20.77067	13.06883	6.84104	3.54807
0.30	37.77285	31.35576	25.68787	20.77071	13.06888	6.84108	3.54809
0.35	37.77288	31.35580	25.68791	20.77076	13.06892	6.84111	3.54812
0.40	37.77292	31.35584	25.68796	20.77080	13.06896	6.84115	3.54814
0.45	37.77296	31.35588	25.68800	20.77085	13.06901	6.84118	3.54817
0.50	37.77300	31.35593	25.68805	20.77089	13.06905	6.84122	3.54819
0.55	37.77303	31.35597	25.68809	20.77094	13.06909	6.84125	3.54822
0.60	37.77307	31.35601	25.68813	20.77098	13.06913	6.84129	3.54824
0.65	37.77311	31.35605	25.68818	20.77103	13.06918	6.84132	3.54827
0.70	37.77315	31.35609	25.68822	20.77108	13.06922	6.84135	3.54829
0.75	37.77318	31.35614	25.68827	20.77112	13.06926	6.84139	3.54832
0.80	37.77322	31.35618	25.68831	20.77117	13.06931	6.84142	3.54834
0.85	37 77326	31.35622	25.68836	20.77121	13.06935	6.84146	3.54836
0.90	37.77330	31.35626	25.68840	20.77126	13.06939	6.84149	3.54839
0.95	37.77334	31.35630	25.68845	20.77130	13.06944	6.84153	3.54841
$c = 1$	37.77337	31.35634	25.68849	20.77135	13.06948	6.84156	3.54844

Table A.6: Call option (on SWISS BANK CO B) prices with $S(0) = 373$ and $T = 122$ days on June 21, 1994

		$K = 350$	360	390	400	425	448
Observed	lpp	71.50	65.50	44.50	39.50	27.00	18.50
Prices	sp	73.50	66.00	45.00	39.00	27.00	18.50
Black Scholes		70.040	61.968	40.813	34.905	22.742	14.643
Linear Approximation		70.114	62.080	41.063	35.201	23.130	15.067
Shifted Gamma		70.114	62.079	41.062	35.200	23.129	15.066
Shifted Inverse Gaussian		70.114	62.079	41.062	35.201	23.130	15.067
$c = 0$		71.29242	63.12283	41.75241	35.79294	23.51978	15.32194
0.05		71.29245	63.12286	41.75246	35.79299	23.51984	15.32199
0.10		71.29249	63.12290	41.75251	35.79305	23.51990	15.32205
0.15		71.29252	63.12294	41.75256	35.79310	23.51995	15.32210
0.20		71.29254	63.12297	41.75261	35.79316	23.52001	15.32216
0.25		71.29257	63.12300	41.75266	35.79320	23.52006	15.32221
0.30		71.29260	63.12304	41.75271	35.79326	23.52012	15.32226
0.35		71.29263	63.12307	41.75276	35.79331	23.52018	15.32231
0.40		71.29266	63.12311	41.75281	35.79337	23.52023	15.32236
0.45		71.29268	63.12314	41.75286	35.79342	23.52029	15.32242
0.50		71.29271	63.12318	41.75291	35.79347	23.52034	15.32247
0.55		71.29274	63.12321	41.75296	35.79352	23.52040	15.32252
0.60		71.29277	63.12325	41.75301	35.79358	23.52046	15.32257
0.65		71.29280	63.12328	41.75306	35.79363	23.52051	15.32263
0.70		71.29283	63.12332	41.75311	35.79369	23.52057	15.32268
0.75		71.29286	63.12335	41.75316	35.79374	23.52063	15.32273
0.80		71.29288	63.12338	41.75321	35.79379	23.52068	15.32278
0.85		71.29291	63.12342	41.75326	35.79384	23.52074	15.32284
0.90		71.29294	63.12346	41.75331	35.79390	23.52079	15.32289
0.95		71.29297	63.12349	41.75336	35.79395	23.52085	15.32294
$c = 1$		71.29300	63.12353	41.75341	35.79400	23.52091	15.32299

Table A.7: Call option (on SWISS BANK CO B) prices with $S(0) = 410$, $T = 158$ days on May 16, 1994

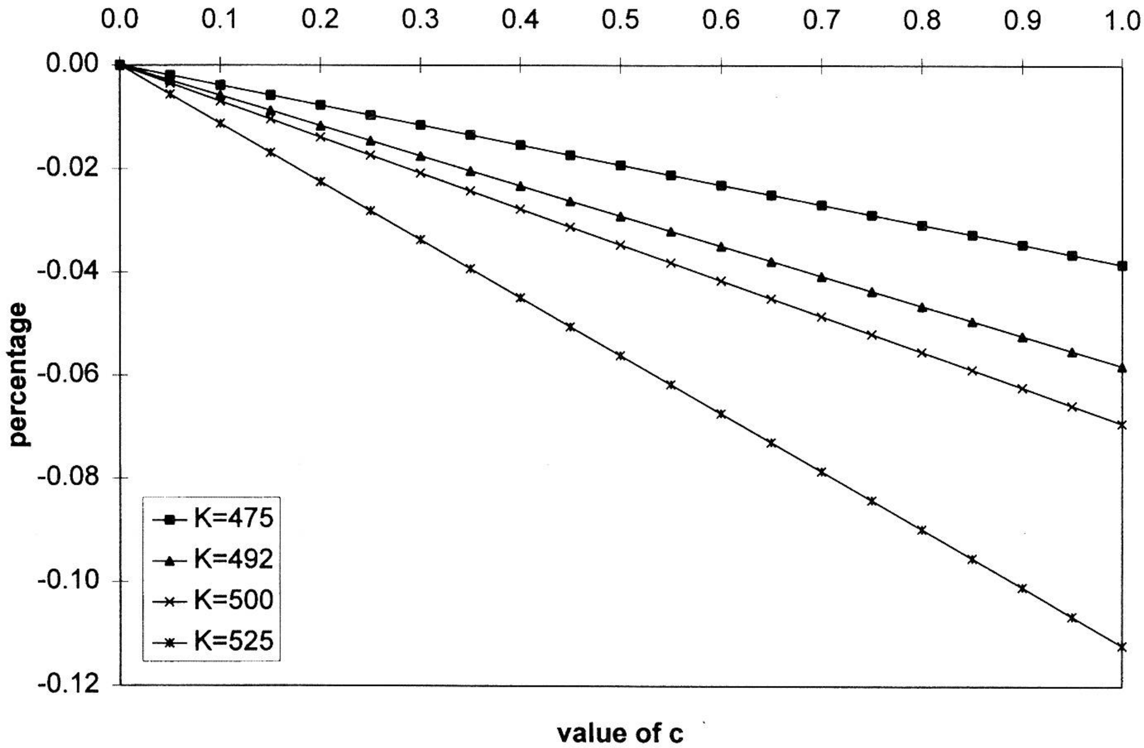

Figure A.1: Change in price from $c = 0$, $S(0) = 508$, $T = 94$ days (ALUSUISSE)

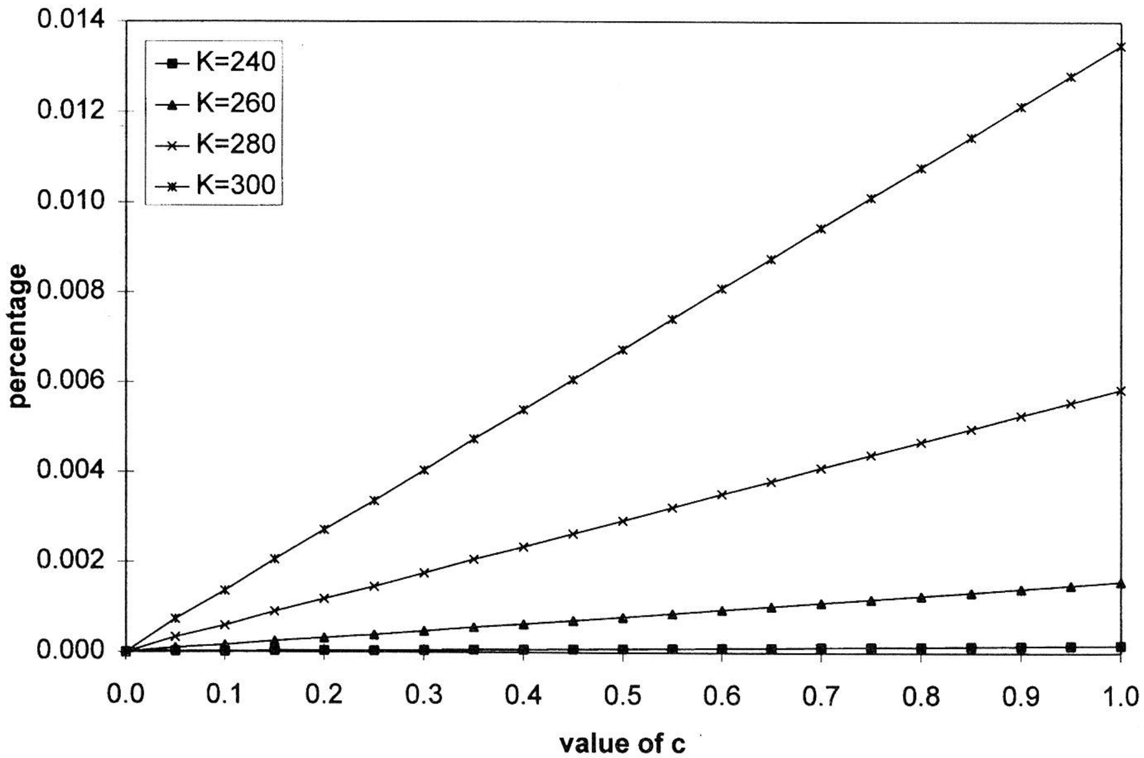

Figure A.2: Change in price from $c = 0$, $S(0) = 263.5$, $T = 28$ days (SBC)

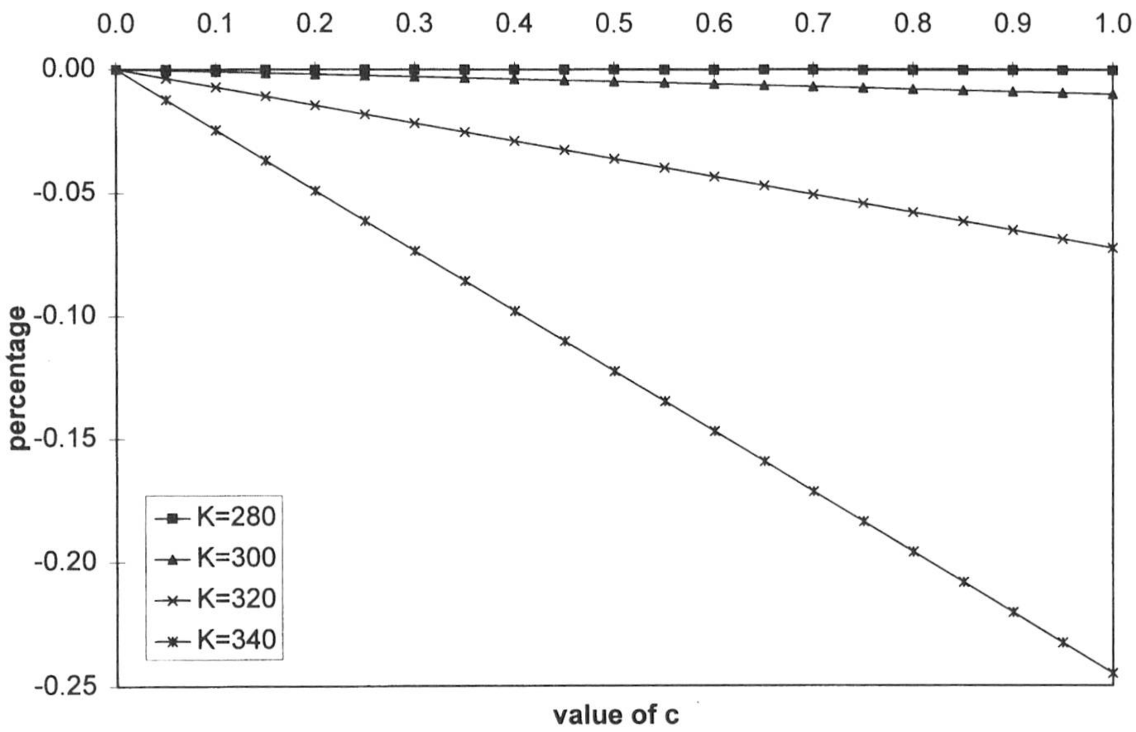

Figure A.3: Change in price from $c = 0$, $S(0) = 325$, $T = 37$ days (SBC)

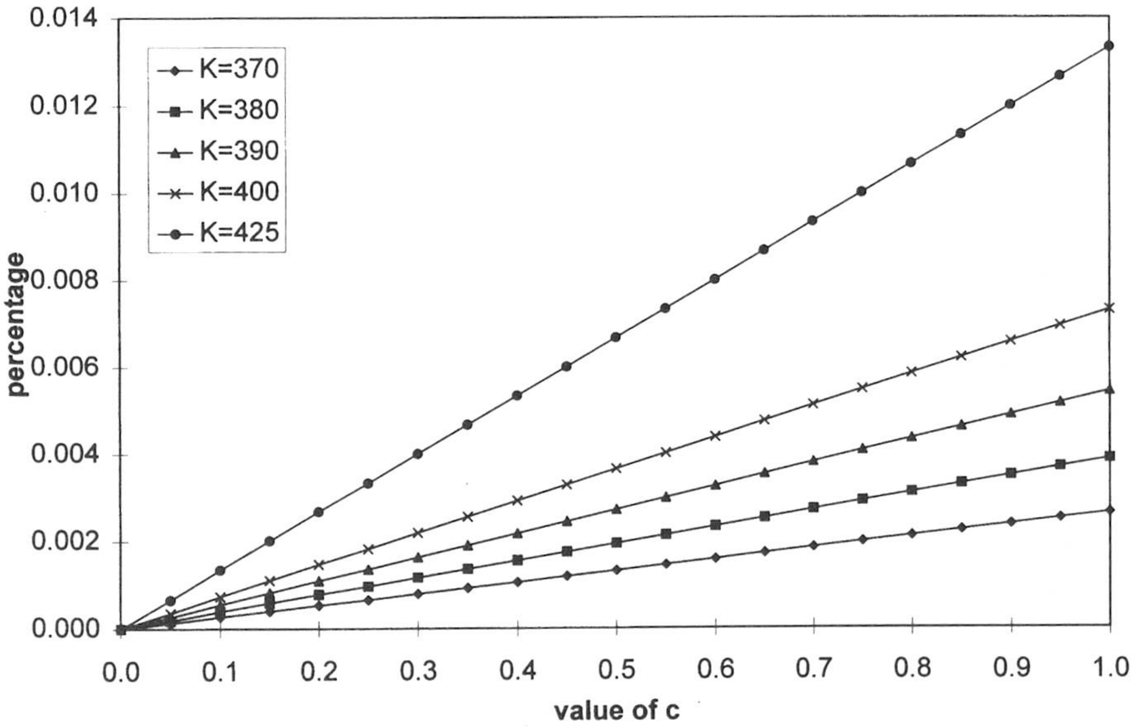

Figure A.4: Change in price from $c = 0$, $S(0) = 380$, $T = 50$ days (SBC)

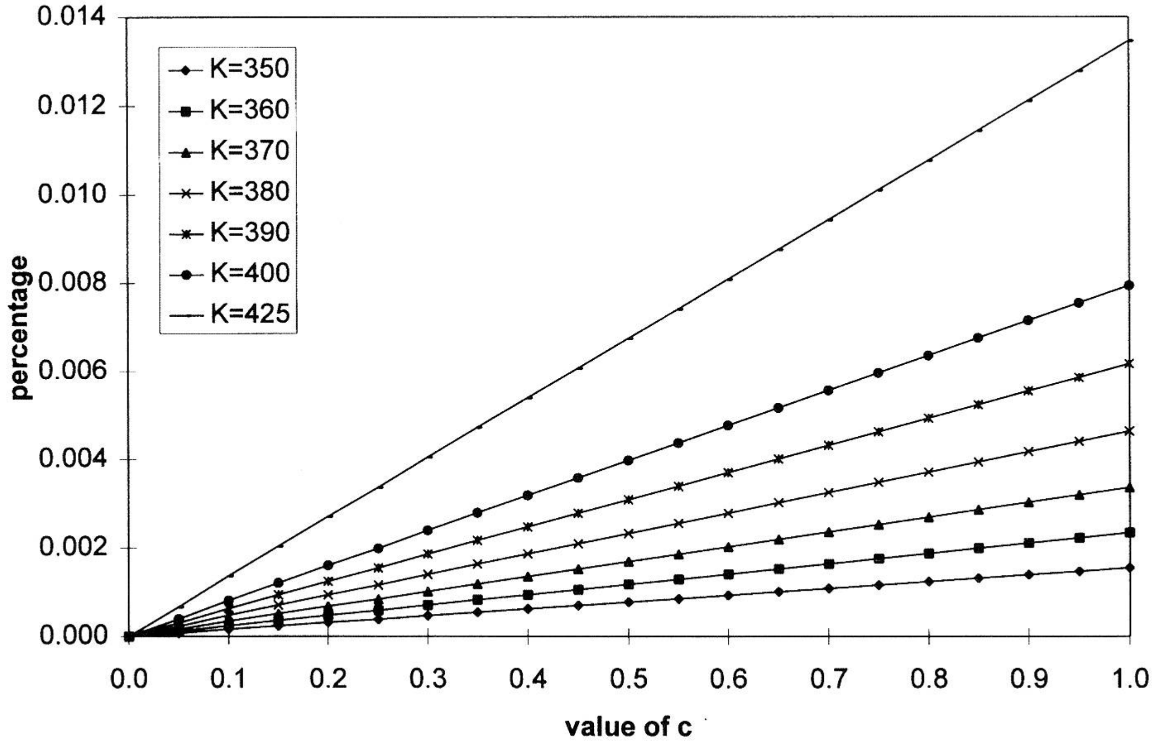

Figure A.5: Change in price from $c = 0$, $S(0) = 374$, $T = 64$ days (SBC)

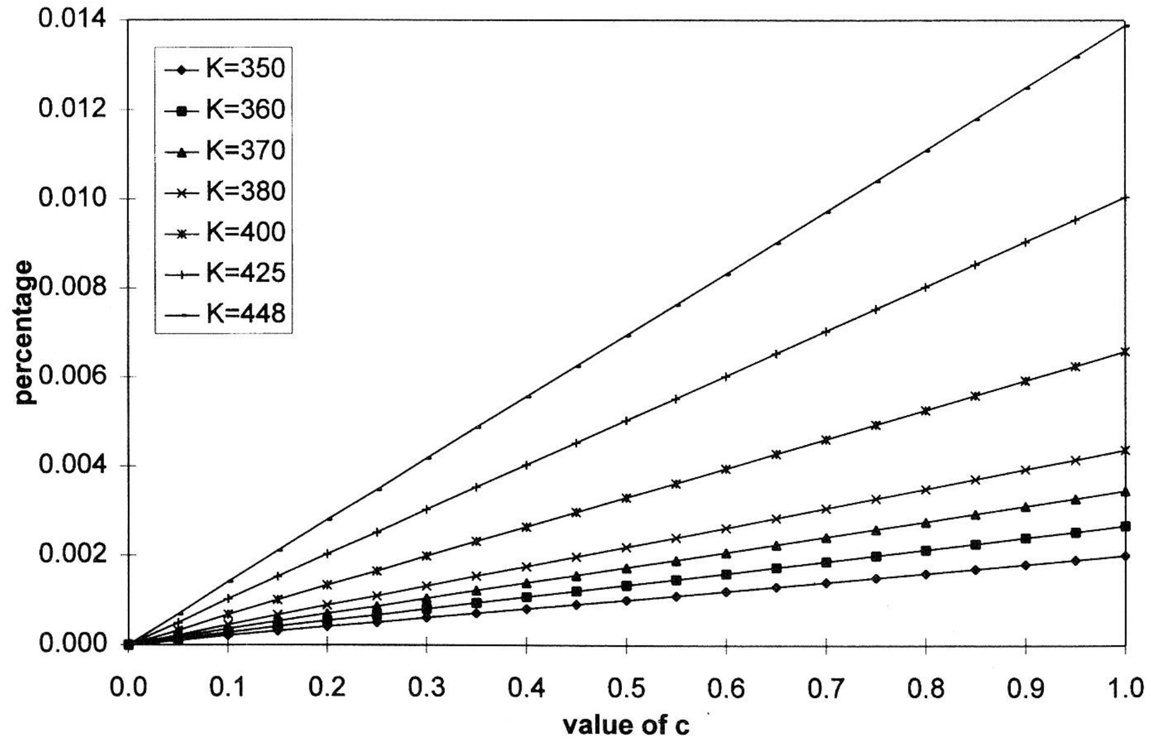

Figure A.6: Change in price from $c = 0$, $S(0) = 373$, $T = 122$ days (SBC)

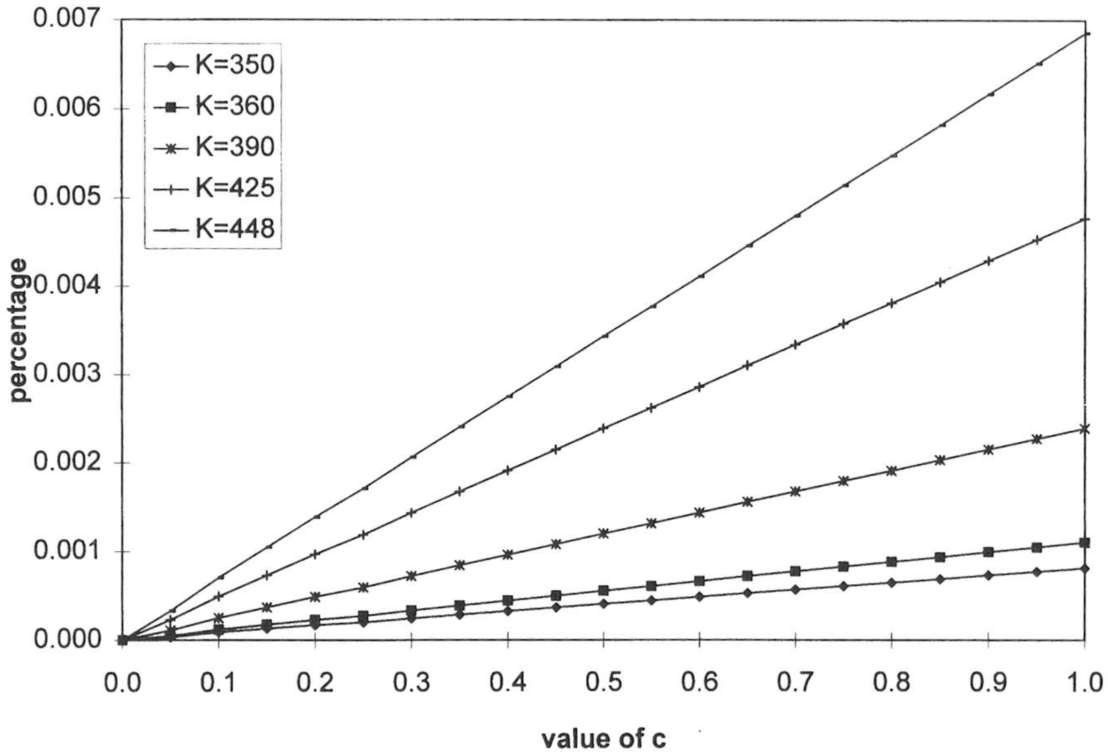

Figure A.7: Change in price from $c = 0$, $S(0) = 410$, $T = 158$ days (SBC)

Appendix B

Exercise Price K	Observed Prices lpp	sp	Black-Scholes Using Implied Volatility (BS-iv)	Linear Approximation Using Implied μ, σ and γ (LA-i)
475	49.00	49.00	49.447	49.050
492	42.50	35.50	38.465	38.093
500	35.50	33.50	33.876	33.527
525	22.00	21.50	21.952	21.703

Table B.1: Call option (on ALUSUISSE R) prices with $S(0) = 508$, $T = 94$ days on July 13, 1993 ($\gamma = 0.000\,001\,906\,308\,2$)

Exercise Price K	Observed Prices lpp	sp	Bs-iv	LA-i
240	26.50	25.50	25.364	25.231
260	9.00	8.50	10.056	9.751
280	2.40	2.40	2.392	2.196
300	1.00	1.00	0.323	0.276

Table B.2: Call option (on SWISS BANK CO B) prices with $S(0) = 263.5$, $T = 28$ days on November 20, 1992 ($\gamma = 0.000\,000\,497\,237\,5$)

Exercise Price K	Observed Prices lpp	sp	Bs-iv	LA-i
280	50.00	45.50	46.929	46.889
300	26.00	26.00	28.175	28.114
320	13.00	13.00	13.153	13.154
340	4.00	4.00	4.415	4.504

Table B.3: Call option (on SWISS BANK CO B) prices with $S(0) = 325$, $T = 37$ days on May 12, 1993 ($\gamma = 0.000\,000\,899\,243\,8$)

Exercise Price K	Observed Prices lpp	sp	Bs-iv	LA-i
370	19.00	19.00	20.906	20.860
380	14.00	14.00	15.337	15.337
390	9.50	9.50	10.867	10.914
400	6.50	6.50	7.431	7.517
425	1.40	1.40	2.461	2.574

Table B.4: Call option (on SWISS BANK CO B) prices with $S(0) = 380$, $T = 50$ days on September 1, 1994 ($\gamma = 0.000\,001\,191\,664\,3$)

Exercise Price K	Observed Prices lpp	sp	Bs-iv	LA-i
350	31.50	33.00	33.963	33.903
360	27.00	26.50	27.381	27.339
370	19.00	21.00	21.649	21.633
380	15.00	15.00	16.783	16.794
390	11.50	11.50	12.756	12.793
400	8.50	8.50	9.506	9.565
425	4.00	3.40	4.191	4.273

Table B.5: Call option (on SWISS BANK CO B) prices with $S(0) = 374$, $T = 64$ days on August 18, 1994 ($\gamma = 0.000\,001\,191\,664\,3$)

Exercise Price K	Observed Prices lpp	sp	Bs-iv	LA-i
350	36.50	41.50	41.749	38.891
360	36.00	36.00	35.782	32.716
370	33.00	30.50	30.417	27.224
380	25.00	25.50	25.647	22.411
400	18.50	14.00	17.809	14.716
425	10.50	10.50	10.822	8.226
448	6.00	6.00	6.588	4.581

Table B.6: Call option (on SWISS BANK CO B) prices with $S(0) = 373$, $T = 122$ days on June 21, 1994 ($\gamma = 0.000\,001\,191\,664\,3$)

Exercise Price K	Observed Prices lpp	sp	Bs-iv	LA-i
350	71.50	73.50	73.436	73.370
360	65.50	66.00	65.938	65.876
390	44.50	45.00	46.215	46.184
400	39.50	39.00	40.617	40.601
425	27.00	27.00	28.756	28.780
448	18.50	18.50	20.367	20.425

Table B.7: Call option (on SWISS BANK CO B) prices with $S(0) = 410$, $T = 158$ days on May 16, 1994 ($\gamma = 0.000\,001\,191\,664\,3$)

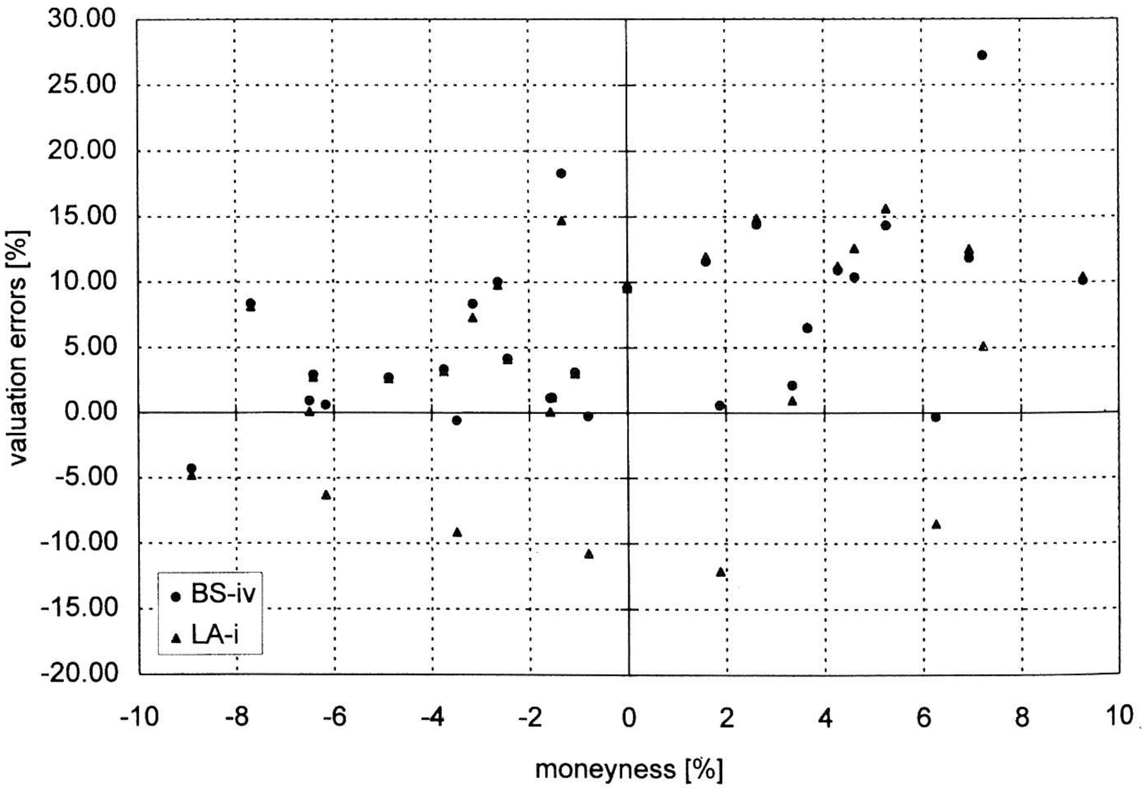

Figure B.1: Valuation errors in percent of the settlement prices for the Black-Scholes and the linear approximation formula, using for both implied parameters

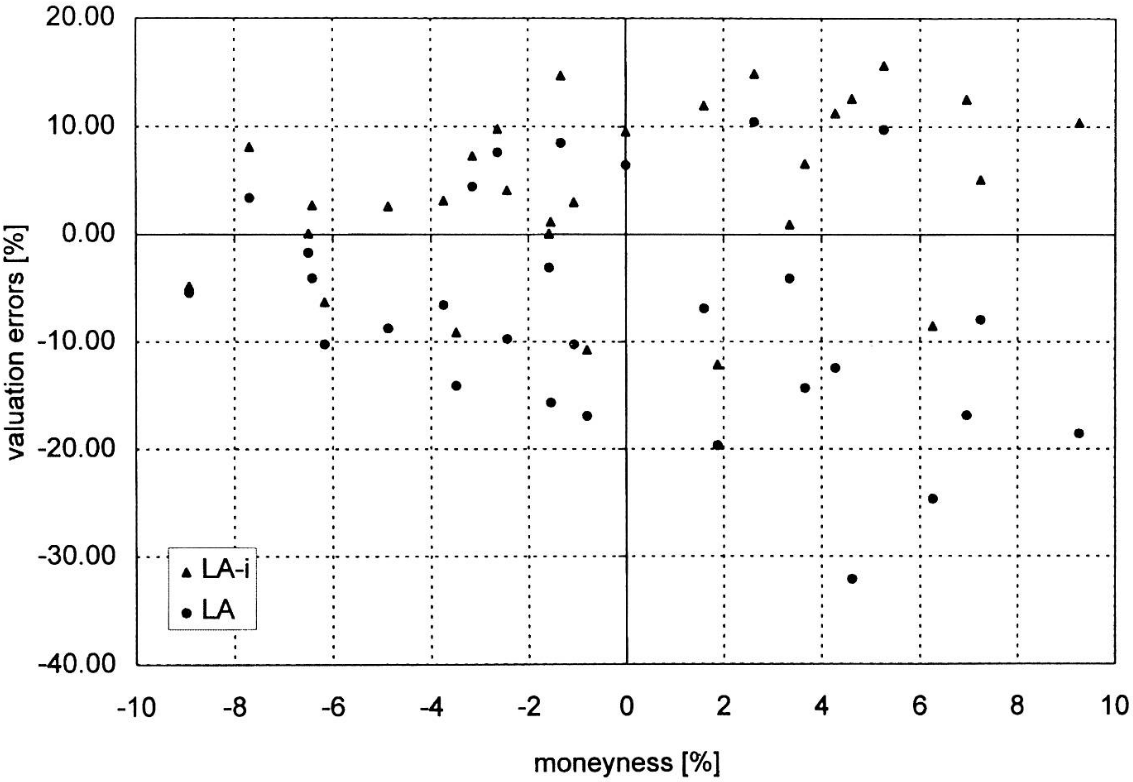

Figure B.2: Valuation errors in percent of the settlement prices for the linear approximation with (LA-i) and without (LA) implied parameters

Summary

This paper studies a one parameter family of equivalent martingale measures to price an option in a particular incomplete model. We also examine by means of real data an approximation formula introduced by Gerber and Landry. We propose to estimate the parameters in an implicit way in order to compute this formula. The study makes use of observed data from the SOFFEX (Swiss Options and Financial Futures Exchange).

Zusammenfassung

Der vorliegende Artikel betrachtet eine einparametrige Familie von äquivalenten Martingalmaßen zur Bewertung von Optionen in einem unvollständigen Markt. Überdies wird die Näherungsformel von Gerber und Landry anhand von wirklichen Daten untersucht. Schließlich wird eine implizite Schätzung der unbekannten Parameter vorgeschlagen. Die Studie stützt sich auf Daten der SOFFEX.

Résumé

Cet article étudie une famille à un paramètre de mesures de martingales équivalentes pour évaluer une option dans un modèle incomplet particulier. On examine aussi à l'aide de données réelles une formule d'approximation introduite par Gerber et Landry. On propose finalement d'estimer les paramètres de manière implicite afin d'évaluer cette formule. L'étude fait usage de données provenant de la SOFFEX.