# On best stop-loss bounds for bivariate sums by known marginal means, variances and correlation

Autor(en): **Hürlimann, Werner**

Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Schweizerische Aktuarvereinigung = Bulletin / Association Suisse des Actuaires = Bulletin / Swiss Association of Actuaries**

Band (Jahr): **- (1998)**

Heft 1

PDF erstellt am: **29.06.2024**

Persistenter Link: https://doi.org/10.5169/seals-551072

WERNER HÜRLIMANN, Winterthur

# On best stop-loss bounds for bivariate sums by known marginal means, variances and correlation

## 1 Introduction

Though there is a huge of statistical methods dealing with all aspects of bivariate and multivariate dependency, their actuarial applications have only scarcely been developed. Various effects of independence assumptions on actuarial calculations have been noticed among others by Heilmann (1986), Norberg (1989), Kling (1993), and Dhaene and Goovaerts(1996).

In the present paper, we determine first the maximal stop-loss bounds for diatomic bivariate sums by given marginal means, variances and fixed positive correlation coefficient. In Section 2 the required structure of bivariate diatomic couples is derived. Section 3 solves the problem under a condition of strict positive dependence. The maximal stop-loss bounds are shown to be strictly less than the corresponding univariate best bounds by Bowers (1969) unless complete dependence is assumed. The obtained formulas can be regarded as the bivariate extension of Bowers' bounds. As Section 4 demonstrates, the independent case depends upon a biquadratic equation and is mathematically more complex. A closed formula can be obtained if the stop-loss deductible equals the mean, a special case known to be important in applications.

In Section 5 one examines if the obtained maximal bivariate diatomic stop-loss bounds also yield the maximum over arbitrary bivariate sums by known first and second order moment structure. For this, a method of predilection is the bivariate version of the widespread univariate quadratic polynomial majorant method, which has been studied systematically by the author (1997a/97b). The maximal stop-loss bounds for diatomic bivariate sums by known positive correlation yield an overall maximum for arbitrary bivariate sums with the same known characteristics if and only if complete dependence holds. In contrast to this, the minimal stop-loss bounds are attained, over large ranges of deductibles, by diatomic bivariate couples under any given negative correlation. As immediate application, the maximal expected value of an exchange option is discussed in Example 5.1. Suggestions for further work are also made.

Finally, note that in the present paper random variables are defined on arbitrary infinite supports $(-\infty, \infty)$. In practical applications, supports are often halfinfinite intervals as $[0,\infty)$ or finite intervals $[a,b]$, $-\infty < a < b < \infty$. These more complex situations have been studied in Hürlimann (1997c/97d/97e).

## 2 Structure of bivariate diatomic couples

Random variables are assumed to take values on the whole real line $(-\infty,\infty)$. Recall the structure in the univariate case.

**Lemma 2.1.** The set $D_2(\mu,\sigma)$ of all non-degenerate diatomic random variables with mean $\mu$ and variance $\sigma^2$ is described by a one-parametric family of supports $\{x_1,x_2\}$, $x_1 < x_2$, and probabilities $\{p_1,p_2\}$ such that

$$\begin{aligned} &x_2 = \mu + \frac{\sigma^2}{\mu - x_1}, \\ &p_1 = \left(\frac{x_2-\mu}{x_2-x_1}\right), \quad p_2 = \left(\frac{\mu-x_1}{x_2-x_1}\right), \quad x_1 < \mu < x_2, \end{aligned} \tag{2.1}$$

or equivalently

$$x_1 = \mu - \sigma\sqrt{\frac{p_2}{p_1}}, \quad x_2 = \mu + \sigma\sqrt{\frac{p_1}{p_2}}, \quad 0 < p_1 < 1. \tag{2.2}$$

**Proof.** The following equations must hold:

$$\begin{aligned} p_1 + p_2 &= 1, \\ p_1 x_1 + p_2 x_2 &= \mu, \\ p_1 \cdot (x_1-\mu)^2 + p_2 \cdot (x_2-\mu)^2 &= \sigma^2. \end{aligned}$$

The first two equations yield

$$p_1 = \left(\frac{x_2-\mu}{x_2-x_1}\right), \quad p_2 = \left(\frac{\mu-x_1}{x_2-x_1}\right).$$

It remains to satisfy the third equation of variance. Inserting the preceding formulas one finds $\sigma^2 = (\mu - x_1)\cdot(x_2-\mu)$. Taken together this shows (2.1). The rest is immediate and left to the reader. □

To clarify the structure of bivariate diatomic couples consider the set

$$BD_2 := \big\{(X,Y)\colon X \in D_2(\mu_x,\sigma_x),\ Y \in D_2(\mu_y,\sigma_y),\\ \operatorname{Cov}[X,Y] = \varrho\sigma_x\sigma_y\big\}\,. \tag{2.3}$$

The marginal $X$ has support $\{x_1, x_2\}$, $x_1 < x_2$ and probabilities $\{p_1, p_2\}$, and $Y$ has support $\{y_1, y_2\}$, $y_1 < y_2$ and probabilities $\{q_1, q_2\}$. By Lemma 2.1 one has the relations

$$\begin{aligned} x_1 &= \mu_x - \sigma_x\sqrt{\frac{p_2}{p_1}}\,, \quad x_2 = \mu_x + \sigma_x\sqrt{\frac{p_1}{p_2}}\,,\\ y_1 &= \mu_y - \sigma_y\sqrt{\frac{q_2}{q_1}}\,, \quad y_2 = \mu_y + \sigma_y\sqrt{\frac{q_1}{q_2}}\,. \end{aligned} \tag{2.4}$$

The bivariate distribution of a couple $(X,Y)$ is uniquely determined by the distribution of $X$ and the conditional distribution of $(Y|X)$. Thus one has to choose a triple $(\alpha,\beta,p_1)$ such that

$$\begin{aligned} \alpha &= P(Y = y_1 \mid X = x_1)\,,\\ \beta &= P(Y = y_1 \mid X = x_2)\,, \qquad 0 < \alpha+\beta < 2, \quad 0 < p_1 < 1\\ p_1 &= P(X = x_1)\,, \end{aligned} \tag{2.5}$$

Then the joint probabilities $p_{ij} = P(X = x_i, Y = y_j)$, $i,j = 1,2$, are given by

$$\begin{aligned} p_{11} &= \alpha p_1\,, \quad p_{12} = (1-\alpha)p_1\,,\\ p_{21} &= \beta p_2\,, \quad p_{22} = (1-\beta)p_2\,. \end{aligned} \tag{2.6}$$

An equivalent representation in terms of the marginal probabilities and the correlation coefficient, that is in terms of the triple $(p_1, q_1, \varrho)$ is obtained as follows.
The marginal probability of $Y$ satisfies the relation

$$\alpha p_1 + \beta p_2 = q_1\,, \tag{2.7}$$

and the correlation coefficient the relation

$$(\alpha-\beta)p_1p_2 = \varrho\sqrt{p_1p_2q_1q_2}\,. \tag{2.8}$$

Solving the linear system (2.7), (2.8) and inserting into (2.6), one gets the following *canonical representation*.

**Lemma 2.2.** The joint probabilities of a diatomic bivariate couple $(X,Y) \in BD_2$, with marginal probabilities

$$p_1 = \left(\frac{x_2 - \mu_x}{x_2 - x_1}\right), \quad q_1 = \left(\frac{y_2 - \mu_y}{y_2 - y_1}\right),$$

variances $\sigma_x^2 = (\mu_x - x_1) \cdot (x_2 - \mu_x)$, $\sigma_y^2 = (\mu_y - y_1) \cdot (y_2 - \mu_y)$, and correlation coefficient $\varrho$, are given by

$$\begin{aligned} p_{11} &= p_1 q_1 + \varrho\sqrt{p_1 p_2 q_1 q_2}\,, \\ p_{12} &= p_1 q_2 - \varrho\sqrt{p_1 p_2 q_1 q_2}\,, \\ p_{21} &= p_2 q_1 - \varrho\sqrt{p_1 p_2 q_1 q_2}\,, \\ p_{22} &= p_2 q_2 + \varrho\sqrt{p_1 p_2 q_1 q_2}\,. \end{aligned} \tag{2.9}$$

For calculations with diatomic couples $(X, Y)$, it suffices sometimes to consider a unique *canonical arrangement* of its atoms.

**Lemma 2.3.** Without loss of generality the atoms of a couple $(X,Y) \in BD_2$ can be rearranged such that $x_1 < x_2$, $y_1 < y_2$, $y_2 - y_1 \leq x_2 - x_1$. Then the atoms of the diatomic bivariate sum $X + Y$ satisfy the condition

$$x_1 + y_1 < x_1 + y_2 \leq x_2 + y_1 < x_2 + y_2\,. \tag{2.10}$$

**Proof.** By Lemma 2.1 one can assume $x_1 < x_2$, $y_1 < y_2$. If $y_2 - y_1 > x_2 - x_1$ then exchange the role of $X$ and $Y$. □

## 3 Optimization by strictly positive dependence

In applications, couples $(X, Y)$ show often "positive dependence" (e.g. the remaining life-times of a husband and his wife). If $(X,Y) \in BD_2$ "positive dependence" is always equivalent with $\varrho \geq 0$. For a fixed $\varrho > 0$ we solve in this Section the optimization problem

$$\pi^*(d) = \max_{(X,Y) \in BD_2} \{\pi(d)\}, \quad \text{where} \quad \pi(d) = E[(X + Y - d)_+]\,. \tag{3.1}$$

In general, given an arbitrary couple $(X,Y)$ with joint probability distribution $H(x,y)$ and fixed marginals $F(x)$, $G(y)$, one has the identity (e.g. Dhaene and Goovaerts (1996)):

$$E[(X+Y-d)_+] = E[X] + E[Y] - d + \int_{-\infty}^{d} H(x, d-x)\, dx\,. \tag{3.2}$$

By Hoeffding (1940) and Fréchet (1951) (e.g. Mardia (1970), p. 31) one knows that for all such $H(x,y)$ one has the best bivariate distribution bounds

$$\begin{aligned} H_*(x,y) &= \max\{F(x) + G(y) - 1, 0\} \\ &\le H(x,y) \le H^*(x,y) = \min\{F(x), G(y)\}\,. \end{aligned} \tag{3.3}$$

Therefore it follows from (3.2) that, by fixed marginals, the maximum of $\pi(d)$ is attained at Fréchet's upper bound $H^*(x,y)$. Varying the marginals under a given "positive dependence" structure, it is possible to obtain divers maximal stop-loss bounds.

In the special situation $(X,Y) \in BD_2$, Fréchet's upper bound is described by the following joint probabilities :

$$p_{11} = p_1\,,\quad p_{12} = 0\,,\qquad p_{21} = q_1 - p_1\,,\quad p_{22} = q_2\,,\quad \text{if } p_1 \le q_1\,, \tag{3.4}$$

$$p_{11} = q_1\,,\quad p_{12} = p_1 - q_1\,,\quad p_{21} = 0\,,\qquad p_{22} = p_2\,,\quad \text{if } p_1 \ge q_1\,. \tag{3.5}$$

Moreover, using (2.9) one sees that the marginal probabilities necessarily satisfy the following constraint:

$$\sqrt{\frac{q_2}{q_1}} = \varrho\sqrt{\frac{p_2}{p_1}} \quad \text{if } p_1 \le q_1\,, \tag{3.6}$$

$$\sqrt{\frac{p_2}{p_1}} = \varrho\sqrt{\frac{q_2}{q_1}} \quad \text{if } p_1 \ge q_1\,. \tag{3.7}$$

Exchanging $p$, $q$ in the results, it suffices to consider the case $p_1 \le q_1$. The identity

$$\begin{aligned} (X+Y-d)_+ &= ((X-\mu_x) + (Y-\mu_y) - (d-\mu))_+ \\ \mu &= \mu_x + \mu_y\,, \end{aligned} \tag{3.8}$$

allows one to reduce calculation to the case $\mu_x = \mu_y = \mu = 0$.

Using the above facts and summarizing, one can restrict the maximization to the subset of all $(X, Y) \in BD_2$ of the form

$$\begin{aligned} x_1 &= -\sigma_x \cdot \theta\,, & x_2 &= \sigma_x \cdot \frac{1}{\theta}\,, & p_1 &= \frac{1}{1+\theta^2}\,, & \theta &> 0\,, \\ y_1 &= -\varrho\sigma_y \cdot \theta\,, & y_2 &= \sigma_y \cdot \frac{1}{\varrho\theta}\,, & q_1 &= \frac{1}{1+(\varrho\theta)^2}\,, \\ p_{11} &= p_1\,, & p_{12} &= 0\,, & p_{21} &= q_1 - p_1\,, & p_{22} &= p_2\,. \end{aligned} \tag{3.9}$$

In particular $\pi(d)$ will be a univariate function of $\theta$, which one denotes with $\pi(d;\theta)$ in what follows. Two situations can occur. One has either

$$\begin{aligned} &x_1 + y_1 < x_2 + y_1 \le x_1 + y_2 < x_2 + y_2 \quad \text{or} \\ &x_1 + y_1 < x_1 + y_2 \le x_2 + y_1 < x_2 + y_2\,. \end{aligned} \tag{3.10}$$

Since $p_{12} = 0$ (no probability on the mass point $x_1 + y_2$) only four subcases are relevant. To see this, one notes that in the first situation the case $x_2 + y_1 \le d < x_1 + y_2$ will be included in the case (C3) below while in the second situation the case $x_1 + y_1 \le d < x_1 + y_2$ will be included in the case (C2). Omitting the elementary but tedious calculations, the following subcases must be considered:

$d \le x_1 + y_1$ :

$$\pi(d;\theta) = -d \tag{C1}$$

$x_1 + y_1 \le d \le x_2 + y_1$ :

$$\begin{aligned} \pi(d;\theta) &= (q_1 - p_1)(x_2 + y_1 - d) + q_2(x_2 + y_2 - d) \\ &= \left(\frac{\theta}{1+\theta^2}\right) \cdot (\sigma_x + \varrho\sigma_y - d\theta) \end{aligned} \tag{C2}$$

$x_2 + y_1 \le d < x_2 + y_2$ :

$$\begin{aligned} \pi(d;\theta) &= q_2(x_2 + y_2 - d) \\ &= \left(\frac{\varrho\theta}{1+(\varrho\theta)^2}\right) \cdot (\varrho\sigma_x + \sigma_y - d\varrho\theta) \end{aligned} \tag{C3}$$

$d \ge x_2 + y_2$ :

$$\pi(d;\theta) = 0\,. \tag{C4}$$

Obviously only (C2) and (C3) can lead to a maximum. Calculation of derivatives yields the first order necessary conditions and their unique solutions in $(0, \infty)$:

$$\begin{aligned} \frac{d}{d\theta}\pi(d;\theta) &= -(1+\theta^2)^{-2} \cdot \left\{(\sigma_x + \varrho\sigma_y)\theta^2 + 2d\theta - (\sigma_x + \varrho\sigma_y)\right\} \\ &= 0\,, \\ \theta_2^* &= (\sigma_x + \varrho\sigma_y)^{-1} \cdot \left\{\sqrt{(\sigma_x + \varrho\sigma_y)^2 + d^2} - d\right\}, \end{aligned} \tag{C2}$$

$$\frac{d}{d\theta}\pi(d;\theta) = -\varrho(1+(\varrho\theta)^2)^{-2} \cdot \{(\varrho\sigma_x + \sigma_y)(\varrho\theta)^2 + 2d\varrho\theta - (\varrho\sigma_x + \sigma_y)\}$$
$$= 0\,, \qquad \text{(C3)}$$
$$\varrho\theta_3^* = (\varrho\sigma_x + \sigma_y)^{-1} \cdot \left\{\sqrt{(\varrho\sigma_x + \sigma_y)^2 + d^2} - d\right\}\,.$$

Since the second derivatives of $\pi(d;\theta)$ are negative at these values, these solutions yield local maxima. The corresponding (local) maximal stop-loss bounds are

$$\pi_2^*(d) = \pi(d;\theta_2^*) = \frac{1}{2} \cdot \left\{\sqrt{(\sigma_x + \varrho\sigma_y)^2 + d^2} - d\right\}\,. \qquad \text{(C2)}$$
$$\pi_3^*(d) = \pi(d;\theta_3^*) = \frac{1}{2} \cdot \left\{\sqrt{(\varrho\sigma_x + \sigma_y)^2 + d^2} - d\right\}\,. \qquad \text{(C3)}$$

To show that the obtained local maxima actually yield global maxima, it suffices to show that the function $\pi(d;\theta)$ is concave over the corresponding domains of definition.

**Lemma 3.1.** One has $\frac{d^2}{d\theta^2}\pi(d;\theta) < 0$ for all values of $\theta$ satisfying the following constraints:

$$x_1 + y_1 = -(\sigma_x + \varrho\sigma_y)\theta \le d < x_2 + y_1 = \frac{\sigma_x - \varrho\sigma_y\theta^2}{\theta} \qquad \text{(C2)}$$
$$x_2 + y_1 = \frac{\sigma_x - \varrho\sigma_y\theta^2}{\theta} \le d < x_2 + y_2 = \frac{\varrho\sigma_x + \sigma_y}{\varrho\theta} \qquad \text{(C3)}$$

**Proof.** First calculate the second order derivatives in both cases

$$\frac{d^2}{d\theta^2}\pi(d;\theta) = -2(1+\theta^2)^{-3} \cdot \{(\sigma_x + \varrho\sigma_y)\theta(3-\theta^2) + d(1-3\theta^2)\}\,, \qquad \text{(C2)}$$
$$\frac{d^2}{d\theta^2}\pi(d;\theta) = -2\varrho^2[1+(\varrho\theta)^2]^{-3} \cdot \Big\{(\varrho\sigma_x + \sigma_y)\varrho\theta[3-(\varrho\theta)^2] + d[1-3(\varrho\theta)^2]\Big\}\,, \qquad \text{(C3)}$$

and then show that the curly brackets are positive by distinguishing between several subcases as follows:

$$1 - 3\theta^2 \ge 0 \quad \Rightarrow \quad \{\ldots\} \ge 2(\sigma_x + \varrho\sigma_y)\theta(1+\theta^2) > 0 \qquad \text{(C2a)}$$

$$1 - 3\theta^2 < 0 \tag{C2b}$$

$$d < 0 \quad \Rightarrow \quad \{\ldots\} > 0 \tag{2b1}$$

$$\begin{aligned} d \geq 0 \quad \Rightarrow \quad \{\ldots\} &\geq (\sigma_x + \varrho\sigma_y)\theta(3 - \theta^2) + \frac{\sigma_x - \varrho\sigma_y\theta^2}{\theta}(1 - 3\theta^2) \\ &= \frac{1 + \theta^2}{\theta} \cdot \{\sigma_x(1 - \theta^2) + 2\varrho\sigma_y\theta^2\} \\ &> 0 \end{aligned} \tag{C2b2}$$

$$1 - 3(\varrho\theta)^2 \geq 0$$

$$d \geq -(\varrho\sigma_x + \sigma_y)\varrho\theta \;\Rightarrow\; \{\ldots\} \geq 2(\varrho\sigma_x + \sigma_y)\varrho\theta[1 + (\varrho\theta)^2] > 0 \tag{C3a}$$

$$1 - 3(\varrho\theta)^2 < 0 \tag{C3b}$$

$$d < 0 \quad \Rightarrow \; \{\ldots\} > 0 \tag{C3b1}$$

$$\begin{aligned} d \geq 0 \quad \Rightarrow \; \{\ldots\} &\geq (\varrho\sigma_x + \sigma_y)\varrho\theta[3 - (\varrho\theta)^2] + \frac{\varrho\sigma_x + \sigma_y}{\varrho\theta}[1 - 3(\varrho\theta)^2] \\ &= \frac{\varrho\sigma_x + \sigma_y}{\varrho\theta}[1 - (\varrho\theta)^4] \\ &> 0. \qquad \square \end{aligned} \tag{C3b2}$$

Using this property we show that if $\sigma_x \geq \sigma_y$, then $\pi_2^*(d)$ is maximal, otherwise it is $\pi_3^*(d)$.

*Case 1:* $\sigma_x \geq \sigma_y$

With $\theta = \theta_2^*$ the constraints on $d$ in (C2) of Lemma 3.1 are fulfilled. This follows by using the relation $d\theta = \frac{1}{2}(\sigma_x + \varrho\sigma_y)(1 - \theta^2)$ and the facts $\sigma_x \geq \sigma_y$ and $0 < \varrho \leq 1$ for the second inequality constraint. Since the function $\pi(d;\theta)$ is concave over the feasible set of $\theta$'s, the local maximum is a global maximum.

*Case 2:* $\sigma_x \leq \sigma_y$

With $\theta = \theta_3^*$ the constraints on $d$ in (C3) of Lemma 3.1 are fulfilled. This follows by using the relation $d\varrho\theta = \frac{1}{2}(\varrho\sigma_x + \sigma_y)[1 - (\varrho\theta)^2]$ and the facts $\sigma_x \leq \sigma_y$ and $0 < \varrho \leq 1$ for the first inequality constraint. By Lemma 3.1 the local maximum is a global maximum.

The inequality

$$
\begin{aligned}
&\max\left\{(\sigma_x+\varrho\sigma_y)^2,(\varrho\sigma_x+\sigma_y)^2\right\}\\
&\quad\le \max\left\{(\sigma_x+\varrho\sigma_y)^2,(\varrho\sigma_x+\sigma_y)^2\right\}+(1-\varrho^2)\min\left\{\sigma_x^2,\sigma_y^2\right\}\\
&\quad=\sigma_x^2+\sigma_y^2+2\varrho\sigma_x\sigma_y\\
&\quad=:\sigma^2\,, \qquad (3.11)
\end{aligned}
$$

whose equality holds only by complete dependence $\varrho = 1$, shows that the maximum is always strictly less (unless $\varrho = 1$) than the corresponding univariate best upper bound by Bowers (1969), which is $\frac{1}{2}\cdot\{\sqrt{\sigma^2+d^2}-d\}$ (note that $\mu = 0$ is still assumed). Let us summarize the obtained result.

**Theorem 3.1.** The maximal net stop-loss premium of a diatomic bivariate sum $X+Y$, $(X,Y)\in BD_2$, with deductible $d$ by given marginal means $\mu_x,\mu_y$, variances $\sigma_x^2,\sigma_y^2$, and correlation coefficient $\varrho > 0$ is given by

$$
\begin{aligned}
&\max_{(X,Y)\in BD_2} E[(X+Y-d)_+]\\
&=\frac{1}{2}\left\{\sqrt{\max\{(\sigma_x^2+\varrho\sigma_y)^2,(\varrho\sigma_x+\sigma_y)^2\}+(d-\mu)^2}-(d-\mu)\right\}. \qquad (3.12)
\end{aligned}
$$

The maximum is attained for a diatomic bivariate distribution, which is determined as follows:

$$
\begin{aligned}
x_1&=\mu_x-\sigma_x\theta\,, & x_2&=\mu_x+\frac{\sigma_x}{\theta}\\
y_1&=\mu_y-\sigma_y\varrho\theta\,, & y_2&=\mu_y+\frac{\sigma_y}{\varrho\theta}\\
p_{11}&=\frac{1}{1+\theta^2}\,, & p_{12}&=0 \qquad (3.13)\\
p_{21}&=\frac{(1-\varrho^2)\theta^2}{(1+\theta^2)(1+(\varrho\theta)^2)}\,, & p_{22}&=\frac{(\varrho\theta)^2}{1+(\varrho\theta)^2}\,.
\end{aligned}
$$

where one has

$$
\theta=\begin{cases}
(\sigma_x+\varrho\sigma_y)^{-1}\cdot\left\{\sqrt{(\sigma_x+\varrho\sigma_y)^2+(d-\mu)^2}-(d-\mu)\right\}, & \text{if } \sigma_x\ge\sigma_y\,,\\
[\varrho(\varrho\sigma_x+\sigma_y)]^{-1}\cdot\left\{\sqrt{(\varrho\sigma_x+\sigma_y)^2+(d-\mu)^2}-(d-\mu)\right\}, & \text{if } \sigma_x\le\sigma_y\,.
\end{cases} \qquad (3.14)
$$

## 4 Optimization in the independent case

In the present Section one assumes that $\varrho = 0$ in the canonical representation (2.9). As shown in Section 3 one can assume $\mu_x = \mu_y = 0$. Then maximization of $\pi(d) = E[(X+Y-d)_+]$ must be done over all $(X,Y) \in BD_2$, of the form

$$\begin{aligned} &x_1 = -\sigma_x \cdot \xi\,, \quad x_2 = \sigma_x \cdot \frac{1}{\xi}\,, \quad p_1 = \frac{1}{1+\xi^2}\,, \quad \xi > 0\,, \\ &y_1 = -\sigma_y \cdot \theta\,, \quad y_2 = \sigma_y \cdot \frac{1}{\theta}\,, \quad q_1 = \frac{1}{1+\theta^2}\,, \quad \theta > 0\,, \\ &p_{11} = p_1 q_1\,, \quad p_{12} = p_1 q_2\,, \quad p_{21} = p_2 q_1\,, \quad p_{22} = p_2 q_2\,. \end{aligned} \tag{4.1}$$

Therefore $\pi(d)$ will be a bivariate function of $(\xi, \theta)$ denoted by $\pi(d;\xi,\theta)$. By Lemma 2.3 and its proof, it suffices to consider the following relevant cases, as well as any solution obtained by permuting $X$ and $Y$. Details of the elementary but somewhat tedious calculations are left to the reader. Three cases are distinguished:

$$x_1 + y_1 \le d < x_1 + y_2$$

$$\pi(d;\xi,\theta) = \left\{ \frac{1}{(1+\xi^2)(1+\theta^2)} \right\} \cdot \{\sigma_x \xi + \sigma_y \theta - d(\xi^2 + \theta^2 + \xi^2\theta^2)\} \tag{C1}$$

$$x_1 + y_2 \le d < x_2 + y_1$$

$$\pi(d;\xi) = \left( \frac{\xi}{1+\xi^2} \right) \cdot (\sigma_x - d\xi) \tag{C2}$$

$$x_2 + y_1 \le d < x_2 + y_2$$

$$\pi(d;\xi,\theta) = \left\{ \frac{\xi\theta}{(1+\xi^2)(1+\theta^2)} \right\} \cdot \{\sigma_x \theta + \sigma_y \xi - d\xi\theta\} \tag{C3}$$

One proceeds case by case in order of simplicity.

*Case (C2):*

$$\frac{d}{d\xi}\pi(d;\xi) = (1+\xi^2)^{-2} \cdot \{(1-\xi^2)\sigma_x - 2d\xi\} = 0$$

$$\xi_2^* = \sigma_x^{-1} \cdot \left\{ \sqrt{\sigma_x^2 + d^2} - d \right\}, \tag{4.2}$$

$$\pi_2^*(d) = \pi(d;\xi_2^*) = \frac{1}{2}\left\{ \sqrt{\sigma_x^2 + d^2} - d \right\}$$

To analyze when (4.2) yields a global maximum, one proceeds as in Section 3.

*Case (C3):*

$$\frac{\partial}{\partial \xi}\pi(d;\xi,\theta) = \left\{\frac{-\theta}{(1+\xi^2)^2(1+\theta^2)}\right\}\cdot\{\sigma_x\xi^2\theta + 2d\xi\theta - 2\sigma_y\xi - \sigma_x\theta\}$$
$$= 0$$
$$\frac{\partial}{\partial \theta}\pi(d;\xi,\theta) = \left\{\frac{-\xi}{(1+\xi^2)(1+\theta^2)^2}\right\}\cdot\{\sigma_y\xi\theta^2 + 2d\xi\theta - 2\sigma_x\theta - \sigma_y\xi\}$$
$$= 0$$

From the first condition one gets

$$\theta = \frac{2\sigma_y\xi}{\sigma_x\xi^2 + 2d\xi - \sigma_x}\,. \tag{4.3}$$

Inserted in the second one, one obtains the biquadratic polynomial condition

$$\sigma_x^2(1+\xi^2)^2 = 4\cdot\left\{(\sigma_x - d\xi)^2 + (\sigma_y\xi)^2\right\} \tag{4.4}$$

*Case (C1):*

$$\frac{\partial}{\partial \xi}\pi(d;\xi,\theta) = \left\{\frac{-1}{(1+\xi^2)^2(1+\theta^2)}\right\}\cdot\{\sigma_x\xi^2 + 2\sigma_y\xi\theta + 2d\xi - \sigma_x\}$$
$$= 0$$
$$\frac{\partial}{\partial \theta}\pi(d;\xi,\theta) = \left\{\frac{-1}{(1+\xi^2)(1+\theta^2)^2}\right\}\cdot\{\sigma_y\theta^2 + 2\sigma_x\xi\theta + 2d\theta - \sigma_y\}$$
$$= 0$$

Setting $\varphi = \frac{1}{\xi}$, $\psi = \frac{1}{\theta}$, $e = -d$, these conditions transform to the system of equations

$$\begin{aligned}\sigma_x\varphi^2\psi + 2e\varphi\psi - 2\sigma_y\varphi - \sigma_x\psi &= 0\,,\\ \sigma_y\varphi\psi^2 + 2e\varphi\psi - 2\sigma_x\psi - \sigma_y\varphi &= 0\,,\end{aligned} \tag{4.5}$$

which is of the same form as in case (C3) with the variables $\xi, \theta, d$ replaced by $\varphi, \psi, e$.

In numerical evaluations of case (C3) (which yields case (C1)), only those $(\xi,\theta) \in (0,\infty)\times(0,\infty)$ derived from (4.3) and (4.4) must be considered, which yield local maxima. As is well-known from standard calculus, a sufficient condition for this is

$$\frac{\partial^2}{\partial \xi^2}\pi(d;\xi,\theta) < 0$$

and

$$\frac{\partial^2}{\partial \xi^2}\pi(d;\xi,\theta)\cdot\frac{\partial^2}{\partial \theta^2}\pi(d;\xi,\theta) - \left[\frac{\partial^2}{\partial \xi\,\partial \theta}\pi(d;\xi,\theta)\right]^2 > 0\,,$$

where the functions are evaluated at the corresponding values. If the function $\pi(d;\xi,\theta)$ is concave over the set of feasible points $(\xi,\theta)$, a local maximum will automatically be a global maximum (two-dimensional analogon of the property used in Section 3).

**Example 4.1:** the mean of $X+Y$ as stop-loss deductible
Without loss of generality one can assume that $\mu_x = \mu_y = \mu = 0$. Thus one has to analyze the above formulas in case $d = 0$. In case (C3) the solution to equation (4.4) can be written as

$$\xi_3^* = \frac{1}{\sigma_x}\cdot\sqrt{2\sigma_y^2 - \sigma_x^2 + 2\sqrt{(\sigma_y^2-\sigma_x^2)^2 + \sigma_x^2\sigma_y^2}}\,, \tag{4.6}$$

and a similar formula holds in case (C1). As illustration, in the most simple situation of equal marginal variances $\sigma_y = \sigma_x$, one obtains herewith:

$$\xi_2^* = 1\,, \qquad \pi_2^*(0) = \frac{1}{2}\sigma_x \qquad \text{Case(C2)}$$

$$\xi_3^* = \theta_3^* = \sqrt{3}\,, \quad \pi_3^*(0) = \frac{3}{8}\sqrt{3}\,\sigma_x \qquad \text{Case(C3)}$$

$$\xi_2^* = \theta_2^* = \frac{\sqrt{3}}{3}\,, \quad \pi_2^*(0) = \frac{3}{8}\sqrt{3}\,\sigma_x \qquad \text{Case(C1)}$$

One checks that in case (C3) the given solution is local maximal, and that $\pi(d;\xi,\theta)$ is concave over the feasible set of points. It follows that the maximum stop-loss premium to the deductible $\mu = \mu_x + \mu_y$ of a diatomic bivariate sum with independent components, given marginal means $\mu_x, \mu_y$, and variances $\sigma_x^2 = \sigma_y^2$ equals $\frac{3}{8}\sqrt{3}\sigma_x$. This is strictly less than $\frac{1}{2}\sqrt{2}\sigma_x$, which is Bowers' univariate best upper bound. This fact has also been mentioned in Hürlimann (1993), without details however.

## 5 Best bounds from the bivariate quadratic polynomial majorant/minorant method

It is natural to ask if the previous upper stop-loss bounds for diatomic bivariate sums are best possible among all bivariate sums with fixed marginal means, variances and positive correlation. Furthermore one would also like to have best lower stop-loss bounds.

In general, one often bounds the expected value of a multivariate risk $f(\mathbf{X}) := f(X_1, \ldots, X_n)$ by constructing a multivariate quadratic polynomial

$$q(\mathbf{x}) := q(x_1, \ldots, x_n) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{i,j=1}^{n} a_{ij} x_i x_j \tag{5.1}$$

such that $q(\mathbf{x}) \geq f(\mathbf{x})$ to obtain a maximum, respectively $q(\mathbf{x}) \leq f(\mathbf{x})$ to obtain a minimum. If a multivariate finite atomic risk $\mathbf{X}$ (usually a multivariate di- or triatomic risk) can be found such that $\Pr(q(\mathbf{X}) = f(\mathbf{X})) = 1$, that is all mass points of the multivariate quadratic risk $q(\mathbf{X})$ are simultaneously mass points of $f(\mathbf{X})$, then $\mathrm{E}[q(\mathbf{X})] = E[f(\mathbf{X})]$, which depends only on the mean-variance-covariance structure, is necessarily the maximum, respectively the minimum. In the univariate case, a systematic study of this approach, which leads to an effective algorithm for the important situation of a piecewise linear function $f(x)$, together with numerous concrete examples from Insurance and Finance, has been offered by the author (1997a/97b). As a next step, we start herewith the analysis of the bivariate case by considering the concrete special bivariate stop-loss sum $f(x, y) = (x + y - T)_+$, where $T$ instead of $d$ denotes now the deductible.

It will be shown in Subsection 5.1 that a bivariate quadratic polynomial majorant is of the separating form $q(x, y) = q(x) + q(y)$, where $q(x)$, $q(y)$ are quadratic polynomials, and thus does not contain the mixed term in $xy$. In particular the maximum does not depend on the given positive correlation and is attained by complete dependence. This provides a further elementary proof of the bivariate version of the inequality of Bowers (1969) given in Hürlimann (1993). Moreover the applied method shows that the extremal stop-loss bound for diatomic bivariate sums of Section 3 for a fixed $0 < \varrho < 1$ cannot be a "global" maximum over all bivariate sums by given $0 < \varrho < 1$. Unfortunately the problem of finding a best upper stop-loss bound remains unsolved in this situation (possibly a solution does not exist at all), a question raised by Gerber at the XXII-th ASTIN Colloquium in Montreux, 1990 (comment after Theorem 2 in Hürlimann (1993)). In contrast to this the minimal stop-loss bound over all bivariate sums by known means, variances and fixed negative correlation $\varrho < 0$ exists, at least over a wide range of deductibles, as shown in Subsection 5.2. However this is less surprising. Our result shows that the trivial best lower stop-loss bound, which is again independent of $\varrho$, is attained by diatomic bivariate sums with any possible negative correlation.

### *5.1 A bivariate quadratic polynomial method*

Without loss of generality one can assume that $\mu_x = \mu_y = \mu = 0$ (see (3.8)). A bivariate quadratic polynomial majorant of $f(x,y) = (x+y-T)_+$, as defined by

$$q(x,y) = ax^2 + by^2 + cxy + dx + ey + f\,, \tag{5.2}$$

depends on 6 unknown coefficients. From Section 3 one knows that the maximum of $E[f(X,Y)]$ over arbitrary couples $(X,Y)$ with distribution $H(x,y)$ by given marginals $F(x)$ and $G(y)$ is attained at the Hoeffding-Fréchet extremal upper bound distribution $H^*(x,y) = \min\{F(x), G(y)\}$. A count of the number of unknowns and corresponding conditions (given below), which must be fulfilled in order to get a bivariate quadratic majorant, shows that the immediate candidates to consider are diatomic couples. By Section 3 restrict first the attention to a Hoeffding-Fréchet extremal diatomic distribution of the form (3.4), that is

$$\begin{gathered} p_{11} = p_1\,, \quad p_{12} = 0\,, \quad p_{21} = q_1 - p_1\,, \quad p_{22} = q_2\,, \quad p_1 \le q_1\,, \\ p_1 = \frac{x_2}{x_2 - x_1}\,, \quad q_1 = \frac{y_2}{y_2 - y_1}\,. \end{gathered} \tag{5.3}$$

Taking into account that $p_{11} = p_1 q_1 + \varrho\sqrt{p_1 p_2 q_1 q_2}$ by (2.9), one finds through comparison the constraint (3.6), which expressed in terms of the atoms yields the relation

$$y_2 = \frac{1}{\varrho} \cdot \frac{\sigma_y}{\sigma_x} \cdot x_2\,, \quad 0 < \varrho \le 1\,. \tag{5.4}$$

On the other side the equations of marginal variances imply the further constraints

$$x_1 x_2 = -\sigma_x^2\,, \quad y_1 y_2 = -\sigma_y^2\,. \tag{5.5}$$

Thus a possible extremal diatomic couple is completely specified by a single unknown atom, say $x_1$. Since $p_{12} = 0$, the relevant bivariate sum mass points are

$x_1+y_1$, $x_2+y_1$, $x_2+y_2$. Furthermore one can suppose that $x_1+y_1 \le T < x_2+y_2$ (see Section 3). Consider $z = q(x,y)$ as a quadratic surface in the $(x,y,z)$-space, and $z = f(x,y) = (x+y-T)$, as a piecewise bivariate linear function with the two pieces $z = \ell_1(x,y) = 0$ defined on the half-plane $H_1 = \{(x,y): x+y \le T\}$ and $z = \ell_2(x,y) = x+y-T$ on the half-plane $H_2 = \{(x,y): x+y \ge T\}$, then one must have $Q_1(x,y) := q(x,y) - \ell_1(x,y) \ge 0$ on $H_1$, and $Q_2(x,y) := q(x,y) - \ell_2(x,y) \ge 0$ on $H_2$. To achieve $\Pr(q(X,Y) = f(X,Y) = 1) = 1$ one must satisfy the 3 conditions

$$\begin{aligned} &Q_1(x,y) = 0 \\ &Q_1(x_2,y_1) = 0\,, \quad (x_2,y_1) \text{ in one of } \quad H_i\,,\ i = 1,2 \\ &Q_2(x_2,y_2) = 0 \end{aligned} \tag{5.6}$$

The inequalities constraints $Q_i(x,y) \ge 0$ imply that $(x_i,y_i)$ must be tangent at the hyperplane $z = \ell_i(x,y)$, $i = 1,2$, hence the 4 further conditions

$$\left.\begin{aligned} &\frac{\partial}{\partial x} Q_i(x,y)\bigg|_{(x_i,y_i)} = 0\,, \\ &\frac{\partial}{\partial y} Q_i(x,y)\bigg|_{(x_i,y_i)} = 0\,, \end{aligned}\right. \quad i = 1,2\,. \tag{5.7}$$

Together (5.6) and (5.7) imply 7 conditions for 7 unknowns (6 coefficients plus one mass point), a necessary system of equations to determine a bivariate quadratic majorant, which can eventually be solved. To simplify calculations, let us replace $q(x,y)$ by the equivalent form

$$\begin{aligned} q(x,y) = {} & a(x-x_1)^2 + b(y-y_1)^2 + c(x-x_1)(y-y_1) \\ & + d(x-x_1) + e(y-y_1) + f\,. \end{aligned} \tag{5.8}$$

The required partial derivatives are

$$\begin{aligned} q_x(x,y) &= 2a(x-x_1) + c(y-y_1) + d \\ q_y(x,y) &= 2b(y-y_1) + c(x-x_1) + e \end{aligned} \tag{5.9}$$

Then the 7 conditions above translate to the system of equations in $x_i, y_i$, $i = 1,2$:

$$q(x_1,y_1) = f = 0 \tag{C1}$$

$$q(x_2,y_1) = a(x_2-x_1)^2 + d(x_2-x_1) = (x_2+y_1-T)_+ \tag{C2}$$

$$q(x_2, y_2) = a(x_2 - x_1)^2 + b(y_2 - y_1)^2 + c(x_2 - x_1)(y_2 - y_1) \\ + d(x_2 - x_1) + e(y_2 - y_1) \\ = x_2 + y_2 - T \tag{C3}$$

$$q_x(x_1, y_1) = d = 0 \tag{C4}$$

$$q_y(x_1, y_1) = e = 0 \tag{C5}$$

$$q_x(x_2, y_2) = 2a(x_2 - x_1) + c(y_2 - y_1) = 1 \tag{C6}$$

$$q_y(x_2, y_2) = 2b(y_2 - y_1) + c(x_2 - x_1) = 1 \tag{C7}$$

In particular one has $d = e = f = 0$. The conditions (C6), (C7) can be rewritten as

$$a(x_2 - x_1) = \frac{1}{2}(1 - c(y_2 - y_1)) \tag{C6}$$

$$b(y_2 - y_1) = \frac{1}{2}(1 - c(x_2 - x_1)) \tag{C7}$$

Insert these values into (C3) to see that the following relation must hold:

$$(x_1 + y_1) + (x_2 + y_2) = 2T\,. \tag{5.10}$$

It says that the sum of the two extreme maximizing couple sums equals two times the deductible. Observe in passing that the similar constraint holds quite generally in the univariate case (cf. Hürlimann (1997a), Theorem 3.1, proof for type (D1), p. 204).

Now try to satisfy (C2). If $(x_2, y_1) \in H_1$, one must have $a = 0$, hence $c(y_2 - y_1) = 1$ by (C6), and

$$b = \frac{1}{2}\left\{\frac{(y_2 - y_1) - (x_2 - x_1)}{(y_2 - y_1)^2}\right\}$$

by (C7). Similarly, if $(x_2, y_1) \in H_2$, one obtains $a(x_2 - x_1)^2 = x_2 + y_1 - T$, hence $c(x_2 - x_1) = 1$ by (C6) using (5.10), and $b = 0$. In the first case, one has $q(x, y) = (y - y_1)(b(y - y_1) + c(x - x_1))$, and in the second one $q(x, y) = (x - x_1)(a(x - x_1) + c(y - y_1))$. In both cases the quadratic form is indefinit, which implies that the majorant constraint $q(x, y) \geq 0$ on $H_1$ or $H_2$ cannot be fulfilled. The only way to get a quadratic majorant is to disregard condition (C2), that is to set $p_{21} = 0$, hence $q_1 = p_1$ (no probability on the couple $(x_2, y_1)$). Calculations using (5.3) to (5.5), or equivalently invoking (3.9), shows that necessarily $\varrho = 1$, which is complete dependence. To get a quadratic majorant one can set $c = 0$ in (C6), (C7). Then one obtains

$$q(x,y) = \frac{1}{2}\left\{ \frac{(x-x_1)^2}{(x_2-x_1)} + \frac{(y-y_1)^2}{(y_2-y_1)} \right\}. \tag{5.11}$$

The discriminant of both $Q_i(x,y)$, $i = 1,2$, equals

$$\Delta = \frac{1}{(x_2-x_1)(y_2-y_1)} > 0\,, \tag{5.12}$$

and furthermore

$$\left.\frac{\partial^2}{\partial x^2} Q_i(x,y)\right|_{(x_i,y_i)} = \frac{1}{x_2-x_1} > 0\,, \quad i = 1,2\,. \tag{5.13}$$

By standard calculus one concludes that $Q_i(x,y)$ is positive definite, hence as required. Solving (5.10) using (5.4) and (5.5), one obtains the explicit maximizing Hoeffding-Fréchet bivariate diatomic couple $(X,Y)$, obtained differently in Hürlimann (1993), Theorem 2. It remains to discuss the form (3.5) of the Hoeffding-Fréchet extremal diatomic distribution. Replacing in the above proof the couple $(x_2, y_1)$ by $(x_1, y_2)$, one obtains similarly that condition (C2) must be disregarded, hence $p_{12} = 0$, $p_1 = q_1$ and thus $\varrho = 1$. The same maximizing couple follows. In fact the applied bivariate quadratic majorant method shows the following stronger result.

**Theorem 5.1.** *(Characterization of the bivariate stop-loss inequality)* The bivariate stop-loss sum maximizing diatomic Hoeffding-Fréchet couple (3.13), (3.14), solves the bivariate quadratic majorant stop-loss sum problem if and only if $\varrho = 1$. Its atoms and probabilities are given by (set $\sigma = \sigma_x + \sigma_y$, $\mu = \mu_x + \mu_y$)

$$\begin{aligned}
x_1 &= -\frac{\sigma_x}{\sigma}\left\{\sqrt{(T-\mu)^2+\sigma^2} - (T-\mu)\right\}, \\
x_2 &= \frac{\sigma_x}{\sigma}\left\{\sqrt{(T-\mu)^2+\sigma^2} + (T-\mu)\right\}, \\
y_1 &= -\frac{\sigma_y}{\sigma}\left\{\sqrt{(T-\mu)^2+\sigma^2} - (T-\mu)\right\}, \\
y_2 &= \frac{\sigma_y}{\sigma}\left\{\sqrt{(T-\mu)^2+\sigma^2} + (T-\mu)\right\}, \\
p_{11} &= \frac{1}{2}\left(1 + \frac{T-\mu}{\sqrt{(T-\mu)^2+\sigma^2}}\right), \quad p_{22} = 1 - p_{11}\,, \quad p_{12} = p_{21} = 0\,,
\end{aligned} \tag{5.14}$$

and the stop-loss sum maximum equals

$$\frac{1}{2}\left\{\sqrt{(T-\mu)^2+\sigma^2}-(T-\mu)\right\} . \tag{5.15}$$

**Proof.** The formulas (5.14) follow from (5.10) as explained in the text, while (5.15) follows from (5.11) by noting that $E[q(X,Y)] = \max\{E[(X+Y-T)_+]\}$. The elementary calculations are left to the reader. □

**Example 5.1:** distribution-free upper bound for the price of an exchange option

Setting $T = 0$ and changing $Y$ to $-Y$, one obtains the maximal expected value of an exchange option as

$$\max_{(X,Y)\in D(\mu_x,\sigma_x,\mu_y,\sigma_y)} \{E[(X-Y)_+]\}$$
$$= \frac{1}{2}\left\{\sqrt{(\sigma_x+\sigma_y)^2+(\mu_x-\mu_y)^2}+(\mu_x-\mu_y)\right\} .$$

In Finance Theory the exchange option has been first priced by Margrabe (1978). To get a distribution-free maximal price over a fixed period, say one-year, discount at the risk-free accumulated rate of interest $r$, and set $\mu_x = r_x$, $\mu_y = r_y$ equal to the expected accumulated returns of the random assets $X$ and $Y$. In an arbitrage-free environment, one would further set $r_x = r_y = r$ to get the maximal exchange option price

$$\max_{(X,Y)\in D(r_x,\sigma_x,r_y,\sigma_y)} \{v\cdot E[(X-Y)_+]\} = \frac{1}{2}(\sigma_x+\sigma_y)\cdot v\,, \quad v=\frac{1}{r}\,.$$

This result, which is a bivariate version of the formula (4.4) in Hürlimann (1991), is the starting point for an extension of previous work by the author (1991/96) to more complex random economics environments.

## 5.2 *Best lower bounds for bivariate stop-loss sums*

We proceed as in Subsection 5.1 with the difference that $q(x,y) \le f(x,y)$ and the fact that the minimum of $E[f(X,Y)]$ should be attained at the Hoeffding-Fréchet extremal lower bound distribution $H_*(x,y) = \max\{F(x)+G(y)-1,0\}$.

For bivariate diatomic couples with negative correlation coefficient $\varrho < 0$, two cases are possible (derivation is immediate):

*Case 1:* $p_1 + q_1 \leq 1$

$$p_{11} = 0\,, \quad p_{12} = p_1\,, \quad p_{21} = q_1 - p_1\,, \quad p_{22} = 1 - q_1 \tag{5.16}$$

*Case 2:* $p_1 + q_1 > 1$

$$p_{11} = p_1 + q_1 - 1\,, \; p_{12} = 1 - q_1\,, \; p_{21} = q_1 - p_1\,, \; p_{22} = 1 - q_1 \tag{5.17}$$

Taking into account (2.9), the form of $p_{11}$ implies the following relations:

$$\begin{aligned} y_1 &= \frac{1}{\varrho}\frac{\sigma_y}{\sigma_x}x_2 \quad \text{in Case 1,} \\ y_2 &= \frac{1}{\varrho}\frac{\sigma_y}{\sigma_x}x_1 \quad \text{in Case 2.} \end{aligned} \tag{5.18}$$

Clearly (5.5) also holds. We show first that there cannot exist a bivariate quadratic minorant with non-zero quadratic coefficients $a$, $b$, $c$, hence the minimum, if it exists, must be attained at a bivariate linear minorant. Since possibly $p_{11} = 0$ (as in Case 1), the non-trivial situation to consider is $x_1 + y_2 \leq T < x_2 + y_2$. We proceed now as in Subsection 5.1. The simplest $q(x, y)$ takes the form

$$\begin{aligned} q(x, y) &= a(x - x_1)^2 + b(y - y_2)^2 + c(x - x_1)(y - y_2) \\ &\quad + d(x - x_1) + e(y - y_2) + f\,. \end{aligned} \tag{5.19}$$

The partial derivatives are

$$\begin{aligned} q_x(x, y) &= 2a(x - x_1) + c(y - y_2) + d \\ q_y(x, y) &= 2b(y - y_2) + c(x - x_1) + e \end{aligned} \tag{5.20}$$

The following 8 conditions must hold (up to the cases where some probabilities vanish):

$$q(x_1, y_1) = a(y_2 - y_1)^2 + e(y_1 - y_2) + f = 0 \tag{C1}$$

$$q(x_1, y_2) = f = 0 \tag{C2}$$

$$\begin{aligned} q(x_2, y_1) &= a(x_2 - x_1)^2 + b(y_2 - y_1)^2 + c(x_2 - x_1)(y_1 - y_2) \\ &\quad + d(x_2 - x_1) + e(y_1 - y_2) \\ &= (x_2 + y_1 - T)_+ \end{aligned} \tag{C3}$$

$$q(x_2, y_2) = a(x_2 - x_1)^2 + d(x_2 - x_1) = x_2 + y_2 - T \tag{C4}$$

$$q_x(x_1, y_2) = d = 0 \tag{C5}$$

$$q_y(x_1, y_2) = e = 0 \tag{C6}$$

$$q_x(x_2, y_2) = 2a(x_2 - x_1) = 1 \tag{C7}$$

$$q_y(x_2, y_2) = (x_2 - x_1) = 1 \tag{C8}$$

In particular one has $d = e = f = 0$. By standard calculus, in order that $Q_i(x, y) \leq 0$ on $H_i$, $i = 1, 2$, the quadratic form $Q_i(x, y)$ must be negative definite. Therefore its discriminant, which is $\Delta = 4ab - c^2$ for both $i = 1, 2$, must be positive, and $a < 0$. But by (C7) one has $a > 0$, which shows that no such $q(x, y)$ can actually be found. Therefore the minimum must be attained for a bivariate linear form. Similarly to the univariate case, the candidates for a linear minorant are $\ell(x, y) = x + y - T$ if $T \leq 0$ and $\ell(x, y) \equiv 0$ if $T > 0$.

*Case(I):* $T \leq 0$, $\ell(x, y) = x + y - T$

Let us construct a bivariate diatomic couple with probabilities (5.16) such that

$$\begin{aligned} &\text{either} \quad x_1 + y_2 = T \leq x_2 + y_1 \leq x_2 + y_2 \,, \\ &\text{or} \quad x_2 + y_1 = T \leq x_1 + y_2 \leq x_2 + y_2 \,. \end{aligned} \tag{5.21}$$

Then one has $\Pr\left(\ell(X, Y) = (X + Y - T)_+\right) = 1$, $\ell(x, y) \leq 0 = (x + y - T)_+$ on $H_1$ and $\ell(x, y) = x + y - T = (x + y - T)_+$ on $H_2$. Together this implies that $\min\{E[(X + Y - T)_+]\} = E[\ell(X, Y)] = -T$, as desired. Let us solve (5.21) using (5.18). Three subcases are distinguished:

$$\sigma_y < \left(\frac{-1}{\varrho}\right)\sigma_x \qquad \text{(hence } \sigma_x + \varrho\sigma_y > 0\text{)} \tag{A}$$

Since $y_2 = \varrho \frac{\sigma_y}{\sigma_x} x_1$ by (5.18), the equation $x_1 + y_1 = T$ has the solution

$$\begin{aligned} x_1 &= \left(\frac{\sigma_x}{\sigma_x + \varrho\sigma_y}\right) \cdot T \,, \quad x_2 = -\frac{\sigma_x(\sigma_x + \varrho\sigma_y)}{T} \,, \\ y_1 &= -\frac{\sigma_y(\sigma_x + \varrho\sigma_y)}{T} \,, \quad y_2 = \left(\frac{\sigma_y}{\sigma_x + \varrho\sigma_y}\right) \cdot T \,. \end{aligned} \tag{5.22}$$

One checks that $T \leq x_2 + y_1 \leq x_2 + y_2$, and that $p_1 + q_1 \leq 1$ (condition for $p_{11} = 0$)

$$\sigma_y > \left(\frac{-1}{\varrho}\right)\sigma_x \tag{B}$$

Exchange $X$ and $Y$ such that $\sigma_x > (\frac{-1}{\varrho})\sigma_y$. Since $\varrho^2 \leq 1$ one gets

$$\sigma_y < (-\varrho)\sigma_x \leq \frac{1}{\varrho^2}(-\varrho)\sigma_x = \left(\frac{-1}{\varrho}\right)\sigma_x ,$$

and one concludes as in Subcase (A).

$$\sigma_x + \varrho\sigma_y = 0 \tag{C}$$

Using (5.18) one gets the relations

$$y_1 = \frac{-1}{\varrho^2}x_2 , \quad y_2 = -x_1 .$$

Setting

$$x_2 = \left(\frac{\varrho^2}{\varrho^2 - 1}\right)T$$

one obtains $x_2 + y_1 = T \leq 0 = x_1 + y_2 \leq x_2 + y_2$, which yields a diatomic couple with the property (5.21).

*Case (II):* $T > 0$, $\ell(x, y) = 0$

One must construct a bivariate diatomic couple with probabilities (5.16) such that $x_2 + y_2 \leq T$. Then all mass couples belong to $H_1$, which implies that $\Pr(\ell(X,Y) = (X + Y - T)_+) = 1$. It follows that $\min\{E[(X + Y - T)_+]\} = 0$. Using that

$$y_2 = \varrho\frac{\sigma_y}{\sigma_x}x_1 ,$$

the equation $x_2 + y_2 = T$ has the solution

$$x_2 = \frac{1}{2}\left(T + \sqrt{T^2 - 4(-\varrho)\sigma_x\sigma_y}\right) \tag{5.23}$$

provided $T \geq 2\sqrt{(-\varrho)\sigma_x\sigma_y}$. Removing the assumption $\mu_x = \mu_y = 0$ (translation of $X$ and $Y$), one obtains the following bivariate extension of the corresponding univariate result (e.g. Kaas et al.(1994), Theorem X.2.4).

**Theorem 5.2.** The minimal stop-loss premium for a bivariate sum $X + Y$ with marginal means $\mu_x, \mu_y$, variances $\sigma_x^2, \sigma_y^2$ and negative correlation $\varrho < 0$ equals $(\mu_x + \mu_y - T)_+$ provided $T \leq \mu_x + \mu_y$ or $T \geq 2\sqrt{(-\varrho)\sigma_x\sigma_y}$. It is attained by a bivariate diatomic couple with atoms as constructed above in Case (I) and Case (II).

**Acknowledgements.** The author is grateful to the referee for his critical comments and for correcting errors in numerous formulas.

## References

*Bowers, N. L.* (1969). An upper bound for the net stop-loss premium. Transactions of the Society of Actuaries XIX, 211–216.

*Dhaene, J. and Goovaerts, M. J.* (1996). Dependency of risks and stop-loss order. ASTIN Bulletin 26(2), 201–212.

*Fréchet, M.* (1951). Sur les tableaux de corrélation dont les marges sont données. Ann. Univ. Lyon Section A 14, 53–77.

*Heilmann, W.-R.* (1986). On the impact of the independence of risks on stop-loss prerniums. Insurance: Mathematics and Economics 5, 197–99.

*Hoeffding, W.* (1940). Massstabinvariante Korrelationstheorie. Schriften des Math. Instituts und des Instituts für Angewandte Mathematik der Universität Berlin 5, 179–233.

*Hürlimann, W.* (1991). Absicherung des Anlagerisikos, Diskontierung der Passiven und Portfoliotheorie. Bulletin Swiss Association of Actuaries, 217–250.

*Hürlimann, W.* (1993). Bivariate distributions with diatomic conditionals and stop-loss transforms of randorn sums. Statistics and Probability Letters 17, 329–35.

*Hürlimann, W.* (1996). Mean-Variance Portfolio Selection under Portfolio Insurance. Proceedings of the 6th International AFIR-Colloquium, vol. 1, 347–374. Verlag Versicherungswirtschaft, Karlsruhe.

*Hürlimann, W.* (1997a). Best bounds for expected financial payoffs I: algorithmic evaluation. Journal of Computational and Applied Mathematics 82, 199–212.

*Hürlimann, W.* (1997b). Best bounds for expected financial payoffs II: applications. Joumal of Computational and Applied Mathematics 82, 213–227.

*Hürlimann, W.* (1997c). Inequalities for the expected value of an exchange option strategy. Proceedings of the 7th International AFIR-Colloquium, Cairns, Australia.

*Hürlimann, W.* (1997d). Bounds for expected positive differences by means of Hoeffding-Fréchet extremal distributions. XXVIII. ASTIN Colloquium, Cairns, Australia.

*Hürlimann, W.* (1997e). Extremal Moment Methods and Stochastic Orders. Application in Actuarial Science. Monograph manuscript (available from the author).

*Kaas, R., Heerwaarden, van A. E., Goovaerts, M. J.* (1994). Ordering of Actuarial Risks. CAIRE Education Series 1, Brussels.

*Kling, B.* (1993). Life insurance, a non-life approach. Ph. D. Thesis, Tinbergen Institute Research Series no. 60, Amsterdam.

*Mardia, K. V.* (1970). Families of Bivariate Distributions. Griffin's Statistical Monographs.

*Margrabe, W.* (1978). The value of an option to exchange one asset for another. Journal of Finance 33, 177–186.

*Norberg, R.* (1989). Actuarial analysis of dependent lives. Bulletin Swiss Association of Actuaries, 243–55.

Werner Hürlimann
Mathematik KB L
"Winterthur"
Paulstr. 9
CH-8401 Winterthur

## Summary

The maximal stop-loss bounds for diatomic bivariate sums by given marginal means, variances and fixed positive correlation coefficient are determined. Based on the bivariate quadratic polynomial majorant/minorant method, it is shown that the bivariate quadratic majorant stop-loss problem can be solved by a bivariate diatomic couple if and only if its components are completely dependent. In particular one obtains a geometric more insightful proof of the bivariate stop-loss inequality of Bowers/Hürlimann (1993), which provides the best upper stop-loss bound for bivariate sums by given marginal means and variances. Some best lower stop-loss bounds are also determined, which in contrast to the upper bounds are attained by bivariate diatomic couples with any possible negative correlation. As immediate application, the maximal price of an exchange option is determined.

## Zusammenfassung

Die maximalen Stop-Loss Schranken für zweipünktigen zweifach veränderlichen Summen mit bekannten Randwerten für Erwartung und Varianz, und festem positivem Korrelationskoeffizient, werden ermittelt. Durch Anwendung einer quadratisch polynomial Majoranten/Minoranten Methode mit zwei Veränderlichen wird gezeigt, dass das zweifach veränderlich quadratische Majoranten Stop-Loss Problem genau dann lösbar ist, falls die Summenkomponenten vollständig abhängig sind. Insbesondere wird ein analytisch geometrischer Beweis der zweifach veränderlichen Stop-Loss Ungleichung von Bowers/Hürlimann (1993) gegeben, welche die beste obere Stop-Loss Schranke für zweifach veränderlichen Summen bei gegebenen Randwerten für Erwartung und Varianz liefert. Einige beste untere Stop-Loss Schranken werden ebenfalls ermittelt, welche, im Gegensatz zu den oberen Schranken, durch zweifach veränderlichen zweipünktigen Paaren rnit beliebiger negativer Korrelation erreicht werden. Als unmittelbare Anwendung wird der maximale Preis einer Austausch-Option ermittelt.

## Résumé

On détermine les bornes maximales stop-loss pour des sommes bivariées biatomiques, dont les valeurs marginales pour la moyenne et la variance, ainsi qu'un coefficient de corrélation positif, sont données. Par application d'une méthode bivariée de construction de majorantes/minorantes polynomiales quadratiques, on montre que le probème de la construction d'une majorante stop-loss quadratique bivariée est résoluble si et seulement si les composantes de la somme sont complètement dépendantes. En particulier, on obtient une démonstration géométrique plus ingénieuse de l'inégalité stop-loss bivariée par Bowers/Hürlimann (1993), qui fournit la meilleure borne supérieure stop-loss lorsque les moyennes et variances marginales de la somme sont données. Quelques meilleures bornes inférieures stop-loss sont également déterminées. Contrairement aux bornes supérieures, celles-ci sont atteintes par des paires biatomiques bivariées à corrélation négative quelconque. Comme application immédiate, le prix maximal d'une option d'échange est déterminé.